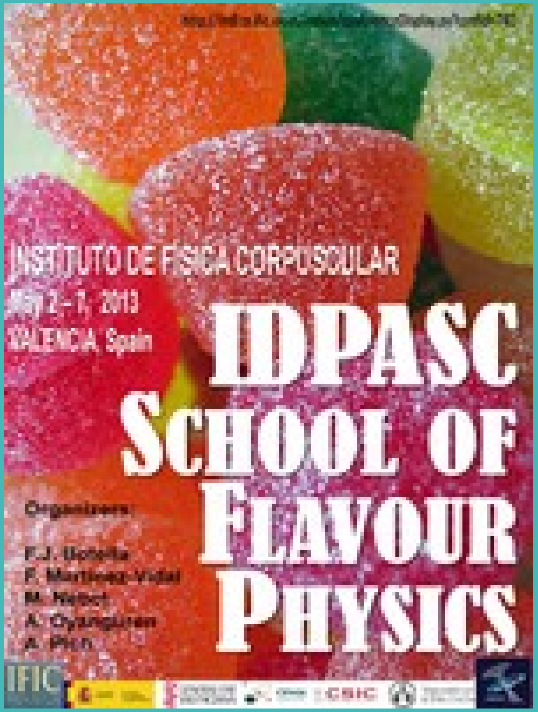
IDPASC School of Flavour Physics



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Introduction to the HQET

Vicent Giménez Gómez

Departament de Física Teòrica and IFIC.

Universitat de València-CSIC

INTRODUCTION

Introduction to the HQET

QCD has two very special properties:

Confinement: quarks are never observed as asymptotic states. *The coupling constant between quarks and gluons increases with the interquark distance (low energy region)*

Asymptotic freedom: quarks are almost free at small distances. *The coupling constant between quarks and gluons decreases with the interquark distance (high energy region)*

Therefore, at small distances PT is a good approximation but at large distances nonperturbative methods must be used.

Introduction to the HQET

A deep understanding of strong interactions is crucial.

For exemple, weak transition amplitudes involving quarks, which theoretical predictions are determined by some matrix elements of the weak hamiltonian between physical hadronic states, are very much affected by strong interaction effects acting at the confinement radius scale and higher.

What can we do ?

Some possible way-outs are:

We can use nonperturbative methods like QCDSR or Lattice.

We can use effective theories exploting the scale separation.

We can use both approaches together.

Introduction to the HQET

In this lectures, we will study an effective theory,

the HQET,

both in the continuum and on the lattice.

EFFECTIVE THEORIES

Introduction to the HQET

Effective theories: basic ideas

What is an effective theory ?

An effective theory is a tool for computing low energy processes, with a prescribed accuracy, without having to use the complete theory, which contains irrelevant degrees of freedom.

In an effective theory, one considers the relevant degrees of freedom only. The "others" are eliminated.

The resulting theory is much simpler that the complete one but still contains the essential charac-teristics of the problem at hand, up to a given energy.

Introduction to the HQET

Effective theories: basic ideas

Such a theory can be constructed if there exist different and well separated mass scales in the problem.

It is generated as an expansion for small momenta of the S-matrix elements of the complete theory.

Example:

Consider the neutron beta decay $n \rightarrow p + e + \nu$

The typical scale of the process is about $E \approx M_n \approx 1\, GeV$.

But the process is mediated by an intermediate W boson with a high virtuality and a large mass, $M_W \approx 80\, GeV$.

Introduction to the HQET

Effective theories: basic ideas

Notice that the process, at the energy scale E, receives contributions from virtual states of much higher energy.

This is a common feature of particle physics processes which is due to their intrinsic quantum nature.

These effects may be quite large because they involve large logarithms of the two very disparate scales. For example, consider the t quark effects on the W and Z mediated processes.

These logarithms can break the perturbative expansion.

Therefore we cannot simply eliminate the virtual excitations with energies higher than E.

Introduction to the HQET

Effective theories: basic ideas

The correct procedure to remove them is the following:

Neglect the W boson field and consider only the particles which appear as asymptotic states: neutron, proton, electron and neutrino.

Compensate the virtual effects of the W boson by introducing new local interactions between these light particles.

Each new interaction is the product of a nonrenormalizable higher dimension operator and a coefficient which determines its strength of the interaction.

The coefficients, or coupling constants, are chosen in such a way that, at an energy equal to the large mass scale, the low energy results of the complete theory are reproduced.

Introduction to the HQET

Effective theories: basic ideas

At tree level, this procedure consists in expanding the propagator:

$$\frac{1}{q^2 - M_W^2 + i\varepsilon} = -\frac{1}{M_W^2} - \frac{1}{M_W^2}\frac{q^2}{M_W^2} + \cdots$$

The series is rapidly convergent because

$$q^2 \approx (M_n - M_p)^2 \ll M_W^2$$

Note that on the right hand side, the W degrees of freedom do not appear anymore. Each term in the expansion can be considered as generated by a new piece in the lagrangian with a coefficient suppressed by inverse powers of the high scale, M_W .

Introduction to the HQET

Effective theories: basic ideas

So, at lowest order, the effective lagrangian is simply:

$$\mathcal{L}_{eff} = -\frac{G}{\sqrt{2}} J^{\mu} J_{\mu} \text{ with } \frac{G}{\sqrt{2}} = \frac{g^2}{8\, M_W^2}$$

where the weak current is given by:

$$J_{\mu} = \bar{u}\, \gamma_{\mu} (1 - \gamma_5)\, d + \bar{\nu}\, \gamma_{\mu} (1 - \gamma_5)\, e$$

This is the old Fermi theory for the neutron beta decay.

Each fermion-W-fermion vertex has been replaced by a four fermion interaction: the W boson has been explicitly eliminated.

The crucial property of this theory is that it is **simpler.**

Introduction to the HQET

Construction of an Effective Theory

Identify the different scales of the problem.

Suppose M is a large mass and the relevant scale of energy is E with $E < M$. We will consider M as an energy cut-off.

Removal of the heavy degrees of freedom.

All fields with masses larger than M, will be eliminated from the original theory. The physical degrees of freedom are then the fields which correspond to external (asymptotic) states.

Construction of the effective interactions.

With only these fields, add the most general possible interactions which are consistent with relativistic invariance, unitarity of the S matrix, CPT invariance and other general symmetries of the original theory. Doing this, we make sure that no quantum effect is missed or omitted.

Introduction to the HQET

Construction of an Effective Theory

We do not require the new interactions to be renormalizable because we know that the effective theory is not complete. In other words, the matrix elements of the S matrix have to be computed at tree level or at a specified number of loops.

The couplings.

Each new interaction has a coupling constant that determines its strength. Strictly speaking, the number of these couplings is infi-nite. Therefore, it seems that we have lost the predictive power of the complete theory. Actually, this is not the case, because:

We can calculate them because we **know** the complete theory.

They are **suppressed** by invers powers of the large scale.

Introduction to the HQET

Construction of an Effective Theory

$$\mathcal{L}_{eff} = \sum_{i=1}^{\infty} c_i(\mu)\, O_i(\mu) \text{ where } c_i(\mu) \sim M^{4-d_i}$$

Therefore, when the large mass M is large enough with respect to the typical energy E, higher contributions are small, the series converges quickly and we can consider the first few terms only.

Consistency and matching.

Finally, we have to impose that the complete and the effective theory give equivalent physical predictions at low energies. This is the consistency requirement.

Introduction to the HQET

Construction of an Effective Theory

The implementation is based on this idea: connect in a suitable way the effective theory couplings just below the cut-off M with those just above and impose that both couplings must match at the cut-off. This procedure is called matching:

- Choose a sufficient number of amplitudes to determine all effective couplings of the effective theory.
- Compute the amplitudes in the complete theory at a given order in perturbation theory.
- Expand them to a given order in the inverse of the large scale.
- Compute the same amplitudes in the effective theory at the scale M.
- Compare both amplitudes and determine the couplings at the scale M.

Introduction to the HQET

Construction of an Effective Theory

Renormalization Group Techniques.

Since the amplitudes are, in general, divergent, one has to renormalize them both in the complete and effective theories using suitable renormalization schemes. Therefore, the effective operators and couplings will depend on the renormalization point μ. This fact allows us to *play* the following game with it:

The requirement that bare quantities are independent of μ, allows us to write differential equations which determine the evolution of the couplings and operators on the renormalization point. These are called the Renormalization Group equations.

The matching is performed at the renormalization point, $\mu \approx M$. Hence, the couplings are obtained at this high scale first.

Introduction to the HQET

Construction of an Effective Theory

Then the RG equations are used to evolve the couplings down to the renormalization point scale $\mu \approx E$, i.e. equal to the typical energy of the processes of interest.

This procedure resums the leading logarithms $\alpha^n \log^n(M/\mu)$ which are the contributions of heavy excitations to low energy physics.

These logarithms are absorbed into the running effective coupling constants, $C_i(\mu)$.

Therefore, if one chooses the renormalization point to be close to the energy scale of interest, no large logarithms appear in the matrix elements of the effective operators.

Notice that the matching involves the computation of the anomalous dimensions of both the complete and effective theory.

