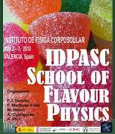

Strong interactions, Lattice and HQET

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May 3, 2013

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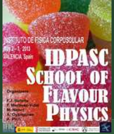
Introduction

Quantum Chromodynamics (QCD), first introduced by Gell-Mann and Frizsch in 1972, is the theory of strong interactions. It is a renormalizable nonabelian gauge theory based on the color $SU(3)$ group which elementary fields are quarks and gluons.

Why is QCD important for Particle Physics?

- Electroweak processes of hadrons necessarily involve strong interactions.
- A quantitative understanding of the QCD background in searches for new physics at present and future accelerators is crucial.

In this lectures we give an introduction to the foundation of perturbative and nonperturbative QCD and some important physical applications: Deep Inelastic Scattering (DIS) and e^+e^- annihilation into hadrons.



History

The history that led to the discovery of QCD is fascinating. We briefly comment on some turning points.

Neutron discovery

Since the discovery of the neutron (Chadwick 1932), the strong interactions have been reconigned as a separate force of nature. It is attractive at intermediate distances and so strong that it overcomes the electric repulsion of the protons in atomic nuclei (typical electromagnetic distance: electrons in an atom, 10^{-10} m to be compared with a typical strong-interaction distance: protons in a nucleus, 10^{-15} m).

Yukawa model

Yukawa in 1934 proposed that the exchange of pions is the source of the forces between protons and neutrons

$$U(r) = \pm \frac{g}{r} e^{-m_\pi r}$$

The mass of the pion is just the inverse of the range of the force.

History (II)

Isospin formalism

Between 1935 and 1938, the charge invariance of strong interactions was experimentally established. Cassen and Condon invented the formalism incorporating this proptery: isospin symmetry. The theory was completed by Kemmer in 1938 by the introduction of a neutral pion ϕ_3 with the same mass of the charged pions $\phi_{1,2}$. The Hamiltonian of the strong interactions incorporating the isospin symmetry reads

$$\mathcal{H} = g\bar{\psi}\vec{\tau}\psi\vec{\phi} \qquad \psi \equiv \begin{pmatrix} \psi_p \\ \psi_n \end{pmatrix}$$

On the experimental side, the $\pi^\pm$ were discovered by Powell in 1947 and the π^0 at Brookhaven in 1950.

History (III)

The discovery of strangeness

The particles discovered in 1947 by Butler and Rochester were almost certainly the decays of K^0 and K^+ to two pions. Soon after this, it was discovered the decay of K^+ to three pions and a Ξ^- to $\pi^- p$. In 1955 Gell-Mann gave them the name of *strange* particles because behaved *strangely*. In fact, they were produced in collisions on cosmic rays with rates comparable to those of pions, but decayed many orders of magnitud slower than as expected for a decay mediated by strong interactions. Gell-Mann in 1953 solved the puzzle of new particle decays by making clever isospin assignements to these particles. Then, isospin conservation prevented them from decaying into the observed decay modes via strong interactions, but allowed them to proceed via weak interactions, which explained their long lifetimes. Isospin assignements worked but there was still the question: is there any deeper reason for it?

History (IV)

Strangeness quantum number

In 1954, Nishijima observed that the peculiar isospin assignement of Gell-Mann can be reformulated in terms of a new quantum number called "strangeness" which is conserved in strong interactions but violated by weak forces. He wrote,

$$Q = I_3 + \frac{B+S}{2} \quad \text{hypercharge } Y \equiv B + S$$

In Gell-Mann's model, each particle is labelled by three quantum numbers I, I_3 and Y or Q.
The introduction of strangeness was crucial because it opened the way to unitary symmetry and and the consequent development of the quark model by Gell-Mann (Nobel prize in Physics in 1969 from the Eight-fold Way). The peculiar isospin assignments of Gell-Mann, Nakano and Nishijima are, within the quark model, simply a consequence of the fact that the strange quark is an isospin singlet.

History (V)

In the annual conference on high energy physics in Pisa in 1955, the properties of the observed strange particles were stablished. Gell-Mann predicted the existence of Σ^0 (found in 1956), Ξ^0 (found in 1959) and, most remarkably, of Ω^- (discovered in 1964), assigned to an isosinglet.

Non abelian gauge theories

In 1954, Yang and Mills created nonabelian gauge theories. QCD,as we will see, is a non abelian gauge theory based on color $SU(3)$.

Sakata model

The Sakata model (1959) postulated that the hadrons could be considered to be composite states of p, n and Λ particles. Ikeda, Ohnuki and Ogawa in 1959 suggested that the triplet of particles transformed in the fundamental representation $\mathbf{3}$ of $SU(3)$. They correctly said that the mesons could be build out bound states of $\mathbf{3}$ and $\overline{\mathbf{3}}$:

$$\mathbf{3} \otimes \overline{\mathbf{3}} = \mathbf{8} \oplus \mathbf{1}$$

but several of their assigments were incorrect.

History (VI)

The Eightfold Way and quark model

In 1961, Gell-Mann and Ne'eman, made the correct $SU(3)$ assignments: baryons and mesons were arranged in what they called the Eightfold Way.

The Gell-Mann and Zweig proposed that these SU(3) assignments could be generated if one postulated the existence of new constituents, called "quarks", which transformed as a triplet $\mathbf{3}$. All other higher representations could be generated beginning by quarks by taking multiple products of the fundamental representation. In the fundamental representation of $SU(3)$, the quarks were called **up**, u, **down**, d, and **strange**, s.

The u and d quarks formed an $SU(2)$ isodoublet. The strange quarks was introduced because in the 1950s, as we said before, it was observed that *strangeness*, a new quantun number in addition to isospin, was conserved by hadronic processes. The $SU(3)$ paradigm explains the new quantum number because it is a rank two Lie group.

History (VII)

Quark model

$SU(3)$ representations are labeled by two numbers: third component of isospin I_3 and hypercharge $Y = B + S$, where B is the baryon number and S the strangeness. But these quantum numbers are not independent. Nishijima and Gell-Mann proposed the relation

$$Q = I_3 + \frac{Y}{2}$$

where Q is the electric charge. To fit the known spectrum, mesons were postulated to be composite states of a quark and an antiquarrk, and baryons are composite of three quarks,

$$\mathbf{3} \otimes \overline{\mathbf{3}} = \mathbf{8} \oplus \mathbf{1} \qquad \mathbf{3} \otimes \mathbf{3} \otimes \mathbf{3} = \mathbf{10} \oplus \mathbf{8} \oplus \mathbf{8} \oplus \mathbf{1}$$

The theory predicted that the mesons should be arranged in terms of octets and singlets, whyle baryons should be in octets and decuplets.