Introduction to the HQET

Construction of an Effective Theory

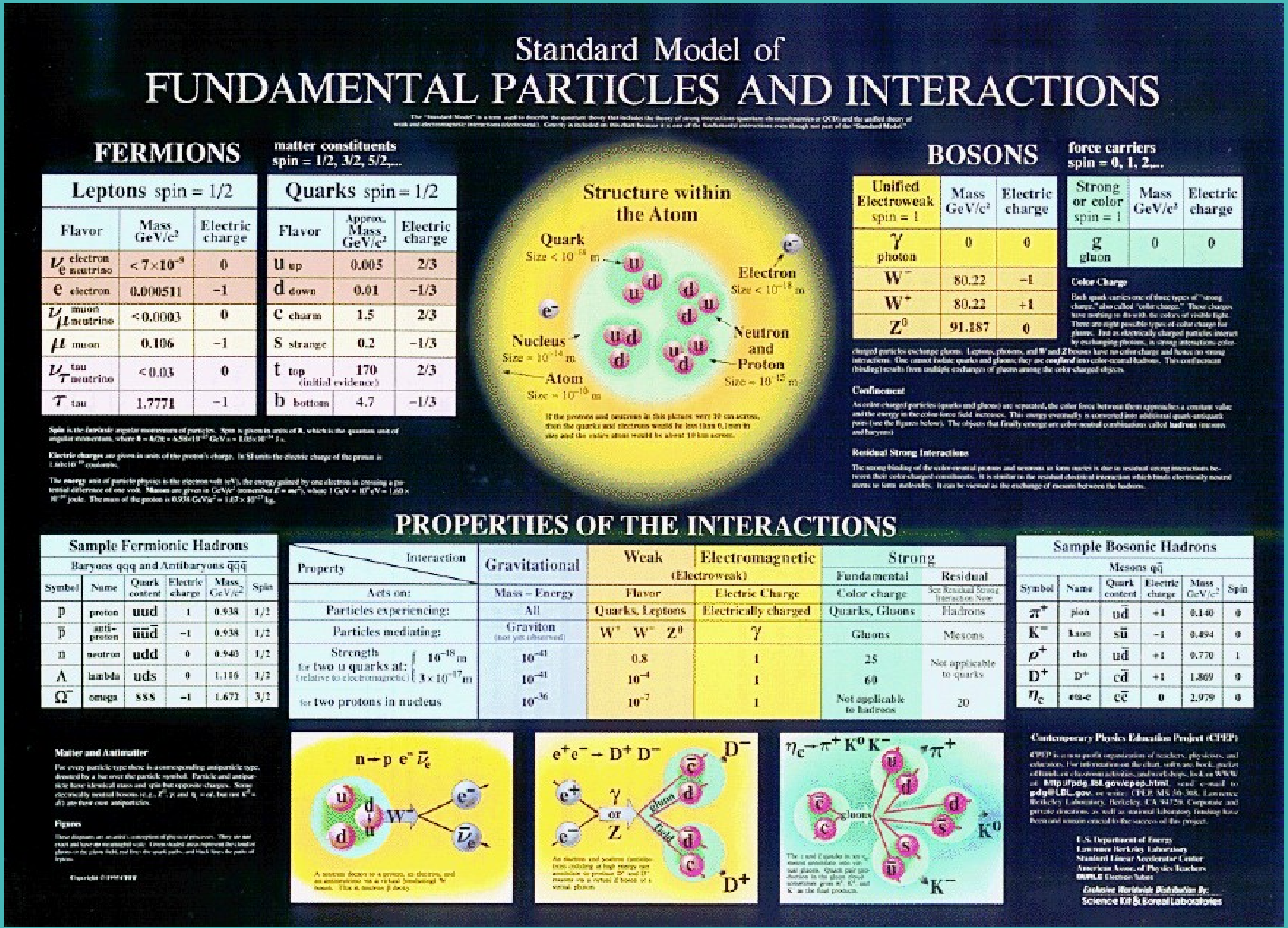
A subtle remark:

In going from the complete theory to the effective one, we have eliminated a virtual heavy particle; i.e. it does not appear in the external states. We can do that because both the energy and the momentum are small, $E \ll M \,, |\vec{p}| \ll M$.

If we want to construct an effective theory for a real heavy particle, we cannot eliminate it completely because it appears in the asymtoptic states. What we do is to remove some degrees of freedom, those that are irrelevant to the process at hand. For instance, a real particle which suffers small momentum transfer with respect to its mass, $E \approx M \,, |\vec{p}| \ll M$.

MOTIVATION OF THE HQET

Introduction to the HQET



Introduction to the HQET

Physical ideas and motivation of the HQET

Quarks can be classified according to their masses in two classes:

Light quarks: $m_q \leq \Lambda_{QCD}$ like m_u, m_d, m_s

Heavy quarks: $m_q \gg \Lambda_{QCD}$ like m_c, m_b, m_t

The properties of hadrons vary according to its quark composition. For a meson, which is a quark-antiquark bound state, we can study the following cases:

Introduction to the HQET

Physical ideas and motivation of the HQET

1) Mesons composed of two light quarks:

The size of such a meson is set by the QCD nonperturbative scale, $R \approx \Lambda_{QCD}^{-1} \approx 1\, Fermi$, which is the order of their Compton wavelength.

By the uncertainty principle, the typical momentum exchange is of the order of the meson size, $p \approx R^{-1} \approx \Lambda_{QCD} \approx 200-300\, MeV$.

This is also indicated by the success of the constituent quark model and by the argument that the exchange of high momentum gluons should be suppressed due to asymtotic freedom.

The light quarks are very far off-shell and its motion is relativistic.

The dynamics is nonperturbative but can be simulated on current lattices $a^{-1} \approx 2 \div 3\, GeV$.

The Chiral ET, an expansion in low momenta, is useful in this case.

Introduction to the HQET

Physical ideas and motivation of the HQET

2) Mesons composed of two heavy quarks:

The reduced mass of the system, that governs the dynamics, is of the order of the heavy quark mass M . This is the first scale.

An estimation of the size of the system is the Bohr radius, $R^{-1} \approx M\,\alpha_s$ where the strong coupling is small by asymptotic freedom. Therefore, the size of these mesons is much smaller than the typical size of a hadron of the type 1).

From the uncertainty principle, the typical momentum exchange can be larger than in case 1).

Since exchange of high momentum gluons is suppressed by asymptotic freedom, the motion of the quarks is nonrelativistic. In fact, the velocity can be estimated by its atomic physics analog, $v \approx \alpha_s$.

Introduction to the HQET

Physical ideas and motivation of the HQET

2) Mesons composed of two heavy quarks:

For example, $\langle v^2 \rangle \sim \frac{1}{3}$ for $c\,\bar{c}$ and $\frac{1}{10}$ for $b\,\bar{b}$. Therefore, $M \gg M\,v$, which is the second relevant scale of these systems.

Finally, based on atomic physics analogs, the Rydberg scale is a measure of the energy spacings. This is of the order of the average kinetic energy, $Rydberg \approx 1/M\,R^2 \approx M\,v^2$ and is much smaller than the other two scales because experimentaly the charmonium mass spacings (100-600 MeV) are small compared to their mass (3-4 GeV).

The dynamics can be studied by nonrelativistic potential methods.

We can also construct an effective theory to deal with the different scales: the so-called NRQCD. It can be considered as an expansion in the relative velocity of the heavy quark pair or, as pointed out by Grinstein, in the parameter $\frac{1}{c}$.

Introduction to the HQET

Physical ideas and motivation of the HQET

3) Mesons composed of a light and a heavy quark:

As in 1) the size of the system is $R \approx \Lambda_{QCD}^{-1} \approx 1\, Fermi$ and the typical momentum transfer is small $p \approx R^{-1} \approx \Lambda_{QCD} \approx 200-300\, MeV$.

The light quark is very far off-shell, by an amount of order Λ_{QCD}, and its motion is relativistic. It is affected very much by nonperturbative effects of strong interactions.

However, the heavy quark is almost on shell, because its mass is very large compared to the momentum transfer, $M \gg \Lambda_{QCD}$.

Moreover, its four velocity is almost constant, $v_Q^\mu = constant$. In fact, the momentum of the heavy quark is $p_Q = m_Q v_Q$. Since soft gluons are exchanged between the two quarks, QCD interactions can only change its momentum by a small amount. Therefore, the change in its velocity $v_Q = p_Q / m_Q$ is negligible in the infinite mass limit.

Introduction to the HQET

Physical ideas and motivation of the HQET

3) Mesons composed of a light and a heavy quark:

So, there is an approximate velocity superselection rule: low energy QCD interactions do not change the velocity of the heavy quark. In other words, heavy quarks move in straight line trajectories and thus the velocity is a good quantum number to label its states.

Furthermore, the velocity of the heavy quark is equal to the velocity of the meson, in the infinite mass limit. Indeed, consider a meson with velocity and momentum $p^\mu = M v^\mu$. Since the heavy quark mass is very large and QCD interactions are soft, $M \approx m_Q$. On the other hand, the heavy quark will carry most but not all the momentum of the meson. There will be a small residual momentum, $-k^\mu$, carried by the light degrees of freedom, low energy QCD interactions: $p_Q^\mu - k^\mu = M v^\mu \approx m_Q v^\mu$.

Introduction to the HQET

Physical ideas and motivation of the HQET

3) Mesons composed of a light and a heavy quark:

Therefore, we get the very important relation:

$$\boxed{p_Q^\mu = m_Q v^\mu + k^\mu \ \text{ with } \ |k^\mu| \sim \Lambda_{QCD} \ll m_Q}$$

Then the velocity of the heavy quark is, $v_Q^\mu = p_Q^\mu / m_Q = v^\mu + k^\mu / m_Q$. In the infinite mass limit, the velocity of the heavy quark and the one of the meson are the same: $v_Q^\mu = v^\mu \; as \; m_Q \to \infty$.

The motion of the heavy quark is nonrelativistic because, in the rest frame of the hadron, the heavy quark is almost at rest. Quark model estimates support this conclusion: $\langle v \rangle \sim \frac{1}{5}$ for D and $\frac{1}{20}$ for B mesons.

Introduction to the HQET

Physical ideas and motivation of the HQET

3) <u>Mesons composed of a light and a heavy quark:</u>

Finally, the physical picture of a heavy-light meson follows:a quark with a very large mass behaves as a source of colour moving with constant velocity (static in the rest frame of the meson) which is followed by the light colour-screening meson-cloud, also called brown muck.

The effective theory that implements this ideas is called the HQET. Formally, the HQET is the limit of QCD where the heavy quark mass goes to infinity with its velocity held fixed.

The HQET is a systematic perturbative expansion in powers of the inverse of the meson o quark mass. Notice that each order in this expansion involves all orders in the strong coupling constant.

Introduction to the HQET

Physical ideas and motivation of the HQET

3) **Mesons composed of a light and a heavy quark:**

The HQET turns out to have more symmetries than QCD. To identify them, we can use an analogy with atomic physics.

In a heavy meson, the role of the nucleus is played by the heavy quark, and the light degrees of freedom correspond to the electron and electromagnetic field.

Flavour symmetry:

If the nucleus is heavy, the atomic wavefunction is independent of the nuclear mass and the chemical properties of the atom are the same for different isotopes. So, in a heavy meson, if one replaces instantaneously the heavy quark with another heavy quark flavour, the properties of the meson does not change.

Introduction to the HQET

Physical ideas and motivation of the HQET

3) <u>Mesons composed of a light and a heavy quark:</u>

<u>Flavour symmetry:</u>

In other words, if the heavy quark is heavy enough, the light degrees of freedom only feel a static source of colour which is independent of the its mass.

<u>Spin symmetry:</u>

If the nucleus is heavy, its velocity is small. Hence, the interaction of the nucleus spin with the atomic magnetic field is very weak. Similarly, in a heavy meson, the interaction between the light degrees of freedom and the spin of the heavy quark decouples in the infinite mass limit and its properties are independent of the spin of the heavy quark.

DERIVATIONS OF THE HQET

Introduction to the HQET

Derivation from the QCD Lagrangian

Consider the quark kinetic energy term of the free action:

$$\int \frac{d^4 p}{(2\pi)^4} \overline{Q}(-p)\,(\not{p} - m_Q)\,Q(p)$$

As we said before, a heavy quark is almost on-shell and hence its momentum is:

$$p_Q^\mu = m_Q v^\mu + k^\mu \;\; \text{with} \;\; |k^\mu| \sim \Lambda_{QCD} \ll m_Q$$

Introduction to the HQET

Derivation from the QCD Lagrangian

Therefore, substituting in its Dirac equation, we have:

$$(\not{p}_Q - m_Q)\, Q(p) = 0 \quad \longrightarrow \quad (\not{v} - 1)\, Q = -\not{k}/m_Q$$
$$(\text{for } m_Q \gg \Lambda_{QCD}) \quad \longrightarrow \quad \not{v}\, Q \approx Q$$

Fields that do not satisfy this relation and momenta far from the one of the heavy quark, give a large contribution to the action and thus make a small contribution to the path integral. We define an effective heavy quark field satisfying exactly:

$$\not{v}\, Q_v = Q_v \quad \text{with} \quad Q_v(p) = Q(p) - O(1/m_Q)$$

Introduction to the HQET

Derivation from the QCD Lagrangian

Then, we can rewrite the action in terms of the effective field and the residual momenta:

$$\int \frac{d^4p}{(2\pi)^4}\,\overline{Q}(-p)\,(\not{p}-m_Q)\,Q(p) \;\approx\; \int \frac{d^4k}{(2\pi)^4}\,\overline{Q_v}(-k)\,[(\not{v}-1)\,m_Q+\not{k}]\,Q_v(k)$$

$$(\not{v}\,Q_v = Q_v) \qquad = \qquad \int \frac{d^4k}{(2\pi)^4}\,\overline{Q_v}(-k)\,[\not{k}]\,Q_v(k)$$

$$(\not{k}\,\not{v} = 2\,(k\cdot v) - \not{v}\,\not{k}) \qquad = \qquad \int \frac{d^4k}{(2\pi)^4}\,\overline{Q_v}(-k)\,[v^\mu k_\mu]\,Q_v(k)$$

Notice that the dirac structure has disappeared !!

Introduction to the HQET

Derivation from the QCD Lagrangian

The QCD interactions can now be easily incorporated by imposing gauge invariance and we get the HQET lagrangian:

$$k^\mu \rightarrow i\,D^\mu \quad \longrightarrow \quad \mathcal{L}_v = i\,\overline{Q}_v\,(v \cdot D)\,Q_v$$

The effective field annihilates heavy quarks of velocity v.

Where are the antiquarks ?

Exercise 1: derive the lagrangian for a heavy antiquark.

$$\mathcal{L}_{\bar{v}} = -i\,\overline{Q}_{\bar{v}}\,(\bar{v} \cdot D)\,Q_{\bar{v}} \quad \text{with} \quad \not{\bar{v}}\,Q_{\bar{v}} = -Q_{\bar{v}}$$

Introduction to the HQET

Derivation from the QCD Feynman rules

We start with the propagator of a heavy quark in QCD:

$$\frac{i\,(\not{p}_Q + m_Q)}{p_Q^2 - m_Q^2 + i\varepsilon} \quad \to \quad \frac{i\,m_Q\,(1+\not{v}) + i\not{k}}{k^2 + 2\,m_Q\,(v\cdot k) + i\varepsilon}$$

$$(p_Q^\mu = m_Q\,v^\mu + k^\mu) \quad \to \quad \frac{1+\not{v}}{2}\,\frac{i}{v\cdot k + i\varepsilon} + O(1/m_Q)$$

This is the propagator of a heavy quark with velocity v.

Notice that the propagator only has one pole in momentum. This fact has physical consequences: there is no pair creation !!

Exercise 2: why ?

Introduction to the HQET

Derivation from the QCD Feynman rules

Consider now the heavy-quark--gluon vertex in QCD. Taking into account that the vertex always appears between heavy-quark propagators, it is easy to obtain the heavy-quark—gluon vertex in the effective theory:

$$
\begin{aligned}
-ig\left(\frac{\lambda^a}{2}\right)_{\alpha\beta}\gamma_\mu \;\rightarrow\; & -ig\left(\tfrac{\lambda^a}{2}\right)_{\alpha\beta}\left(\tfrac{1+\not{v}}{2}\right)\gamma_\mu\left(\tfrac{1+\not{v}}{2}\right) \\
= \; & -ig\left(\frac{\lambda^a}{2}\right)_{\alpha\beta}\left[v_\mu+\gamma_\mu\left(\frac{1-\not{v}}{2}\right)\right]\left(\frac{1+\not{v}}{2}\right) \\
= \; & -ig\left(\frac{\lambda^a}{2}\right)_{\alpha\beta} v_\mu
\end{aligned}
$$

Introduction to the HQET

Derivation from the QCD Feynman rules

Notice that the effective vertex does not couple Dirac indices because the gamma matrix has disappeared.

Exercise 3: calculate the Feynman rules for heavy antiquarks. What is the origin of the change of sign with respecte to quarks?

$$\text{Propagator} \quad \rightarrow \quad \frac{1-\not{\bar{v}}}{2}\,\frac{i}{\bar{v}\cdot k + i\varepsilon}$$

$$\text{Vertex} \quad \rightarrow \quad +i\,g\left(\frac{\lambda^a}{2}\right)_{\beta\alpha}\bar{v}_\mu$$

Introduction to the HQET

Derivation from position space

Consider now the Dirac equation for the position space propagator of a heavy quark in QCD:

$$(\not{D} - m_Q)\, S_Q(x,y;A) = \delta^4(x-y)$$

According to our physical ideas, the velocity of a heavy quark is almost constant beacuse the low energy QCD interactions cannot change it. Going to the its rest frame, the heavy quark is almost at rest and its Dirac equation simplifies:

$$(D_0\gamma^0 - m_Q)\, S_h(x,y;A) = \delta^4(x-y)$$

Introduction to the HQET

Derivation from position space

In other words, we have neglected the spatial components of the momentum with respect to its energy.

Exercise 4: solve the heavy quark Dirac equation exactly.

$$
\begin{aligned}
S_h(x,y;A) \;=\; & -i\,\theta(x^0-y^0)\,e^{-im_Q(x^0-y^0)}\left(\frac{1+\gamma^0}{2}\right)P_{\vec{x}}\left[x^0,y^0\right]\delta^3(\vec{x}-\vec{y}) \\
& -\; i\,\theta(y^0-x^0)\,e^{-im_Q(y^0-x^0)}\left(\frac{1-\gamma^0}{2}\right)P_{\vec{x}}\left[x^0,y^0\right]\delta^3(\vec{y}-\vec{x})
\end{aligned}
$$

where

$$
P_{\vec{x}}\left[x^0,y^0\right] = P\exp\left[ig\int_{y^0}^{x^0} dt\,A^0(t,\vec{x})\right]
$$

Introduction to the HQET

Derivation from position space

This propagator just describes a particle sitting still and propagating only in time.

The first term represents a particle propagating forward in time (a heavy quark), while the second piece describes a particle propagating backward in time (a heavy antiquark).

There is no interference between the forward and the backward propagation. They are independent !

Both the heavy quark and the antiquark behave as a static source of colour (see the Dirac delta).

The whole interaction with the gluon field is contained in the time ordered exponential. This parallel transport the quarks.

Introduction to the HQET

Derivation from position space

Exercise 5: generalize this result to constant velocity v.

$$S_{h_v}(x,y;A) = P_{\vec{u}(x^0-y^0)}\left[x^0,y^0\right]\frac{1}{v_0}\delta^3(\vec{x}-\vec{y}-\vec{u}(x^0-y^0))\times$$
$$\left\{-i\theta(x^0-y^0)\, e^{-im_Q v\cdot(x-y)}\left(\frac{1+\not{v}}{2}\right)\right.$$
$$\left.-i\theta(y^0-x^0)\, e^{-im_Q v\cdot(y-x)}\left(\frac{1-\not{v}}{2}\right)\right\}$$

where $\vec{u}=\vec{v}/v_0$ and $P_{\vec{x}}\left[x^0,y^0\right] = P\exp\left[ig\int_{y^0}^{x^0}\frac{dt}{v_0}v\cdot A(\vec{x},t)\right]$

Introduction to the HQET

Derivation from the path integral approach

The HQET, as any effective theory, can be constructed by integrating out some irrelevant degrees of freedom but not the complete particle because it appears in external states.

To identify the irrelevant heavy degrees of freedom, is convenient to remember that a heavy quark moves with constant velocity because the interactions with the light degrees of freedom are soft. In other words, it moves as if it were free. Therefore, the space-time dependence of its field will be very close to the one of a free particle propagating with constant velocity v; that is, a plain wave.

On the other hand, we know that in the heavy (anti)quark field only the upper (lower) components are relevant.

Introduction to the HQET

Derivation from the path integral approach

These remarks suggest to introduce two new heavy quark fields to scale out the rapidly varying space-time dependence and to split, using the velocity vector, the field into upper and lower components:

$$h_v^{\pm}(x) = e^{\pm i m_Q (v\cdot x)} \left(\frac{1 \pm \not{v}}{2}\right) Q(x)$$

$$H_v^{\pm}(x) = e^{\pm i m_Q (v\cdot x)} \left(\frac{1 \mp \not{v}}{2}\right) Q(x)$$

Introduction to the HQET

Derivation from the path integral approach

Using these fields, we can reparametrize the heavy quark field in QCD as follows:

$$
\begin{aligned}
Q(x) &= e^{-im_Q(v\cdot x)}\left(h_v^+(x) + H_v^+(x)\right) \quad \text{for} \quad \text{quarks}\\
Q(x) &= e^{+im_Q(v\cdot x)}\left(H_v^-(x) + h_v^-(x)\right) \quad \text{for} \quad \text{antiquarks}
\end{aligned}
$$

Notice that this descomposition is exact. No approximation has been made yet. The first parametrization is useful to describe a **heavy quark** in a hadron because the h field has a small space-time dependence and contains the upper components only. Similarly, the second parametrization is useful for **heavy antiquarks**.

Introduction to the HQET

Derivation from the path integral approach

Consider, for example, the quark parametrization. Inserting it in the QCD lagrangian, we get:

$$\begin{aligned} \mathcal{L}_{QCD}(x) &= \overline{Q}\,(i\not{D} - m_Q)\,Q \\ &= \bar{h}_v^+\, i(v\cdot D)\, h_v^+ - \bar{H}_v^+\, \{i(v\cdot D) + 2m_Q\}\, H_v^+ \\ (V_\mu^\perp = V_\mu - (v\cdot V)\,v^\mu) \quad &+ \bar{h}_v^+\, i\not{D}^\perp\, H_v^+ + \bar{H}_v^+\, i\not{D}^\perp\, h_v^+ \end{aligned}$$

The upper component field has no mass term while the lower one has a mass twice the mass of the heavy quark. Therefore, the lower component field will give a large contribution, proportional to the heavy quark mass, to the action and thus a small contribution to the path integral. **So, the H field are the irrelevant degrees of freedom that we will integrate out.**

Introduction to the HQET

Derivation from the path integral approach

This can be done analiticaly because the integral is gaussian. The effective lagrangian turns to be:

$$\begin{aligned}\mathcal{L}'_v &= \bar{h}_v^+ \, i(v\cdot D)\, h_v^+ + \bar{h}_v^+ \, i\not{D}^\perp \left(\frac{1}{i(v\cdot D) + 2m_Q - i\varepsilon} \right) i\not{D}^\perp \, h_v^+ \\ &- \; i/2 \, \mathrm{Tr} \ln \left[i(v\cdot D) + 2m_Q - i\varepsilon \right]\end{aligned}$$

Notice that the lagrangian is not local !!