History (VIII)

In order to reproduce the known charges of mesons and baryons, the quarks are postulated to have fractional charges

$$Q_u = \frac{2}{3} \quad Q_d = -\frac{1}{3} \quad Q_s = -\frac{1}{3}$$

Since three quarks make up a baryon, quarks have baryon number $1/3$. More in the next section.

Turning points to be disccussed in the lectures

- The observation of scaling in deep-inelastic scattering (DIS).
- The proposal of color a symmetry of the strong interactions.
- Asymptotic Freedom

DIS made Quantum Chromodynamics (QCD), a quantum theory of fields with color as a local symmetry, the unique explanation of the strong interactions, through its property of asymptotic freedom.

State of particle physics in the early sixties

In the early sixties of the last century, QED had been formulated as a Quantum Field Theory (QFT) to describe electromagnetic interactions. Many precise QED predictions were confirmed experimentally.
But, the renormalization procedure, an essential ingredient of the perturbative treatement of QED, was not universally accepted even among founding fathers of QFT:

- ”Sweeping the infinities under the rug”
- ”I do not subscribe to the philosophy of renormalization”, R. Feynman, Solvay Conference 1961.

For weak interactions, the Fermi theory that succesfully described weak decays, was non renormalizable and hence it was inconsistent at high energies. This fundamental problem was solved by the invention of the electroweak theory, a renormalizable gauge theory, at the end of the sixties.

State of particle physics in the early sixties

The state of strong interactions was even worst. A rapidly increasing number of hadrons could be classified by the quark model of Gell-Mann and Zweig but its dynamics was completely unknown. As perturbation theory (PT) cannot be used here, many physicists thought that QFT might not be adequate for the strong interactions.

1. S-matrix approach (Chew *et al.*): do not look for more fundamental constituents of hadrons, all hadrons are equal, forget QFT, work directly with the S-matrix for strong interactions.
2. QFT is a toy model (Gell-Mann): use algebraic relations between currents (current algebra), suggested/deduced from a Lagrangian field theory model but do not take it seriously. Quarks are purely mathematical entities without any physical reality.

The light at the end of the tunnel

Deep Inelastic Scattering and Asymptotic Freedom

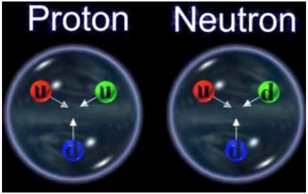
The light at the end of the tunnel

Experiment showed the right path at the end of the sixties: the MIT-SLAC collaboration found unexpected results in deep inelastic scattering (DIS) of leptons on nucleons. At low energies, the cross sections were characterized by baryon resonances, at large energies and momentum transfer the nucleons seem to consist of noninteracting partons (proposed by Feynman). The identification of partons as quarks is natural, but then how could quarks be free at high energies and yet be permanently bound in hadrons?

We now know that the strength of the strong (an other) interactions depends on energy because pair creation converts the vacuum in a polarisable medium that screens the (color) charge.

There is, however, an all important difference between QED and QCD. In the former, the effective charge increases with energy but in the later, due to its nonabelian character, decreases with energy. This is called asymptotic freedom.

Quark Model



Quark model

Introduced by Gell-Mann and Zweig in 1963, it was an attempt to explain the increasingly complex list of hadrons and resonances discovered in the new particle accelerators during the 1950s and 1960s.

Previously, it has been well stablished that isospin, based on $SU(2)$, was a good symmetry of the strong interactions and that strangeness, a $U(1)$ symmetry, was conserved by them.

These two symmetries was combined in a larger symmetry group, $SU(3)$ of flavour which was found to be conserved to a good approximation by the strong interactions.

The quark model is a realization of this symmetry.

In the quark model, mesons and baryons are bound states of quarks,

Mesons (π, K, ρ, ...) are bound states of the form $q\bar{q}$

Baryons (p, n, Λ, ...) are bound states of the form qqq

Quark model (II)

To reproduce the integer spin of mesons and the half-integer spin of baryons, it was postulated that quarks have spin $1/2$. Assuming, as it is the case for non pathological potentials, that the lowest states have zero spatial angular momentum, mesons can be classified as

Mesons $J^P = 0^-$: spin singlet states with $L = 0$.

Mesons $J^P = 1^-$: spin triplet states with $L = 0$.

Baryons can in principle have all combinations of three $1/2$-spins: $1/2 \otimes 1/2 \otimes 1/2 = 1/2 \oplus 1/2 \oplus 3/2$. The experimental spectrum contains only one spin-1/2 set of baryons. To eliminate one of the spin-1/2 term in the decomposition above, it was postulated that the baryon spin-flavour wave-function is totally symmetric. Therefore, the classification of baryons is

Baryons $J^P = 1/2^+$ with $L = 0$.

Baryons $J^P = 3/2^+$ with $L = 0$.

Quark model (III)

In addition, three types of quarks were necessary to explain the hadron spectrum: u, up, d down and s, strange. The heavy quarks c, charm, b, bottom, and t, top, could not be produced in the accelerators of the 1960s. Since their mass is larger than the $\Lambda_{QCD} \sim 300$ MeV, the scale that characterizes the strong interactions, their behaviour is significantly different from the one for light quarks. We will come back to this aspect of QCD later.

Now, one can start assigning a quark content to each hadron. For example, the proton is uud, the neutron udd, $\ldots$, etc. In order to explain the charge of baryons, the quarks must have a fractional charge: $2/3$ for the u and $-1/3$ for the d. The quark content of light mesons is $u\bar{d}$ for the π^-, $d\bar{u}$ for the π^+, $u\bar{u} - d\bar{d}$ for the π^0 and so on. Pseudoscalar mesons with strangeness must include the strange quark or its antiquark. Thus, the quark content of the K^+ is $u\bar{s}$ and of the K^0 is $d\bar{s}$, so the s must have charge $-1/3$.

Quark model: flavour $SU(3)$

Remember that $U(N)$ is the group of $N \times N$ unitary matrices, $U\,U^\dagger = I$. It is easy to see that $\det(U) = e^{i\phi}$, a pure phase. The subset of matrices of $U(N)$ with $\det(U) = 1$ form a subgroup called $SU(N)$. It follows that

$$U(N) = SU(N) \times U(1)$$

The flavour symmetry is associated to $SU(3)$ because the additional $U(1)$ is an exact symmetry of the strong interactions, the baryon number. Therefore, we arrange the three quarks in a column vector and define that it transforms under $SU(3)_f$ as its $\mathbf{3}$ irreducible representation, i.e. as a triplet. Both, isospin and strangeness are included in $SU(3)_f$ as subgroups: isospin is the $SU(2)$ subgroup of transformations acting on doublets of u and d quarks while strangeness is a $U(1)$ transformation for the s quark,

$$\begin{pmatrix} u \\ d \end{pmatrix} \to U_I \begin{pmatrix} u \\ d \end{pmatrix}, \;\; s \to U_s\, s \qquad \text{with } \; U_I \in SU(2), U_s \in U(1)$$

Quark model: flavour $SU(3)$ (II)

The hadrons fall into irreducible representations of $SU(3)_f$. In fact, taking into account that quarks are members of the $\mathbf{3}$ representation and antiquarks of the $\bar{\mathbf{3}}$, the mesons are members of the product

$$\mathbf{3} \otimes \bar{\mathbf{3}} = \mathbf{8} \oplus \mathbf{1}$$

This accounts for the two mesons octets,

$$\mathbf{J^P = 0^-}: \quad \pi^{\pm}, \pi^0, K^{\pm}, K^0, \bar{K}^0, \eta$$

$$\mathbf{J^P = 1^-}: \quad \rho^{\pm}, \rho^0, K^{*\pm}, K^{*0}, \bar{K}^{*0}, \omega$$

and two singlets: the η' ($\mathbf{J^P = 0^-}$) and ω' ($\mathbf{J^P = 1^-}$). The flavour wave-function of the mesons are easily obtained. The only subtlety is with the neutral states $s\pi^0$ and η that are choosen to make the former a member of the isospin triplet with $\pi^{\pm}$ and the later and isospin singlet. The η' is also a singlet under $SU(3)_f$.

Quark model: flavour $SU(3)$ (III)

We can collect all the quark wave-functions in a mesonic matrix wave-function

$$
\begin{aligned}
M &= \begin{pmatrix} u \\ d \\ s \end{pmatrix} \otimes \begin{pmatrix} \bar{u} \\ \bar{d} \\ \bar{s} \end{pmatrix} = \begin{pmatrix} u\bar{u} & u\bar{d} & u\bar{s} \\ d\bar{u} & d\bar{d} & d\bar{s} \\ s\bar{u} & s\bar{d} & s\bar{s} \end{pmatrix} \\
&= \begin{pmatrix} \frac{1}{3}(2\,u\bar{u} - d\bar{d} - s\bar{s}) & u\bar{d} & u\bar{s} \\ d\bar{u} & \frac{1}{3}(2\,d\bar{d} - u\bar{u} - s\bar{s}) & d\bar{s} \\ s\bar{u} & s\bar{d} & \frac{1}{3}(2\,s\bar{s} - u\bar{u} - d\bar{d}) \end{pmatrix} \\
&+ \frac{1}{3}(u\bar{u} + d\bar{d} + s\bar{s}) \\
&= \begin{pmatrix} \frac{1}{\sqrt{2}}\pi^0 + \frac{1}{\sqrt{6}}\eta & \pi^+ & K^+ \\ \pi^- & -\frac{1}{\sqrt{2}}\pi^0 + \frac{1}{\sqrt{6}}\eta & K^0 \\ K^- & \bar{K}^0 & -\frac{2}{\sqrt{6}}\eta \end{pmatrix} + \eta'
\end{aligned}
$$

Quark model: flavour $SU(3)$ (IV)

The baryons are members of the product

$$\mathbf{3} \otimes \mathbf{3} \otimes \mathbf{3} = \mathbf{10} \oplus \mathbf{8} \oplus \mathbf{8} \oplus \mathbf{1}$$

plus the constraint that the wave-function is totally symmetric. The states remaining are

$\mathbf{J^P = 1/2^+}$: Baryon octet: p, n, $\Sigma^{\pm}$, Σ^0, Ξ^-, Ξ^0, Λ

$\mathbf{J^P = 3/2^+}$: Baryon decouplet: Δ^{++}, Δ^+, Δ^0, Δ^-, $\Sigma^{*\pm}$, Σ^{*0}, Ξ^{*-}, Ξ^{*0}, Ω^-

Their quark content is easily worked out given the charges and strangeness of each particle. We can collect all the quark wave-function in a barionic one

$$B = \begin{pmatrix} \frac{1}{\sqrt{2}}\Sigma^0 + \frac{1}{\sqrt{6}}\Lambda^0 & \Sigma^+ & p \\ \Sigma^- & -\frac{1}{\sqrt{2}}\Sigma^0 + \frac{1}{\sqrt{6}}\Lambda^0 & n \\ \Xi^- & \Xi^0 & -\frac{2}{\sqrt{6}}\Lambda^0 \end{pmatrix}$$

Quark model: advantages and disadvantages

The quark model appears to be a very useful Periodic Table of Hadrons because using it one can nicely classify the entire hadronic spectrum. A remarkable success of the quark model was the prediction, including the mass, of the $\Omega^- = sss$ baryon. But the model has several problems:

1. a free quark has never been observed. Therefore, is it a real particle or simply a mathematical entity that realizes the $SU(3)_f$ symmetry?
2. the absence of antisymmetric combinations of spin and flavour representations in the baryon sector which could led to problems with the Fermi-Dirac statistics.
3. there are no signs of exotics states like $q\bar{q}q\bar{q}$ or $qqqq\bar{q}$.
4. nothing is said about the dynamics of strong interactions. What is the binding force that hold quarks together? If the glue is a vector meson, to be renormalizable must be massless but being massless it should generate a long-range force, like gravity and electromagnetism.

The color degree of freedom

After years of black confusion, color came to rescue.

Evidences for color

Before the invention of QCD there were a number of indications for the existence of color degrees of freedom.

- spin-statistics problem
- Absence of exotics
- e^+e^- annihilation
- $\pi^0 \to 2\gamma$
- Anomaly cancellation in the SM
- hadronic decays of the lepton τ

Evidences for color: Spin-statistics

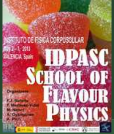
As we said before, the spin-flavour wave-function of baryons is postulated to be fully symmetric. But quarks are fermions and their wave-functions should therefore be antisymmetric with respect to the interchange of all degrees of freedom, spacetime and internal. It follows that the spatial wave-function of a baryon should be antisymmetric.