The last term is the determinant of the gaussian integration.

Exercise 6: using that the determinant is gauge invariant, show that it is just a constant and thus can be omitted.

Introduction to the HQET

Derivation from the path integral approach

Note also that the non local piece is just the solution of the classical equation of motion of the lower component field:

$$(i(v\cdot D)+2m_Q)\,H_v^+ = i\not{D}^\perp\, h_v^+ \;\longrightarrow\; H_v^+ = \frac{1}{(i(v\cdot D)+2m_Q))}\, i\not{D}^\perp\, h_v^+$$

The lowest order of the expansion in invers powers of the heavy quark mass, is the effective lagrangian of the HQET in terms of the effective upper (lower) component fields:

$$\begin{aligned}
\mathcal{L}_v &= \bar{h}_v^+ \quad i\,(v\cdot D)\,h_v^+ \quad \text{for} \quad \text{quarks} \\
\mathcal{L}_v &= \bar{h}_v^-\,(-i)(v\cdot D)\,h_v^- \quad \text{for antiquarks}
\end{aligned}$$

Introduction to the HQET

Derivation from the path integral approach

Notice that the effective fields for heavy quarks and antiquarks are independent. They are separately conserved because no interaction in the effective theory can create a pair of heavy quark-antiquark. They live in completely separated worlds. Therefore, the lagrangian is the sum of both pieces.

We have shown that the different approaches for the construction of the effective theory of a heavy quark are in perfect agreement among them. This is a check of our results.

SYMMETRIES OF THE HQET

Introduction to the HQET

Lorentz invariance of the HQET

In the HQET, the heavy quark velocity is a good quantum number because it is fixed; i.e. no low energy interaction can modify it.

In this case, we have an independent effective field for each velocity. This is the Georgi's velocity superselection rule which divides the Hilbert space in unrelated orthogonal subspaces.

The effective lagrangian is not invariant under a general Lorentz transformation due to the presence of a fixed vector.

As pointed out by Grinstein: "*This is not a surprise, since we have expanded the Green functions about one particular velocity: in boosted frames, the expansion becomes invalid because the boosted residual momentum can become arbitrarily large*"

Introduction to the HQET

Lorentz invariance of the HQET

From Grinstein's point of view, the break of Lorentz invariance is a convenient property of the effective theory. We can recover Lorent invariance simply by boosting **also** the velocity. In other words, the effective fields of different velocities are unrelated but connected by Lorentz transformations:

$$v_\mu \to \Lambda_{\mu}{}^{\nu} v_\nu \quad \text{and} \quad h_v^+(x) \to \mathcal{D}(\Lambda^{-1})\, h^+_{\Lambda^{-1} v}(\Lambda^{-1} x)$$

Georgi proposed to recover the Lorentz invariance by summing over all velocities. This does not lead to overcounting of states because the sectors of different velocity do not couple to each other so far as the low energy theory is concerned.

Introduction to the HQET

Flavour symmetry of the HQET

Consider N different heavy quark flavours, i.e. heavy quarks of different but very large masses and **all moving with the same velocity v**. We can assign an effective field to each flavour and construct the complete lagrangian as the sum

$$\mathcal{L}_{HQET} = \sum_{j=1}^{N} \bar{h}_v^{(j)+}\, i(v \cdot D)\, h_v^{(j)+}$$

To determine the symmetries of this lagrangian, it is convenient to put the fields of different flavours together into a column vector of 4N components,

Introduction to the HQET

Flavour symmetry of the HQET

$$\vec{h}_v^{+} \equiv \begin{pmatrix} h_v^{(1)+} \\ h_v^{(2)+} \\ \vdots \\ h_v^{(N)+} \end{pmatrix} \longrightarrow \mathcal{L}_{HQET} = \bar{\vec{h}}_v^{+}\, i(v\cdot D)\,\vec{h}_v^{+}$$

Since the HQET lagrangian does not contain any mass term, we can rotate all fields **simultaneously** by a unitary transformation that does not change the lagrangian,

Introduction to the HQET

Flavour symmetry of the HQET

$$\begin{array}{lcl} \vec{h}_v^{+} & \to & U\,\vec{h}_v^{+} \\ \bar{\vec{h}}_v^{+} & \to & \bar{\vec{h}}_v^{+}\,U^{\dagger} \end{array} \qquad \text{with} \quad U \in SU(N)$$

This is the so-called **flavour symmetry of the HQET**.

<u>Some remarks are in order here</u>:

This is not a symmetry of QCD, which is invariant under a U(1) abelian transformation for each flavour.

If the velocity of the heavy quarks is different, this transformation is not a symmetry.

The masses of the flavours can be very different but large.

Introduction to the HQET

Spin symmetry of the HQET

Consider the lagrangian of the HQET

$$\begin{aligned} \mathcal{L}_v &= \bar{h}_v^+ \quad i \,(v\cdot D)\, h_v^+ \quad \text{for} \quad\quad \text{quarks} \\ \mathcal{L}_v &= \bar{h}_v^- \,(-i)(v\cdot D)\, h_v^- \quad \text{for antiquarks} \end{aligned}$$

Since there is no Dirac structure, this lagrangian is diagonal in the spinorial indices. Therefore, a unitary transformation acting on the spinorial indices will be a symmetry of the lagrangian,

$$\begin{aligned} \left(h_v^+\right)_\alpha(x) &\to S_{\alpha\beta}\left(h_v^+\right)_\beta(x) \\ \left(\bar{h}_v^+\right)_\alpha(x) &\to \left(\bar{h}_v^+\right)_\beta(x)\, S^\dagger_{\alpha\beta} \end{aligned}$$

Introduction to the HQET

Spin symmetry of the HQET

What is this unitary transformation? It cannot be a complete Lorentz transformation because the lagrangian is not invariant under a general boost. However, any restricted Lorentz transformation that does not change the velocity v, will be a symmetry of the HQET Lagrangian.

To get a clue to the form of S, it is convenient to go to the rest frame of the heavy quark. The most general unitary transformation of the spinorial indices that keep the velocity unchanged, is a three-dimensional rotation which generators are,

$$\left(\vec{S}^{\pm}_{v_r}\right)_i = \frac{1}{2}\begin{pmatrix} \sigma_i & 0 \\ 0 & \sigma_i \end{pmatrix}\left(\frac{1\pm\gamma^0}{2}\right) = \frac{1}{2}\vec{\Sigma}_i \quad with \quad v_r = (1,\vec{0})$$

Introduction to the HQET

Spin symmetry of the HQET

Therefore, the HQET static lagrangian has a SU(2) spin symmetry. Going back, by a Lorent transformation, to the frame in which the heavy quark is moving with velocity v,

$$\left(\vec{S}_v^{\pm}\right)_i = \frac{i}{4}\varepsilon_{ijk}\left[\not{\varepsilon}_j, \not{\varepsilon}_k\right]\left(\frac{1\pm\not{v}}{2}\right) \quad \text{with } \varepsilon_{i\mu}\varepsilon_j^{\mu} = -\delta_{ij},\ v_\mu\varepsilon_i^{\mu} = 0$$

These are the generators of a SU(2) symmetry of the HQET lagrangian **for each velocity v** and for quarks and antiquarks. This symmetry is called the spin symmetry of the HQET.

Notice that it is an internal symmetry because no transformation of the coordinates is performed (or needed).

Introduction to the HQET

Tensor analysis and covariant states

Symmetries are useful to obtain relations between matrix elements and physical quantities.

To exploit the combination of the flavour and spin symmetries of the HQET, a tensor calculus have been devised. This method is powerful and permits to obtain the relations due to the symmetry in a simple way.

In this short course, we do not have time to explain it. Therefore, we choose to present an alternative method, by Aglietti and Martinelli, to obtain the symmetry results which is closer to lattice ideas. In this method, we will use the propagator in coordinate space of a heavy (anti)quark.

1/m CORRECTIONS TO THE HQET

Introduction to the HQET

1/m corrections to the Lagrangian

The following equation allows us to systematicaly find higher order corrections in 1/m to the lagrangian of the HQET,

$$\begin{aligned}\mathcal{L}'_v &= \bar{h}_v^+\, i(v\cdot D)\, h_v^+ + \bar{h}_v^+\, i\not{D}^\perp \left(\frac{1}{i(v\cdot D) + 2m_Q - i\varepsilon}\right) i\not{D}^\perp\, h_v^+ \\ &- i/2\,\mathrm{Tr}\ln\left[i(v\cdot D) + 2m_Q - i\varepsilon\right]\end{aligned}$$

In fact, expanding the non local piece in powers of 1/m, we get

$$\mathcal{L}_{HQET} = \bar{h}_v^+\, i(v\cdot D)\, h_v^+ + \bar{h}_v^+\, i\not{D}^\perp \left(\frac{1}{2m_Q}\right) i\not{D}^\perp\, h_v^+ + O(1/m_Q^2)$$

Introduction to the HQET

1/m corrections to the Lagrangian

Exercise 7: demostrate the identity,

$$\bar{h}_v^+ \not{D}^\perp \not{D}^\perp h_v^+ = \bar{h}_v^+ D_\perp^2 h_v^+ + \frac{g}{2} \bar{h}_v^+ \sigma_{\mu\nu} G^{\mu\nu} h_v^+$$

Using the exercise 6, we get the HQET lagrangian up to 1/m

$$\mathcal{L}_{HQET} = \bar{h}_v^+ i(v \cdot D) h_v^+ - \frac{1}{2m_Q} \bar{h}_v^+ D_\perp^2 h_v^+ - g \bar{h}_v^+ \left(\frac{\sigma^{\mu\nu} G_{\mu\nu}}{4m_Q} \right) h_v^+$$

The first correction is the heavy quark kinetic energy while the second one is the chromomagnetic moment interaction.

Introduction to the HQET

1/m corrections to the Lagrangian

The first term is the covariant expression of the kinetic energy of the heavy quark. Indeed, in the rest frame it corresponds to the non relativistic kinetic energy.

It breaks the flavour symmetry but not the spin symmetry.

The second term is the covariant expression of the magnetic moment interaction because, in the rest frame, it reduces to the Pauli (chromomagnetic) term.

It breaks both the flavour and spin symmetries because couples the heavy quark spin to the chromomagnetic field.

Introduction to the HQET

1/m corrections to the currents

Consider first a heavy-heavy current in QCD,

$$O(x) = \bar{Q}_1(x)\,\Gamma\, Q_2(x)$$

To obtain the corresponding effective current, we split each field into its upper and lower components and scale out its space-time plain wave dependence,

$$O(x) = e^{-ix\cdot(m_2 v_2 - m_1 v_1)} \left(\bar{h}_{v_1}^{(Q_1)+}(x) + \bar{H}_{v_1}^{(Q_1)+}(x)\right) \Gamma \left(h_{v_2}^{(Q_2)+}(x) + H_{v_2}^{(Q_2)+}(x)\right)$$

Introduction to the HQET

1/m corrections to the currents

Then, we formally write the lower component fields in terms of the upper component ones using the equation,

$$H_{v_{1,2}}^{(Q_{1,2})+}(x) = \left(\frac{1}{i(v_{1,2}\cdot D) + 2m_{1,2} - i\varepsilon}\right) i\not{D}^{\perp}\, h_{v_{1,2}}^{(Q_{1,2})+}(x)$$

This is equivalent to integrate out the lower component fields in the path integral. Finally, we expand the current and obtain,

$$\begin{aligned} O(x) &= e^{-ix\cdot(m_2 v_2 - m_1 v_1)}\left[\bar{h}_{v_1}^{(Q_1)+}(x)\,\Gamma\, h_{v_2}^{(Q_2)+}(x) - \frac{i}{2m_1}\bar{h}_{v_1}^{(Q_1)+}(x)\,\overleftarrow{\not{D}}\,\Gamma\, h_{v_2}^{(Q_2)+}(x)\right. \\ &+ \left.\frac{i}{2m_2}\bar{h}_{v_1}^{(Q_1)+}(x)\,\Gamma\,\not{D}\, h_{v_2}^{(Q_2)+}(x)\right] + O\left(\frac{1}{m_{1,2}^2}, \frac{1}{m_1 m_2}\right) \end{aligned}$$

Introduction to the HQET

1/m corrections to the currents

Consider now a light-heavy current,

$$O(x) = \bar{q}(x)\,\Gamma\, Q(x)$$

Exercise 8: show that the effective current is

$$O(x) = e^{-ix\cdot m_Q v_Q}\left[\bar{q}(x)\,\Gamma\, h_{v_Q}^{(Q_1)+}(x) + \frac{i}{2m_Q}\bar{q}(x)\,\Gamma\, \not{D}^{\perp}\, h_{v_Q}^{(Q)+}(x)\right] + O\!\left(\frac{1}{m_Q^2}\right)$$

DECAY CONSTANTS AND FORM FACTORS

Introduction to the HQET

Two-point Green functions

Consider the euclidean correlation function,

$$C_2(t) = \int d^3x\, e^{\vec{P}\cdot\vec{x}} < 0 | \, T \left\{ O_B(\vec{x},t)\, O_B(\vec{0},0)^\dagger \right\} |0>$$

where

$$O_B(0)^\dagger \ \text{ creates from the vacuum a state } \ |B> \ \rightarrow \ < B | \, O_B(0)^\dagger \, |0> \neq 0$$

The quantum numbers of the state are the same as those of the interpolating operator.

Introduction to the HQET

Two-point Green functions

Inserting a complete set of intermediate states,

$$\sum_{\alpha} \int \frac{d^3 p}{(2\pi)^3\, 2E_p} |\vec{p},\alpha><\vec{p},\alpha| = I$$

and taking into account that at large time distances, only the lightest state with the quantum numbers of the interpolating operator, gives a sizable contribution, we have

$$C_2(t) \to \frac{e^{-E_P t}}{2E_P} |<0|\, O_B(0)\, |B(P)>|^2 + \text{exponentially suppressed}$$

Introduction to the HQET

Two-point Green functions

On the other hand, by using the Feynman Path Integral formalism, the vacuum expectation value of a product of operators is equal to the average,

$$< O_1(x_1) \cdots O_n(x_n) > \equiv \frac{1}{Z} \int \mathcal{D}A\, \mathcal{D}q\, \mathcal{D}\bar{q}\, O_1(x_1) \cdots O_n(x_n)\, e^{-S(A,q,\bar{q})}$$

Writting the vacuum matrix element in the two-point Green function as an average, we get the important relation,

$$\frac{e^{-E_P t}}{2E_P} \left| < 0 | O_B(0) | B(P) > \right|^2 \rightarrow \int d^3x\, e^{\vec{P}\cdot\vec{x}} < O_B(\vec{x},t)\, O_B(\vec{0},0)^\dagger > \quad \text{as} \;\; t \rightarrow \infty$$

Introduction to the HQET

Two-point Green functions

This equation is the master relation we will use as a theoretical tool to study the masses and decay constants of mesons and the relations among them that the flavour and spin symmetries of the HQET generate.

It is also used as a numerical tool in numerical simulations. The idea is to compute the right hand side on a lattice space-time for large enough time slices. Comparing (actually, by fitting) the numerial data to the exponential behaviour of the left hand size, we can obtain both the mass of the state and its operator matrix element. This procedure can be applied not only to QCD but also to the HQET and other effective theories.

Introduction to the HQET

Heavy meson binding energy

In the rest frame of the heavy meson, the heavy quark is almost at rest, too. Therefore, we can use the static effective propagator in coordinate space we have calculated before,

$$< O_B(x) O_B(0)^\dagger > = \frac{1}{Z} \int \mathcal{D}A \, \mathrm{Tr} \left(S_q((\vec{0},t),(\vec{0},0)) \, \Gamma \, \frac{1-\gamma^0}{2} \, \Gamma \, P_{\vec{0}}[0,t] \right) e^{-m_Q t} \, e^{-S(A)}$$

Substituting this into the master relation, it follows that

$$\frac{e^{-(M_B - m_Q)t}}{2M_B} \left| < 0 | \, O_B(0) \, | B > \right|^2 = \frac{1}{Z} \int \mathcal{D}A \, \mathrm{Tr} \left(S_q((\vec{0},t),(\vec{0},0)) \, \Gamma \, \frac{1-\gamma^0}{2} \, \Gamma \, P_{\vec{0}}[0,t] \right) e^{-S(A)}$$

Notice that the rhs is independent of the heavy masses !!

Introduction to the HQET

Heavy meson binding energy

Let us consider now two types of heavy mesons: pseudoscalars, B, and vectors, B*.

The key observation is that **the left hand side is independent of both the heavy meson and the heavy quark mass.** However, it depends, in a very complicated way, on the light degrees of freedom and its interactions; the brown muck.

Since the temporal dependence must be the same on both sides, the binding energies are the same and are also independent of the heavy quark mass,

$$\begin{array}{lcl} \Gamma = \gamma^0\gamma^5 & \to & M_B - m_Q \equiv \overline{\Lambda} \\ \Gamma = \gamma^i & \to & M_{B^*} - m_Q \equiv \overline{\Lambda}^* \end{array} \qquad \text{where } \overline{\Lambda} = \overline{\Lambda}^* \text{ and } \overline{\Lambda} \neq \overline{\Lambda}(m_Q)$$

Introduction to the HQET

Heavy meson decay constants

We define the pseudoscalar and vector decay constants, calculated in the HQET and denoted with a tilde, as

$$\begin{aligned} < 0|\, O_B(0)\, |B > &= M_B\, \tilde{f}_B \\ < 0|\, O^{(i)}_{B^*}(0)\, |B^*, \varepsilon > &= \varepsilon^i\, \frac{M^2_{B^*}}{\tilde{f}_{B^*}} \end{aligned}$$

where ε is the polarization of B^*

Again we apply that the right hand side is independent of the heavy quark and meson masses to arrive at the scaling laws

$$| < 0|\, O_B(0)\, |B(B^*) > |^2 / M_{B(B^*)} = \Lambda^{(*)} \neq \Lambda^{(*)}(m_Q) \rightarrow \begin{cases} \tilde{f}_B = \dfrac{\Lambda}{\sqrt{M_B}} \\ \tilde{f}_{B^*} = \Lambda^{(*)} M^{3/2}_{B^*} \end{cases}$$

Introduction to the HQET

Heavy meson decay constants

Of course, this scaling laws are also valid for the D and D* mesons, at least approximately. Since the constants that appear in the scaling laws are independent of the heavy quark mass, we have

$$\frac{\tilde{f}_B}{\tilde{f}_D} = \sqrt{\frac{M_D}{M_B}} \quad \text{and} \quad \left(\frac{\tilde{f}_{B^*}}{\tilde{f}_{D^*}}\right)^2 = \left(\frac{M_{B^*}}{M_{D^*}}\right)^3$$

Furthermore, using the trivial identity

$$\left(\gamma^0\gamma^5\right)\frac{1-\gamma^0}{2}\left(\gamma^0\gamma^5\right) = \gamma^i\,\frac{1-\gamma^0}{2}\gamma^i = -\frac{1+\gamma^0}{2}$$

Introduction to the HQET

Heavy meson decay constants

we obtain, from the relation between pseudoscalar and vector,

$$\frac{e^{-(M_B-m_Q)t}}{2M_B}\,|<0|\,O_B(0)\,|B>|^2=\sum_{r=1}^{3}\frac{e^{-(M_{B^*}-m_Q)t}}{2M_{B^*}}\,|<0|\,O_{B^*}^{(i)}(0)\,|B^*,\varepsilon>|^2$$

Comparing the time dependent and independent terms, we get the following complementary relations,

$$M_B=M_{B^*}\ \text{ and }\ \frac{\tilde{f}_B\cdot\tilde{f}_{B^*}}{M_B}=1$$

Introduction to the HQET

Semileptonic decays of heavy quarks

The semileptonic decays of D and B mesons are a rich source of information on the properties of the weak interaction; for exemple and specially, on the CKM matrix elements.

Consider this very important example:

$$B(p_B) \to D^{(*)}(p_{D^{(*)}})\, e(p_e)\, \bar{\nu}_e(p_{\bar{\nu}_e})$$

In order to see if the HQET is appropriate, we study the kinematics of the decay.

Introduction to the HQET

Semileptonic decays of heavy quarks

Exercise 10: show that the momentum transfer is in the range

$$q^2 \equiv (p_e + p_{\bar{\nu}_e})^2 = (p_B - p_{D^{(*)}})^2 \;\; \rightarrow \;\; 0 \leq q^2 \leq (M_B - M_{D^{(*)}})^2$$

The maximum value of the square of the momentum transfer corresponds to the zero recoil kinematic point where the D is at rest in the rest frame of the B meson.