This is, however, problematic. For every resonable potential, the ground state of the system of three quarks is expected to be symmetric under the exchange of any pair of coordinates ($L = 0$). For example, if we model the potential with a symmetric harmonic oscillator, the ground state is a Gaussian invariant under rotations, i.e. with $L = 0$. Therefore the total spin of the baryon $\vec{J} = \vec{L} + \vec{S} = \vec{S}$. The baryons have half-integer spin which implies that their wave-funtion must be antisymmetric.

Now, consider a classical example: the baryon Δ^{++}. Since it has $S_z = 3/2$ and its quark content is uuu, its spin-flavour wave function is

$$|\Delta^{++}\rangle = |u\uparrow, u\uparrow, u\uparrow\rangle$$

completely symmetric. Therefore, the complete wave-function is also totally symmetric which violates the spin-statistics theorem.



Evidences for color: Spin-statistics (II)

A natural solution to this serious problem, proposed by Han, Nambu, Greenberg and Gell-Mann, independently, consists in assuming that the quarks possess an additional degree of freedom named color. Each quark comes in three colors $q_i, i = 1,2,3$ (red, blue and green). Now it is easy to construct a baryon wave-function antisymmetric under the exchange of the color of any two quarks,

$$|\Delta^{++}\rangle = \frac{1}{\sqrt{6}}\varepsilon^{ijk}|u_i, u_j, u_k\rangle$$

where ε_{ijk} is the totally antisymmetric tensor with $\varepsilon_{123} = 1$. The total wave-function is now antisymmetric because is symmetric under space, spin and flavour but antisymmetric in color.

Similar problems exist also for the statistics of Δ^- and Ω^-, composed of three identical quarks, d and s respectively. The problem is solved the same way by the introduction of the color degree of freedom.

Evidences for color: Spin-statistics (III)

For mesons, the color wave-function is

$$|M(q'\bar{q})\rangle = \frac{1}{\sqrt{3}}\,\delta^{ij}\,|q'_i\bar{q}_j\rangle$$

Notice that for this scheme to work at least three colors are necessary. Therefore, each quark field is a triplet,

$$q = \begin{pmatrix} q_1 \\ q_2 \\ q_3 \end{pmatrix}$$

and it is resonable to expect a new internal symmetry that rotates the color degrees of freedom among themselves. The associated symmetry group is $SU(3)_c$. It is important to distinguish it from the flavour $SU(3)_f$ that rotates the flavours among themselves but with the same color.

A quark field carries both indices: $q_{f\,i}$, with $f = u, d, s$ and $i = 1, 2, 3$ and transforms as a $(3, 3)$ under $SU(3)_f \otimes SU(3)_c$.

Evidences for color: Spin-statistics and confinement

Color solves in an elegant and natural way the statistics problem of the quark model but introduces new states: the colored hadrons. Indeed, if each quark/antiquark comes in three colors, this allows for 81 $q\bar{q}$ combinations only 9 of which have been found. The remaining combinations are not bound states.
These extra states have not been observed and hence one needs to postulate that all asymptotic states must be colorless, i.e. singlets under color rotations. In other words, unobserved hadrons are white, colorless, combinations of quarks. This is the confinement assumption: quarks, colored objects, are not observable as asymptotic states. They are confined in color-singlet bound states.

Notice that the confinement hypothesis rules out the existence of diquark states but not of states like $qqqq\bar{q}$. On the other hand, this hypothesis leads to another question: why are there only color singlet states in nature?

Evidences for color: Spin-statistics and confinement

A first step towards the answer was taken by Y. Nambu in 1965. Nambu proposed that quark confinement follows from strong forces acting between quarks mediated by an **octet of gauge fields**, G^a, $a = 1, \cdots, 8$, coupled to the color $SU(3)$ generators, λ^a. This is the essence of QCD.

Unfortunately, Nambu assignment of electric charges to quarks was completely wrong, the Hahn-Nambu model: different colors were assigned different integer electric charges in such a way that their color-averaged charge were equal to the fractional electric charges of the quark model.

The Han-Nambu model should be considered as a precursor of QCD.

Evidences for color: Absence of exotics

All of the nine mesons predicted by the quark model as bound states of the form $q\bar{q}$ have been observed. This suggests an attractive force between quarks and antiquarks. Also baryons fit nicely in qqq bound states. If the strong force is purely attractive, why objects of the form $\bar{q}qqq$, with fractional charge, have never been observed? The quark model does not explain why exotic states such $\mathbf{3} \otimes \mathbf{3}$, ..., etc are not observed. Notice that this question cannot be answered within the quark model because it does not give any indication of what is the binding force between quarks.

In QCD there is a simple reason: confinement. Only color singlets, members of the $\mathbf{1}$ irreducible representation of the color $SU(3)$, are physical states. We know that $q\bar{q}$ and qqq states are invariant under the color group because the color indices are contracted by invariant tensors. Exotic states are not invariant under the color group and hence they are absent of the spectrum.

Evidences for color: e^+e^- annihilation

A direct test, actually a quark counting, of the color degree of freedom can be obtained from the ratio

$$R = \frac{\sigma(e^+e^- \to \text{hadrons})}{\sigma(e^+e^- \to \mu^+\mu^-)}$$

The hadronic production is mediated by a virtual photon or a virtual Z,

$$e^+e^- \to (\gamma^*, Z^*) \to q\bar{q} \to \text{hadrons}$$

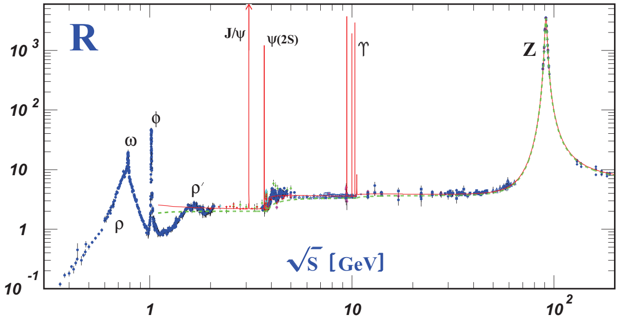
We will show that at energies well below the Z peak, the cross-section is dominated by the photon exchange and the R ratio is simply proportional the sum of the squares of the quark charges,

$$R = N \sum_{f=1}^{N_F} Q_f^2$$

where N is the number of colors and Q_f is the electric charge, in units of e, of the active quarks at the process energy.

Evidences for color: e^+e^- annihilation (II)

At low energies, below the charm threshold, and assuming $N = 3$, the ratio should be $R \approx 2$. When we hit the $c - \bar{c}$ threshold, the ratio rises to $10/3$. When we include the b quark, $R \approx 11/3$. This pattern of steps agrees rather well between thresholds with experiment.