The minimum value occurs when the D is recoiling maximally.

As can be seen, the momentum transfer can be large. But this is not the important parameter to decide the applicability of the HQET.

Introduction to the HQET

Semileptonic decays of heavy quarks

What we have to verify is if the momentum tranfer **to the light degrees of freedom** is much less than the heavy quark mass.

Exercise 11: show that the velocities of the B and D mesons satisfy the constraints

$$1 \leq v_B \cdot v_D \leq \frac{M_B^2 + M_{D^{(*)}}^2}{2 M_B M_{D^{(*)}}} \longrightarrow 0 \leq (v_B - v_D)^2 \leq 1.2$$

Introduction to the HQET

Semileptonic decays of heavy quarks

The momentum transfer to the light degrees of freedom can be estimated to be:

$$q_l^2 \sim (\Lambda_{QCD}\, v_D - \Lambda_{QCD}\, v_B)^2 = 2\Lambda_{QCD}^2\,((v_B \cdot v_D) - 1) \sim \Lambda_{QCD}^2$$

Therefore, we can use the HQET to study this decay.

Let us calculate its width. The process is mediated by a W boson that can be integrated out and thus we can describe the decay, as far as weak interactions is concerned, by the weak four fermion lagrangian.

Introduction to the HQET

Semileptonic decays of heavy quarks

The width of this decay is proportional to the square of the folowing B to D matrix element,

$$\langle D^{(*)}(p_{D^{(*)}})\,|\,\bar{c}\gamma^{\mu}(1-\gamma^{5})\,b\,|\,B(p_B)\rangle$$

Exercise 12: show that using Lorentz invariance, parity and time reversal symmetries, the matrix elements of the vector current and the axial vector current, can be written in terms of six unknown form factors.

Introduction to the HQET

Semileptonic decays of heavy quarks

$$\frac{< D, v_D|\,\bar{c}\gamma^\mu b\,|B, v_B >}{\sqrt{M_B M_D}} = f_+(\omega)\,(v_B + v_D)^\mu + f_-(\omega)\,(v_B - v_D)^\mu$$

$$< D, v_D|\,\bar{c}\gamma^\mu\gamma^5 b\,|B, v_B > = 0$$

$$\frac{< D^*, v_D, \varepsilon|\,\bar{c}\gamma^\mu b\,|B, v_B >}{\sqrt{M_B M_D}} = i\,\varepsilon^{\mu\alpha\beta\gamma}\varepsilon^*_\alpha\,(v_D)_\beta\,(v_B)_\gamma\,f_V(\omega)$$

$$\begin{aligned}\frac{< D^*, v_D, \varepsilon|\,\bar{c}\gamma^\mu\gamma^5 b\,|B, v_B >}{\sqrt{M_B M_D}} &= -f_{A_2}(w)\,(v_B)^\mu(\varepsilon^*\cdot v_B) - f_{A_3}(\omega)\,(v_D)^\mu(\varepsilon^*\cdot v_B)\\ &+ f_{A_1}(\omega)\,\varepsilon^{\mu *}\,(1 + v_B\cdot v_D)\end{aligned}$$

where $\omega = v_B \cdot v_D$..

Introduction to the HQET

Semileptonic decays of heavy quarks

Using these form factors, it is easy to see that the differential decay width is proportional to a complicated combination of them,

$$\frac{d\Gamma}{d\omega} = \frac{G_F^2 M_B^5}{192\pi^3} r^3 (1-r)^2 (\omega^2-1)^{\frac{1}{2}} (1+\omega)^2 \times \left[1 + \frac{4\omega}{1+\omega} \frac{1-2\omega r + r^2}{(1-r)^2}\right] |V_{cb}|^2 \, |F_{B\to D^*}(\omega)|^2$$

Where $r = m_D / m_B$. At this point the HQET can be used to find relations between the form factors and simplify this equation.

We anticipate the result, all the non zero form factors are equal !!

Introduction to the HQET

Semileptonic decays of heavy quarks

$$
\begin{aligned}
f_+(\omega) = f_V(\omega) = f_{A_1}(\omega) &= f_{A_3}(\omega) = \xi(\omega) \\
f_-(\omega) = f_{A_2}(\omega) = 0 &\rightarrow F_{B\to D^*}(\omega) = \xi(\omega) \text{ and } \xi(1) = 1
\end{aligned}
$$

The single form factor is the so-called the Isgur-Wise function.

Moreover, this unknown function has an absolute normalization that can be used to calculate the CKM matrix element V_{cb}.

In fact, by extrapolating the experimental data for the differential decay width to the non recoil point, we can calculate this CKM matrix.

Introduction to the HQET

Three-point Green functions

Consider the three-point euclidean correlation function,

$$C_3(t_1,t_2) = \int d^3x_1\, d^3x_2\, e^{-\vec{P}_1\cdot\vec{x}_1}\, e^{\vec{P}_2\cdot\vec{x}_2} < 0|\, T\left\{ O_{B_1}(\vec{x}_1,t_1)\, J(0)\, O^{\dagger}_{B_2}(\vec{x}_2,t_2)^{\dagger} \right\} |0>$$

where the operators $O_{B_{1,2}}$ are the interpolating operators which annihilate the states $B_{1,2}$ and J is some current with the correct quantum numbers.

On the one hand, we can insert two complete sets of intermediate states in the T-ordered product, take the limits $t_1 \to \infty$ and $t_2 \to -\infty$ and use that excited states are exponentially suppressed. On the other hand, the T-product can be written as a path integral average. Putting all together, we have

Introduction to the HQET

Three-point Green functions

$$\frac{e^{-E_{P_1}t_1+E_{P_2}t_2}}{2E_{P_1}\,2E_{P_2}} < 0|\,O_{B_1}(0)\,|B_1(P_1)>< B_1(P_1)|\,J(0)\,|B_2(P_2)>< B_2(P_2)|\,O_{B_2}(0)^\dagger\,|0> \;\rightarrow$$
$$\int d^3x_1\,d^3x_2\,e^{\vec{P}_1\cdot\vec{x}_1-\vec{P}_2\cdot\vec{x}_2} < O_{B_1}(\vec{x}_1,t_1)\,J(0)\,O_{B_2}(\vec{x}_2,t_2)^\dagger > \; t_1\rightarrow\infty,\, t_2\rightarrow-\infty$$

This will be our master equation for the study of three-point functions in the HQET.

Notice also that this equation is basically what we use in the analysis of Lattice data.

Introduction to the HQET

Form factors of heavy-heavy currents

Consider two heavy mesons, a D-like meson moving with velocity v_D and a B-like meson moving with velocity v_B . The corresponding interpolating operators and the heavy-heavy current which mediates the transition are

$$\tilde{J}(x) = \bar{h}_{v_D}^{(c)+}(x)\,\Gamma\, h_{v_B}^{(b)+}(x), \quad O_D(x) = \bar{q}(x)\,\Gamma_D\, h_{v_D}^{(c)+}(x), \quad O_B(x) = \bar{q}(x)\,\Gamma_B\, h_{v_B}^{(b)+}(x)$$

The Dirac matrices Γ_D and Γ_B are chosen to describe pseudoscalar or vector mesons, (ε is the polarization of vector $B\,(D)$)

$$\Gamma_{B,D} = \gamma^0\gamma^5 \ \text{ for pseudoscalar} \qquad \Gamma_{B,D} = \not{\varepsilon} \ \text{ for vector}$$

Introduction to the HQET

Form factors of heavy quark currents

As usual, the right hand side of the master equation can be written in terms of heavy and light-quark propagators by funtional integrating over the fermionic variables,

$$\begin{aligned} < O_D(\vec{x}_D, t_D)\, \tilde{J}(0)\, O_B(\vec{x}_B, t_B)^\dagger > \;&=\; - < \mathrm{Tr}\left\{ S^{(c)}_{h_{v_D}}((\vec{x}_D, t_D), (\vec{0}, 0))\, \Gamma \right. \\ &\times\; S^{(b)}_{h_{v_B}}((\vec{0}, 0), (\vec{x}_B, t_B))\, \gamma^0 \Gamma_B^\dagger \gamma^0 \\ &\times\; \left. S_q((\vec{x}_B, t_B), (\vec{x}_D, t_D))\, \Gamma_D \right\} > \end{aligned}$$

Exercise 13: using the propagator for an effective heavy quark moving with four-velocity v, rewrite the master equation for three-point functions as,

Introduction to the HQET

Form factors of heavy quark currents

$$< D, v_D|\tilde{J}(0)|B, v_B> = K\,\mathrm{Tr}\left[\frac{1+\not{v}_B}{2}\gamma^0\Gamma_B^\dagger\gamma^0 L(v_D, v_B)\,\Gamma_D\,\frac{1+\not{v}_D}{2}\Gamma\right]$$

where

$$\begin{aligned}
K &\equiv \frac{4M_B M_D}{< 0|O_D(0)|D, v_D >< 0|O_B(0)|B, v_B >^*} \\
L(v_D, v_B) &\equiv -\exp\left[\overline{\Lambda}\left(\frac{t_D}{\gamma_D} - \frac{t_B}{\gamma_B}\right)\right] < P_b\,S_q((\vec{v}_B t_B, t_B), (\vec{v}_D t_D, t_D))\,P_c > \\
P_c &= P\exp\left[ig\int_0^{t_D} v_D\cdot A(\vec{v}_D t, t)\,\frac{dt}{\gamma_D}\right] \\
P_b &= P\exp\left[ig\int_{t_B}^{0} v_B\cdot A(\vec{v}_B t, t)\,\frac{dt}{\gamma_B}\right]
\end{aligned}$$

Introduction to the HQET

Form factors of heavy quark currents

The dirac-color matrix L is a very complicated functional integral of the light degrees of freedom over the gauge field.

The crucial observation is that, acording to the equation above, L must be independent of both times.

Therefore, L can only be a function of the velocities. Lorentz invariance tells us that the most general form of L is

$$L = L_1(v_D \cdot v_B) + L_2(v_D \cdot v_B)\not{v}_B + L_3(v_D \cdot v_B)\not{v}_D + L_4(v_D \cdot v_B)\not{v}_B\not{v}_D$$

where the L's form factors are unknown color matrices which depend on the only scalar we can form with the quantities we have at our disposal.

Introduction to the HQET

Form factors of heavy quark currents

Exercise 14: show that higher powers of the velocity do not introduce linearly independent form factors. Why do not appear terms proporcional to γ^5 ?

$B \to D$ Decays:

In this case we have a pseudoscalar B meson which decays into a pseudoscalar D meson. The currents we are interested in are the vector and the axial heavy-heavy current. Inserting the corresponding dirac matrices in the equation for L, we get

$$
\begin{aligned}
< D, v_D | \bar{h}_{v_D}^{(c)+} \gamma^\mu h_{v_B}^{(b)+} | B, v_B > &= \sqrt{M_B M_D} (v_B + v_D)^\mu \xi(v_B \cdot v_D) \\
< D, v_D | \bar{h}_{v_D}^{(c)+} \gamma^\mu \gamma^5 h_{v_B}^{(b)+} | B, v_B > &= 0
\end{aligned}
$$

Introduction to the HQET

Form factors of heavy quark currents

$$\xi(v_B \cdot v_D) \equiv \frac{K}{\sqrt{M_B M_D}} \mathrm{Tr}(L_1 - L_2 - L_3 + L_4)$$

This is the famous Isgur-Wise function.

$B \rightarrow D^*$ **Decays:**

In this case we have a pseudoscalar B meson which decays into a vector D meson. The currents we are interested in are the vector and the axial heavy-heavy current. Inserting the corresponding dirac matrices in the equation for L, we get

Introduction to the HQET

Form factors of heavy quark currents

$$
\begin{aligned}
< D^*, v_D, \varepsilon | \bar{h}_{v_D}^{(c)+} \gamma^\mu h_{v_B}^{(b)+} | B, v_B > &= \sqrt{M_B M_D}\, i\, \varepsilon^{\mu\alpha\beta\gamma} \varepsilon_\alpha (v_D)_\beta (v_B)_\gamma \xi(v_B \cdot v_D) \\
< D^*, v_D, \varepsilon | \bar{h}_{v_D}^{(c)+} \gamma^\mu \gamma^5 h_{v_B}^{(b)+} | B, v_B > &= \sqrt{M_B M_D}\, \xi(v_B \cdot v_D) \\
&\times \left[\varepsilon^\mu (1 + v_B \cdot v_D) - (v_D)^\mu (\varepsilon \cdot v_B) \right]
\end{aligned}
$$

The conclusion is that, in the infinite mass limit, the six form factors which determine the B into D and D* semileptonic decays, can all be written in terms of a single form factor, the Isgur-Wise function, due to the additional symmetries of HQET.

This function, which contains the nonperturbative interaction between the light quarks and gluons, is unknown.

However, the HQET allows us to fix its normalization.

Introduction to the HQET

Form factors of heavy quark currents

To see this, consider both mesons at rest.

Exercise 16: show that, in this case,

$$< D, v_D = (1,\vec{0}) | \bar{h}_{v_D}^{(c)+} \gamma^0 h_{v_B}^{(b)+} | B, v_B = (1,\vec{0}) > = \sqrt{4 M_B M_D}$$

On the other hand, we can express this matrix element in terms of the Isgur-Wise function,

$$< D, v_D = (1,\vec{0}) | \bar{h}_{v_D}^{(c)+} \gamma^0 h_{v_B}^{(b)+} | B, v_B = (1,\vec{0}) > = 2\sqrt{M_B M_D}\,\xi(1)$$

Comparing, we obtain the absolute normalization,

$$\xi(v_B \cdot v_D = 1) = 1$$

Introduction to the HQET

Heavy hadron spectrum structure

Consider hadrons containing a single heavy quark. We write the total hadron spin as the sum of the spin of the light degrees of freedom and the spin of the heavy quark, $\vec{S} \equiv \vec{S}_l + \vec{S}_h$.

The total spin of the hadron, which is the generator of rotations, is conserved due to the invariance under rotations.

The spin of the heavy quark is conserved in the heavy mass limit, as we have shown before.

Therefore, the spin of the light degrees of freedom is conserved too !!

This fact has important physical consequences on the structure of hadrons containing a single heavy quark.

Introduction to the HQET

Heavy hadron spectrum structure

Since the spin of the heavy quark is 1/2, the total spin of the hadron has only two permitted values: $s = s_l \pm \frac{1}{2}$.

Therefore, the heavy hadrons come in degenerate doublets, at lowest order in the HQET.

The heavy hadron states can be classified with the quantum numbers $\left(s_l, m_l; s_h = 1/2, m_h = \pm 1/2\right)$.

For heavy-light mesons, and assuming that in the ground state the angular momentum of the light antiquark is $L = 0$ as in the constituent quark model, the total spin of the light degrees of freedom is $s_l = 1/2$. The parity of this states is negative because they are quark-antiquark systems in an S-wave.

The first excitation corresponds to $L = 1$. Therefore, we have two possibilities: $s_l = 1/2 ; s_l = 3/2$ and the parity is positive.

Introduction to the HQET

Heavy hadron spectrum structure

Is this pattern present in nature?

D^0	1864.5(5)	$\frac{1}{2}(0^-)$		B^0	5279.2(1.8)	$\frac{1}{2}(0^-)$
$D^\pm$	1869.3(5)	$\frac{1}{2}(0^-)$		$B^\pm$	5278.9(1.8)	$\frac{1}{2}(0^-)$
$D^{*(0)}$	2006.7(5)	$\frac{1}{2}(1^-)$		$B^{*(0)}$	5324.8(1.8)	$\frac{1}{2}(1^-)$
$D^{*(\pm)}$	2010.0(5)	$\frac{1}{2}(1^-)$		$B^{*(\pm)}$	5324.8(1.8)	$\frac{1}{2}(1^-)$

Therefore, the doublet of spin 0 and spin 1, corresponding to the spin of the light degrees of freedom equal to ½, are the heavy-light mesons (D, D*) and (B, B*). As you can see, they are almost degenerate in mass.

Introduction to the HQET

Heavy hadron spectrum structure

Consider now the first radial excitacion L=1.

D_1^0	2422.1(1.8)	$\frac{1}{2}(1^+)$	$\|\|$	--	-----	$\frac{1}{2}(0^+)$
$D_2^{*(0)}$	2458.9(2.0)	$\frac{1}{2}(2^+)$	$\|\|$	--	-----	$\frac{1}{2}(2^+)$
$D_2^{*(\pm)}$	2459(4)	$\frac{1}{2}(2^+)$	$\|\|$	--	-----	$\frac{1}{2}(2^+)$

Therefore, this is the predicted doublet of spin 1 and spin 2, corresponding to the spin of the light degrees of freedom 3/2 As you can see, they are almost degenerate in mass.

However, the doublet ½ is not observed. Wise has suggested that the reason is that the members of this doublet decay in an S-wave and are expected to be too difficult to observe because they are quite broad (width greater than 100 MeV).

Introduction to the HQET

The mass formulae

Can we make quantitative predictions? Start with the HQET lagrangian in the rest frame of the heavy quark, $v = v_r = (1, \vec{0})$,

$$\begin{aligned}\mathcal{L}_{HQET} &= \bar{h}^+_{v_r}\, iD_0\, h^+_{v_r} \\ &+ \frac{1}{2m_Q}\bar{h}^+_{v_r}\, \vec{D}^2\, h^+_{v_r} + c_{mag}(\mu)\, \bar{h}^+_{v_r} \left(\frac{\vec{S}_h \cdot g\vec{B}}{m_Q}\right) h^+_{v_r}\end{aligned}$$

where the chromomagnetic field and the heavy quark spin are

$$\begin{aligned}B_i &= 1/2\,\varepsilon_{ijk}\, G_{jk} \\ \vec{S}_h &= \vec{\Sigma}/2 = \frac{1}{2}\begin{pmatrix}\vec{\sigma} & 0 \\ 0 & \vec{\sigma}\end{pmatrix}\end{aligned}$$

Introduction to the HQET

The mass formulae

The origin of the renormalization point dependent coefficient of the chromomagnetic operator in the HQET lagrangian at order 1/m, is the following:

So far, all our calculations have been performed at lowest order in the QCD coupling constant. At this order, we have obtained that the coefficients of both the kinetic and the chromomagnetic operators are, simply, 1.

However, these coefficients change, in general, when we include QCD corrections.

The chromomagnetic operator is divergent, so its coefficient will depend of the renormalization point and on the heavy quark mass through weak logarithms of their ratio.

Introduction to the HQET

The mass formulae

The dependence of the coefficient of the chromomagnetic operator on the renormalization point is necessary to cancel the dependence on the latter of the operator. In this way, the lagrangian is independent of the renormalization point, as it should be.

However, the kinetic energy operator **does not get renormalized in the HQET** and thus its coefficient in the lagrangian of the HQET is always 1, at any order in perturbation theory.

This result, which is refered to as the non renormalization theorem of the kinetic energy, is a consequence of the so-called reparametrization invariance of the HQET.

Introduction to the HQET

The reparametrization invariance

In the basis of the construction of the HQET there is the observation that the momentum of the heavy quark must be

$$p_Q^\mu = m_Q v^\mu + k^\mu \quad \text{with} \quad |k^\mu| \sim \Lambda_{QCD} \ll m_Q$$

But this descomposition is not unique. In fact, we can always make the following transformation, which leaves the heavy quark momentum unchanged and the residual one is still small

$$\begin{array}{lcl} v^\mu & \to & v^\mu + \varepsilon^\mu / m_Q \\ k^\mu & \to & k^\mu - \varepsilon^\mu \end{array} \quad \text{with} \quad \varepsilon^\mu \sim O(\Lambda_{QCD})$$

Introduction to the HQET

The reparametrization invariance

The physical idea behind this transformation is that it is equivalent to a low energy interaction of the heavy quark with the light degrees of freedom, which changes its velocity by a negligible amount and thus it does not have any effect in HQET.

Exercise 17: find the general form of the field transformation under a velocity reparametrization (impose that $v^2=1$ and that the new field must be an upper component field).

$$\begin{aligned} v^\mu &\rightarrow v^\mu + \varepsilon^\mu / m_Q \\ k &\rightarrow k^\mu - \varepsilon^\mu \\ h_v^+ &\rightarrow e^{i(\varepsilon \cdot x)} \left(1 + \frac{\not{\varepsilon}}{2m_Q} \right) h_v^+ \end{aligned}$$

where $\varepsilon \sim O(\Lambda_{QCD})$

Introduction to the HQET

The reparametrization invariance

Exercise 18: Apply the reparametrization transformation to the HQET lagrangian including the 1/m corrections with an unknown coefficient for the kinetic energy operator. Show that the invariance under reparametrizations implies that the coefficient of the kinetic energy operator must be 1.

Non renormalization theorem:

The coefficient of the kinetic energy correction to the HQET is 1 at any order in perturbation theory. Moreover, the kinetic energy operator is finite and thus its anomalous dimension is 0.

Introduction to the HQET

The mass formulae

We will use the 1/m terms to obtain the 1/m correction to the lowest order mass formulae for a heavy hadron H we have already derived using the flavour-spin symmetry,

$$M_H = m_Q + \bar{\Lambda} \quad \text{with} \quad \Lambda \neq \Lambda(m_Q)$$

We can consider the kinetic and the chromomagnetic moment interaction as perturbations to the lowest order lagrangian, because they are suppressed by the mass of the heavy quark.