Evidences for color: $\pi^0 \to 2\gamma$

The neutral pion π^0 has the following quark content

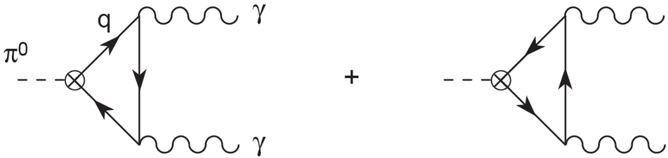
$$|\pi^0\rangle = \frac{1}{\sqrt{2}}\left(|u\bar{u}\rangle - |d\bar{d}\rangle\right)$$

Therefore, we can take the divergence of the isospin $I = 1, I_3 = 0$ axial current as an interpolation field for the π^0,

$$A_\mu = \frac{1}{\sqrt{2}}\left(\bar{u}\gamma_\mu\gamma_5 u - \bar{d}\gamma_\mu\gamma_5 d\right)$$

The Feynman diagram for this decay consists of an internal quark triangle loop, with the pion and the photon attached to the corners of the triangle. Due to the so-called triangle anomaly (to be discussed below), the coupling of the axial current with two electromagentic currents does not vanish, producing the decay of the π^0 to two photons.

Evidences for color: $\pi^0 \to 2\gamma$ (II)



Triangle diagrams for $\pi^0 \to 2\gamma$

Evidences for color: $\pi^0 \to 2\gamma$ (III)

The result of the calculation is

$$\Gamma(\pi^0 \to \gamma\gamma) = N\,(Q_u^2 - Q_d^2)^2\,\frac{\alpha^2\,m_\pi^3}{64\,\pi^3 f_\pi^2} = 7.73\,\mathrm{eV} \leftrightarrow \Gamma_{exp} = 7.7(6)\,\mathrm{eV}$$

for $N = 3$; $Q_u = 2/3$; $Q_d = -1/3$ and where $f_\pi = 92.4$ MeV is the coupling of the π^0 to the A_μ. The agreement is very good. Notice that for $N = 1$, the two values would disagree by a factor of 9.

Moreover, the triangle anomaly is a global flavour symmetry broken by quantum effects, precisely the triangle loops. In other words, the symmetry is present at the classical level but not at the quatum level. It can be shown that the decay amplitude does not get corrected by strong interactions.

Evidences for color: Anomaly cancellation in the SM

This is very nice theoretical argument, not based on any experiment, only on the internal consistency of the SM.

An anomaly is a symmetry of the classical Lagrangian that has not survived in the passage to the quantum theory. To start with, consider QED. As a consequence of the gauge invariance of the theory, there exist some identities valid up to all orders in Perturbation Theory: the Ward-Takahashi identities. The vector (electromagnetic) and axial currents,

$$V^\mu(x) = \bar{\psi}(x)\gamma^\mu\psi(x) \qquad A^\mu(x) = \bar{\psi}(x)\gamma^\mu\gamma_5\psi(x)$$

satisfy the equations of motion

$$\partial_\mu V^\mu(x) = 0 \qquad \partial_\mu A^\mu(x) = 2\,i\,m\,P(x) \quad \text{with} \quad P(x) = \bar{\psi}(x)\gamma_5\psi(x)$$

For massless fields, also the axial current is conserved and the action is invariant under the vector and axial transformations

$$\psi \to e^{i\theta}\psi \qquad \psi \to e^{i\theta\gamma_5}\psi$$

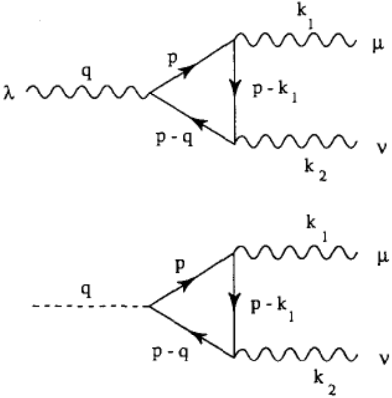
Evidences for color: Anomaly cancellation in the SM

All this is correct at the classical level. At the quantum level, since the coupling of the electromagnetic current to the photon field, B_μ is $J_\mu B^\mu$, it is easy to show that the amplitudes involving the vector current only are gauge invariant. But this is not the case for some amplitudes involving the axial current. To be specific, consider the triangle diagrams associated to a VVA vertex and a VVP one. The correspondig amplitudes are denoted by $T_{\mu\nu\lambda}$ and $T_{\mu\nu}$, respectively. Taking derivatives and using the equations for the divergences of the currents, we get

$$k_1^\nu T_{\mu\nu\lambda} = 0 \qquad k_2^\nu T_{\mu\nu\lambda} = 0 \qquad q^\lambda T_{\mu\nu\lambda} = 2m T_{\mu\nu}$$

The Feyman integral for $T_{\mu\nu\lambda}$ is divergent but when it is inserted in the third equation, the LHS term is finite and equal to the RHS term. However, when it is inserted in the first and second equations, the result is divergent and the renormalized expression for $T_{\mu\nu\lambda}$ satisfies the first and second equations but not the third. It can be shown that there is no regularization method compatible with the three Ward Identities.

Evidences for color: Anomaly cancellation in the SM



The V-V-A and V-V-P triangle graphs, which give rise to the anomaly.

Evidences for color: Anomaly cancellation in the SM

This is a disaster for the SM because the Ward Identities are used to prove the renormalizability of the theory. They allow the cancellation of the infinities. Any Quantum Field theory should be anomaly free. In the non abelian case, the regularization of triangle diagrams lead to correct results for the WI of the vectorial current but not for the axial current. Therefore, we must construct our theory in such a way that the axial WI is algebraically satified. In a theory with one or more vector gauge bosons coupled to the lefthanded and righthanded fermions as

$$\bar{\psi}_L \gamma^\mu M^a \psi_L W_\mu^a \qquad \bar{\psi}_R \gamma^\mu N^a \psi_R W_\mu^a$$

where M^a and N^a are the (matrices representing the) generators of the gauge transformations, the analysis of the triangular diagrams for $W^a \to W^b + W^c$, shows that the cancellation of the axial anomalies imposes the condition

$$0 = \mathrm{Tr}\left[M^a \left\{M^b, M^c\right\}\right] - \mathrm{Tr}\left[N^a \left\{N^b, N^c\right\}\right]$$

Evidences for color: Anomaly cancellation in the SM

In the case of the SM, it is not difficult to arrive at the condition

$$\sum_{f_L} Q_f = 0$$

For leptons $e^-, \mu^-, \nu_e, \nu_\mu$, the sum of the charges is -2 andf for the 4 quarks, u, d, s and c, is $2/3$. It follows that the complete sum does not vanish. If, however, each quark contributes with 3 colors, we have

$$\sum_{f_L} Q_f = -2 + 3 \cdot \frac{2}{3} = 0 \;\leftrightarrow\; \sum_{f_L} Q_f = -2 + 1 \cdot \frac{2}{3} = -4/3 \neq 0$$

If the τ lepton is added, we need new quarks, the bottom and the top.