Therefore, we calculate their contributions in first order in perturbation theory. These are simply the matrix elements of the kinetic and chromomagnetic operators between lowest HQET meson states, that is composed of a static heavy quark .

Introduction to the HQET

The mass formulae

In order to extract the complete heavy mass dependence from the matrix elements, we will use a mass independent normalization of states.

In QCD the covariant normalization is,

$$\langle H(\vec{k}') \,|\, H(\vec{k})\rangle = 2E\,(2\pi)^3\,\delta^3(\vec{k}-\vec{k}')$$

But the energy E contains a hidden mass factor.

However, it is easy to define a normalization of states which is independent of the meson mass,

$$\langle \widehat{H}(\vec{k}', v') \,|\, \widehat{H}(\vec{k}, v)\rangle = 2v^0\,(2\pi)^3\,\delta_{vv'}\,\delta^3(\vec{k}-\vec{k}') \;\;\rightarrow\;\; \widehat{H}(\vec{k})\rangle = \frac{1}{\sqrt{m_H}}\,|H(\vec{k})\rangle$$

Introduction to the HQET

The mass formulae

With this normalization, we can study the mass and spin structure of the matrix elements.

Kinetic energy matrix element

Since the square of the covariant derivative does not contain any heavy quark mass and it is rotational invariant, we parametrize the kinetic matrix element as a function of λ_1,

$$2\lambda_1 \equiv \langle \widehat{H}_{v_r} | \bar{h}^+_{v_r} \left(\vec{D}^2 \right) h^+_{v_r} | \widehat{H}_{v_r} \rangle = \langle \widehat{H}_{v_r} | \bar{h}^+_{v_r} \, (iD_\perp)^2 \, h^+_{v_r} | \widehat{H}_{v_r} \rangle$$

This parameter is independent both of the mass and spin of the heavy quark. Moreover, it is of the order of Λ^2_{QCD} .

Introduction to the HQET

The mass formulae

Chromomagnetic matrix element

The matrix element of the chromomagnetic operator must be proportional to the scalar product of the spin operators of the heavy and light quarks. In fact, these are the only vectors we have at our disposal to construct a Lorentz invariant.

It is convenient to parametrize the chromomagnetic matrix element as a function of λ_2,

$$\begin{aligned}\left(\vec{S}_h \cdot \vec{S}_l\right) 16\lambda_2 \;&\equiv\; -c_{mag}(\mu)\, \langle \widehat{H}_{v_r} \,|\, \bar{h}^{+}_{v_r} \left(g\sigma_{\mu\nu} G^{\mu\nu}\right) h^{+}_{v_r} \,|\, \widehat{H}_{v_r} \rangle \\ &=\; 4\, c_{mag}(\mu)\, \langle \widehat{H}_{v_r} \,|\, \bar{h}^{+}_{v_r} \left(\vec{S}_h \cdot g\vec{B}\right) h^{+}_{v_r} \,|\, \widehat{H}_{v_r} \rangle\end{aligned}$$

Introduction to the HQET

The mass formulae

Chromomagnetic matrix element

Since the chromomagnetic operator is divergent, it has a weak logarithmic dependence, $\ln(m_Q/\mu)$, on the mass of the heavy quark.

However, λ_2 is independent both of heavy and light quark spin, as can be seen from its definition.

By using the total hadron spin, $\vec{S} \equiv \vec{S}_l + \vec{S}_h$, we can write

$$
\begin{aligned}
\vec{S}_h \cdot \vec{S}_l &= \left(\vec{S}^2 - \vec{S}_h^2 - \vec{S}_l^2\right)/2 \\
&= \left(s(s+1) - \frac{3}{4} - s_l(s_l+1)\right)/2
\end{aligned}
$$

HEAVY HADRON SPECTROSCOPY

Introduction to the HQET

The mass formulae

Chromomagnetic matrix element

On the other hand, quantum mechanics tells us that the possible values of the total hadron spin are,

$$\left.\begin{array}{lcl} s \equiv s_- & = & s_l - 1/2 \\ s \equiv s_+ & = & s_l + 1/2 \end{array}\right\} \rightarrow \left(\vec{S}_h \cdot \vec{S}_l\right)_\pm = \pm\frac{1}{4}(2\,s_\mp + 1)$$

Putting all together we get the mass formulae

$$M_{H(\pm)} = m_Q + \bar{\Lambda} - \frac{\lambda_1}{2\,m_Q} \mp \frac{(2\,s_\mp + 1)\,\lambda_2}{2\,m_Q} + O(1/m_Q^2)$$

Introduction to the HQET

The heavy meson spectrum

It is useful to form some combinations of the masses of the doublet which are independent of the kinetic energy or of the chromomagnetic corrections,

$$\begin{aligned}\bar{M}_H &\equiv \frac{(2s_+ + 1)M_{H(+)} + (2s_- + 1)M_{H(-)}}{(2s_+ + 1) + (2s_- + 1)} \\ &= m_Q + \bar{\Lambda} - \frac{\lambda_1}{2m_Q}\end{aligned}$$

$$M_{H(-)} - M_{H(+)} = \frac{(2s_l + 1)\lambda_2}{m_Q}$$

Introduction to the HQET

The heavy meson spectrum

Let us study the doublet (D, D*), (B, B*) of spin 0 and 1.

The mass equations for the members of the doublet are:

$$
\begin{aligned}
M_{P_Q} &= m_Q + \bar{\Lambda} - \frac{\lambda_1}{2m_Q} - \frac{3\lambda_2}{2m_Q} + \cdots \\
M_{P_{Q^*}} &= m_Q + \bar{\Lambda} - \frac{\lambda_1}{2m_Q} + \frac{\lambda_2}{2m_Q} + \cdots
\end{aligned}
$$

The mass degeneracy in the doublet is a good approximation:

$$
\frac{M_{D^*} - M_D}{M_D} \sim 8\% \qquad \frac{M_{B^*} - M_B}{M_B} \sim 0.9\%
$$

Introduction to the HQET

The heavy meson spectrum

Exercise 19: improve the prediction above by using the formula for the gravity center and the mass difference in the doublet

$$\frac{1}{4}\left[M_D + 3M_{D^*}\right]\left[M_{D^*} - M_D\right] \approx \frac{1}{4}\left[M_B + 3M_{B^*}\right]\left[M_{B^*} - M_B\right]$$

which is in very good agreement with the experimental data:
$1.115\ \mathrm{GeV}^2 = 0.978\ \mathrm{GeV}^2$

Let us study the heavy-light mesons containing a strange quark

We can use exactly the same mass formula but notice that the values of the parameters are different because the light quarks are different .

Introduction to the HQET

The heavy meson spectrum

Exercise 20: obtain the following relations between the masses of the D and B mesons in the doublet of spin 0 and 1, with q=s.

$$\left[M_{D_s} + 3M_{D_s^*}\right] - \left[M_D + 3M_{D^*}\right] \approx \left[M_{B_s} + 3M_{B_s^*}\right] - \left[M_B + 3M_{B^*}\right]$$

$$\frac{1}{4}\left[M_{D_s} + 3M_{D_s^*}\right]\left[M_{D_s^*} - M_{D_s}\right] \approx \frac{1}{4}\left[M_{B_s} + 3M_{B_s^*}\right]\left[M_{B_s^*} - M_{B_s}\right]$$

We can use these equations to predict the mass of the B meson doublet with a strange quark.

Introduction to the HQET

The heavy meson spectrum

$$
\begin{aligned}
M_{B_s} &\approx 5375 \text{ MeV} \\
M_{B_s^*} - M_{B_s} &\approx 55 \text{ MeV}
\end{aligned}
$$

To be compared to the experimental data:

$D_s^{\pm}$	$1968.5(0.6)$	$0(0^-)$	B_s^0	$5369.3(2.0)$	$?(?^?)$
$D_s^{*(\pm)}$	$2112.4(0.7)$	$?(?^?)$	B_s^*	$----$	$--$

Similar predictions can be obtained for excited mesons and for baryons. The agreement with the experimental data is good.

RENORMALIZATION AND MATCHING

Introduction to the HQET

Relation between Green functions

We have constructed an effective theory for a heavy quark in a hadron affected by low energy QCD interactions only.

The construction has been made at tree level, that is without including radiative corrections.

At tree level, the HQET is an effective theory of QCD in this region of energies because any Green function in QCD can be approximated by a series of effective Green functions

$$G(p;q) = \sum_{i=0}^{n-1} \frac{1}{m_Q^i} \sum_{j=1}^{N_i} \tilde{G}^{(i,j)}(k;q) + \left(\frac{\Lambda^n{}_{QCD}}{m_Q^n}\right) \quad \text{with} \quad p = m_Q v + k$$

Introduction to the HQET

Relation between Green functions

When we include radiative corrections, this relation is not valid anymore.

The reason is the following: QCD and the HQET have the same infrared structure but different ultraviolet behaviour.

In fact, in QCD and for a large quark mass, the Green functions contain large logarithms $\ln\left(m_Q/\mu\right)$. However, in the HQET the heavy quark mass does not appear explicitly because we are in the infinit mass limit. Therefore, the HQET does not have these large logarithms. Quantum fluctuations in the HQET give rise to logarithms $\ln\left(\left(v\cdot k\right)/\mu\right)$. Then, the tree level relation between Green functions is incomplete; the QCD logarithms must be added to the effective theory in some way.

Introduction to the HQET

Wilson coefficients

The correct relationship between Green functions of QCD and the HQET includes some coefficients that correct the different ultraviolet behaviour of the theories: the Wilson coefficients,

$$G(p;q;\mu) = \sum_{i=0}^{n-1} \frac{1}{m_Q^i} \sum_{j=1}^{N_i} \tilde{C}^{(i,j)}(m_Q/\mu, \alpha_s)\, \tilde{G}^{(i,j)}(k;q;\mu) + O\left(\frac{\Lambda^n{}_{QCD}}{m_Q^n}\right) \quad \text{with} \quad p = m_Q v + k$$

The Wilson coefficients contain the QCD logarithms that the HQET cannot reproduce.

Notice that they do not depend on external momenta of gluons and light quarks and masses. They are a function of the ratio of the heavy quark mass and the renormalization point only.

Introduction to the HQET

Calculation of the Wilson coefficients

Calculate a sufficient number of suitable Green functions in QCD at a given order in the coupling constant.

Calculate the same Green functions in the effective theory.

Expand the QCD Green functions in invers powers of 1/m up to a given order.

Renormalize the Green functions using, if you like, different renormalization schemes in QCD and in HQET.

Compare them and obtain the Wilson coefficient.

Use the Renormalization Group Equations to sum the dominant and, if you know the two-loop anomalous dimensions, the subdominant logarithms of the large mass.

Introduction to the HQET

The End