To sum, the leptonic part of the SM contain anomalies that arise when a classical symmetry of an action does not survive the process of quantization. In particular, there are certain divergent fermionic triangle diagrams that can destroy the WI and hence ruin the renormalizability of the theory. When quarks are included in the SM, they also produce anomalies but of the opposite sign. The charge assignments in the SM are precisely the ones that cancel the anomaly.

Evidences for color: hadronic decays of the lepton τ

The lepton τ can decay through the emission of a W into

$$\tau^- \to \nu_\tau + \left\{ \begin{array}{ccc} e^- & + & \bar{\nu}_e \\ \mu^- & + & \bar{\nu}_\mu \\ d & + & \bar{u} \end{array} \right.$$

Defining the ratio

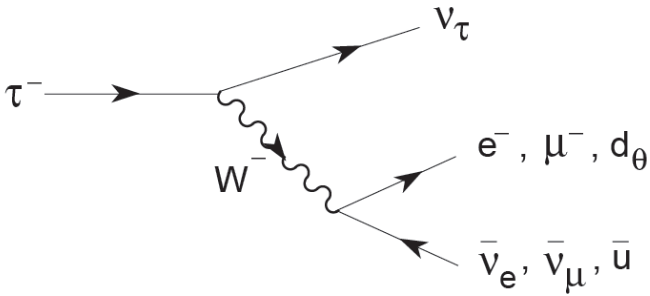
$$R_\tau = \frac{\Gamma(\tau^- \to \nu_\tau + \text{hadrons})}{\Gamma(\tau^- \to \nu_\tau e^- \bar{\nu}_e)}$$

A similar analysis to the one for $R_{e^+e^-}$ but replacing the electromagnetic current by the charged weak current, gives

$$R_\tau \approx N \left(|V_{ud}|^2 + |V_{us}|^2 \right) S_{EW} \left\{ 1 + \delta'_{EW} + \delta_{pert} + \delta_{nonpert} \right\}$$

where the leading and nonleading electroweak perturbative corrections are $S_{EW} = 1.0194$ and $\delta'_{EW} = 0.0010$.

Evidences for color: hadronic decays of the lepton τ



Evidences for color: hadronic decays of the lepton τ

The QCD perturbative corrections are known, as an expansion in $\alpha_s(m_\tau)$, at the NNNLO, $O(\alpha_s^3)$

$$\delta_{pert} = \frac{\alpha_s(m_\tau)}{\pi} + 5.2\left(\frac{\alpha_s(m_\tau)}{\pi}\right)^2 + 26.4\left(\frac{\alpha_s(m_\tau)}{\pi}\right)^3 + \cdots$$

and the nonperturbative contributions are estimated to be $\delta_{nonpert} = -0.014(5)$. Taking $V_{ud} = 0.9773$ and $V_{us} = 0.2246$ and including QCD corrections with $\alpha_s(m_\tau) = 0.35(3)$, we get

$$R_\tau \approx 3.69 \quad \text{to be compared with} \quad R_\tau^{exp} = 3.63(1)$$

in very good agreement with the experimental value.

Notice that this quantity can also be used to precisely determine $\alpha_s(m_\tau)$, assuming $N = 3$.

Deep Inelastic Scattering

Deep Inelastic Scattering (DIS) of leptons on hadrons has had an enormous impact on teh development of the ideas that led to QCD. The aim of the SLAC-MIT DIS experiment of the 1960s was to understand the structure of the proton and the neutron. They measured the electric and magnetic form factors by sending electrons with an energy up to 20 GeV against a target of hydrogen or deuterium. The energy and direction of the scattered electron are measured in the detector but the final hadronic state (denoted by X) is not measured,

$$e^-(k,E) + N(p) \rightarrow e^-(k',E') + X(p_X)$$

The lepton interacts with the target through the exchange of a virtual photon. In the deep inelastic region, the photon absorbed by the target hadron, blows it apart and it is fragmented into many particles.

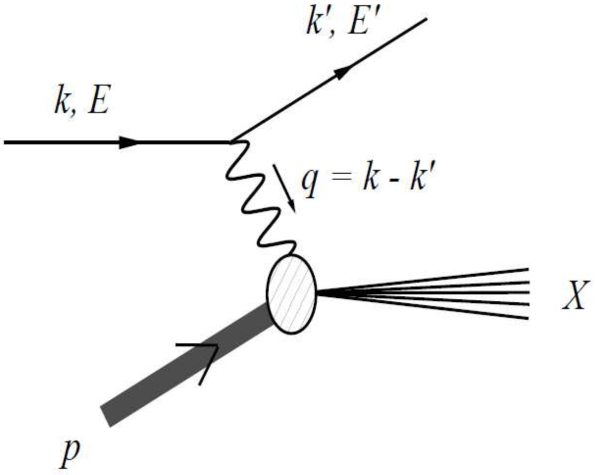
They observed a larger number than expected if the proton were elementary of large angle deflections of the electron. This is a reminiscent of Rutherford's experiment.

Deep Inelastic Scattering

To explain the hard behaviour in DIS, Feynman and Bjorken and Paschos, independently, proposed the parton model:

the nucleon has to be considered in deep inelastic collisions as a gas of non interacting pointlike particles, the partons. The electron simply suffers an elastic collision with a parton. Since a pointlike cross-section has not the form factor supression of an extended object, we have hard interactions with large electron angle deflections.

Deep Inelastic Scattering



DIS kinematics

The kinematic variables used in the DIS are M the nucleon mass, E and E' the energies of the incoming and outgoing lepton, lepton masses are neglected, and the four momenta are

$$
\begin{aligned}
k &= E\,(1,0,0,1) \ \text{incoming lepton moving in the } z \text{ direction} \\
k' &= E'\,(1,\, \sin\theta\cos\phi,\, \sin\theta\sin\phi,\, \cos\theta) \ \text{outgoing lepton moving along } (\theta,\phi) \\
P &= (M,0,0,0) \ \text{momentum of the target in the rest-frame of the hadron} \\
q &= k-k' \ \text{momentum transfer, i.e. momentum of the virtual photon} \\
\nu &= E-E' \,\text{(target rest frame)} = (P\cdot q)/M \ \text{energy loss of the lepton} \\
y &= \nu/E = 1-E'/E = (P\cdot q)/(P\cdot k) \ \text{fractional energy loss of the lepton} \\
Q^2 &= -q^2 = 2EE'\,(1-\cos\theta) = 4EE'\,\sin^2\theta/2 \\
x &= Q^2/2M\nu = Q^2/(2(P\cdot q)) = Q^2/(2MEy) \ \text{Bjorken variable} \\
\omega &= 1/x
\end{aligned}
$$

DIS kinematics

The invariant mass of the final hadronic state must be at least that of the nucleon (conservation of the baryon number). Thus, we have the inequality

$$M^2 \leq M_X^2 = (P+q)^2 = M^2 + 2P\cdot q q^2 \rightarrow 0 \leq x \leq 1$$

The lepton energy loss $E - E'$ must be between zero and E. The physically allowed kinematical region is

$$0 \leq x \leq 1 \qquad 0 \leq y \leq 1$$

In the case of a elastic scattering ($X = N$), we have $M_X^2 = M^2$, $Q^2 = 2M\nu$ and $x = 1$.
The formal definition of the Deep Ineslatic Scattering is

$$Q^2 \rightarrow \infty \ \nu \rightarrow \infty \quad \text{with } x \text{ fixed}$$

DIS kinematics

Therefore, in the deep inelastic limit, all nucleon resonances get pushed to $x = 1$,

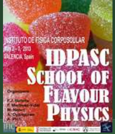
$$x = \frac{1}{1 + (M_X^2 - M^2)/Q^2} \rightarrow x \rightarrow 1 \text{ as } Q^2 \rightarrow \infty$$

For a fixed value of Q^2, a small region around $x = 1$ of width Λ^2/Q^2 contains hadronic resonances with masses around that of the nucleon. Notice that the resonance region is absent in the formal deep inelastic limit but present in any real experiment. As $x \rightarrow 0$ the invariant mass of the hadronic state increases. Outside the resonance region, M_X^2 is of order Q^2.

It is also useful the behaviour of the scattering angle θ and of the different components of q, as $Q^2 \rightarrow \infty$

$$\theta \propto \frac{M}{Q} \longrightarrow \theta \rightarrow 0 \quad q^0 = \frac{Q^2}{2Mxy} \quad q^3 = \frac{Q^2}{2Mxy} + Mxy \quad q^\perp \approx Q\sqrt{1-y}$$

It is interesting to note that $q^\perp$ is of order Q, q^0 and q^3 are both of order Q^2/M and $q^0 - q^3$ is of order M.



DIS cross section

The matrix element (scattering amplitude) has the structure

$$i\mathcal{M} = (-ie)^2 \left(\frac{-g^{\mu\nu}}{q^2} \right) \langle k' | J_j^\mu(0) | k, s_l \rangle \, \langle X | J_h^\nu(0) | P, \lambda \rangle$$

where $J_{l,h}$ are the leptonic and hadronic electromagnetic currents, s_l is the polarization of the incoming lepton and λ of the target hadron. The polarizations of the outgoing lepton and hadron states are not measured. By squaring $\mathcal{M}$, taking into account the phase space factors and summing over the polarisations of the outgoing states, we get

$$d\sigma = \frac{e^4}{Q^4} \int \frac{d^3k'}{(2\pi)^3\, 2E'} \, \frac{4\pi}{(2E)(2M)} \, L_{\mu\nu} H^{\mu\nu}$$

where we have defined the leptonic, $L_{\mu\nu}$ and hadronic, $H_{\mu\nu}$ tensors. It is useful to write de double differential scattering cross section in terms of x and y,

$$\frac{d^2\sigma}{dx\,dy} = \frac{\alpha^2}{Q^4} \, 2\pi y \, L_{\mu\nu} H^{\mu\nu}$$

The leptonic tensor

The leptonic tensor is defined by

$$L^{\mu\nu}(k,k') = \sum_{finalspin} \langle k'|J_l^{\nu}(0)|k,\, s_l\rangle\, \langle k,\, s_l|J_l^{\mu}(0)|k'\rangle$$

Since the leptons are pointlike fermions, $L^{\mu\nu}$ is simply

$$L^{\mu\nu} = \sum_{\text{finalspin}} \bar{u}(k')\,\gamma^{\nu}\,u(k,s_l)\;\bar{u}(k,s_l)\,\gamma^{\mu}\,u(k')$$

The sum over final polarisations is performed with $\sum_{\text{finalspin}} u(k')\,\bar{u}(k') = \not{k}' + m$ with spinors normalised to $2E$. The spinor product for the incoming lepton can be written in terms of the spin projector operator

$$u(k,s_l)\,\bar{u}(k,s_l) = (\not{k} + m)\,\frac{1}{2}\,\left(1 + \gamma_5\,\not{S}_l/m\right)$$

where S_l^{μ} is the spin vector defined by $2\,S_l^{\mu} = \bar{u}(k,s_l)\,\gamma^{\mu}\gamma_5\,u(k,s_l)$

The leptonic tensor (II)

Notice the unusual normalizatization of S_l by a factor of the fermion mass m. In the rest frame with spin up, the spin vetor is $\vec{S}_l = m\hat{z}$. It is very useful in the extreme relativistic limit, where all fermion masses can be neglected. With this normalization, longitudinally polarised fermions in the extrem relativistic limit have $S_l = h_l k$, where k is the momentum and $h_l = \pm 1$ is the fermion helicity.

The final expression for the lepton tensor is (neglecting the lepton masses)

$$L^{\mu\nu} = 2\left(k^\mu k'^\nu + k^\nu k'^\mu - g^{\mu\nu}(k\cdot k') - i\,\varepsilon^{\mu\nu\alpha\beta} q_\alpha S_{l\beta}\right)$$

The hadronic tensor

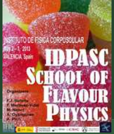
The hadronic tensor is defined by

$$H^{\mu\nu}(P,q) = \frac{1}{4\pi}\int d^4z\, e^{iq\cdot z}\, \langle P,\lambda|\left[J_h^\mu(z), J_h^\nu(0)\right]|P,\lambda\rangle$$

It cannot be computed directly, even in QCD PT, because of nonperturbative effects in the strong interactions. What we can due is to decompose it using all the symmetries of the theory: parity, time reversal, charge conjugation and, of course, Lorentz.
The most general polarisation state of a spin $1/2$ target in its redt frame can be written as a density matrix described by an axial vector $\vec{S}_h$

$$\rho = \frac{1}{2}\left(1 + \vec{\sigma}\cdot\frac{\vec{S}_h}{M}\right)$$

Notice the additional mass factor in the definition of $\vec{S}_h$. The four vector S_h^μ is defined to be $(0, \vec{S}_h)$ in the rest frame of the target and is defined in other reference frames by a Lorents boost.



The hadronic tensor: structure functions

The tensor $H^{\mu\nu}(P, q, S_h)$ for a spin-$1/2$ target is defined by

$$H^{\mu\nu}(P, q, S_h) = \mathrm{Tr}\,\rho\, H^{\mu\nu}$$

The most general tensor decomposition of $H^{\mu\nu}$ (polarised deep ineslatic scattering from spin-$1/2$ targets) using parity, time reversal invariance, hermicity of currents, antisymmetry of conmutator and translational invariance, can be written in terms of the so-called structure functions,

$$\begin{aligned}
H^{\mu\nu}(P, q, S_h) &= F_1(x, Q^2)\left(-g^{\mu\nu} + \frac{q^\mu q^\nu}{q^2}\right) \\
&+ \frac{F_2(x, Q^2)}{(P\cdot q)}\left(P^\mu - \frac{(P\cdot q)\, q^\mu}{q^2}\right)\left(P^\nu - \frac{(P\cdot q)\, q^\nu}{q^2}\right) \\
&+ \frac{i g_1(x, Q^2)}{(P\cdot q)}\, \varepsilon^{\mu\nu}{}_{\lambda\sigma}\, q^\lambda S_h^\sigma \\
&+ \frac{i g_2(x, Q^2)}{(P\cdot q)^2}\, \varepsilon^{\mu\nu}{}_{\lambda\sigma}\, q^\lambda \left((P\cdot q)\, S_h^\sigma - (S_h\cdot q)\, P^\sigma\right)
\end{aligned}$$

Unpolarized cross section and structure functions

Notice that as the leptonic current is conserved $q_\mu L^{\mu\nu} = q_\nu L^{\mu\nu} = 0$, we can omit the terms proportional to q^μ and q^ν before contracting with $L_{\mu\nu}$,

$$
\begin{aligned}
H^{\mu\nu}(P, q, S_h) &= -F_1(x, Q^2)\, g^{\mu\nu} \\
&+ \frac{F_2(x, Q^2)}{(P \cdot q)} P^\mu P^\nu \\
&+ \frac{i g_1(x, Q^2)}{(P \cdot q)} \varepsilon^{\mu\nu}{}_{\lambda\sigma} q^\lambda S_h^\sigma \\
&+ \frac{i g_2(x, Q^2)}{(P \cdot q)^2} \varepsilon^{\mu\nu}{}_{\lambda\sigma} q^\lambda \left((P \cdot q) S_h^\sigma - (S_h \cdot q) P^\sigma \right)
\end{aligned}
$$

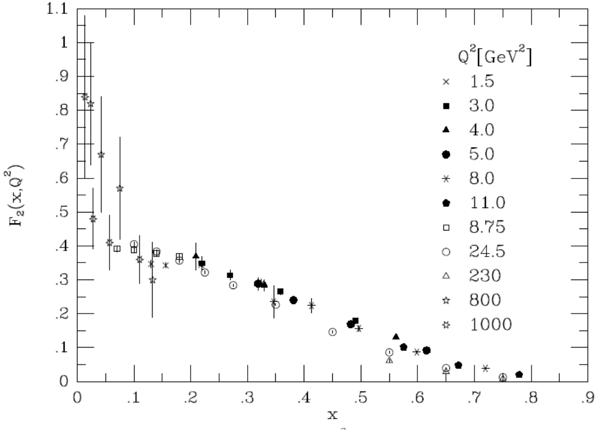
The cross section for unpolarized target and incoming lepton is

$$
\frac{d^2\sigma}{dx\,dy} = \frac{\alpha^2}{Q^4} 8\pi M E \left[x y^2 F_1(x, Q^2) + (1-y) F_2(x, Q^2) \right]
$$

Bjorken scaling

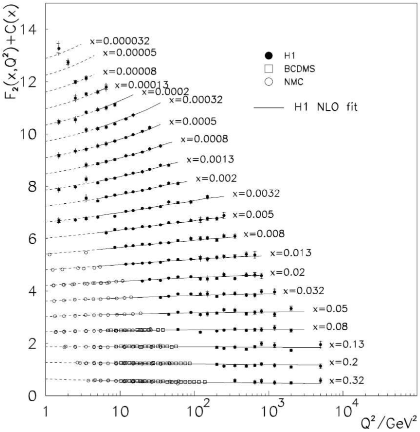
The early SLAC data indicates that if $F_1(x, Q^2)$ and $F_2(x, Q^2)$ are plotted as a function of x, they are nearly independent of Q^2. This behavior is called Bjorken scaling.

Experimental data for F_2



Experimental data for $F_2(x, Q^2)$ clearly shows Bjorken scaling

Scaling violation in F_2

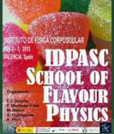


Q^2 dependence of $F_2(x,Q^2)$ is weak but visible

Naive parton model

To explain Bjorken scaling, Feynman introduced the so-called parton model in which the nucleon is made of free independent point-like partons (quarks and gluons) of spin $1/2$. In the deep inelastic limit, the inelastic lepton-nucleon scattering, viewed from a frame in which the nucleon has a very large (infinite) momentum, the photon scatters off these free on-shell partons. Partons can be any particles with no internal structure.

Notice that any frame is as good as the infinite momentum one but the parton model cannot easily formulated in other frames, as the proton rest frame. The lepton-nucleon center of mass system is at high energies a good approximation to this frame. In a infinite momentum frame the transverse momentum of the partons, their masses and the nucleon mass can be neglected with respect to the longitudinal one. Therefore the partons carry a fraction ξ of nucleon momentum, $p = \xi P$. Moreover, the corresponding cross section is the incoherent sum of individual parton cross sections. This is all we need to calculate differential cross sections.



The Breit frame

The parton model is, of course, Lorentz invariant but is more easily formulated in a frame in which the nucleon moves very fast. The most convenient system is the so-called Breit frame in which the nucleon and the photon collide head-on along, say, the X axis. The Breit frame of the hadron is defined by the requirements that the photon energy $q_0 = 0$ and that $\vec{q}$ be antiparallel to the hadron momentum, $\vec{P}$. Therefore, for $q \gg M$,

$$P = (\sqrt{p^2+M^2}, p, 0, 0) \approx (p, p, 0, 0) \qquad q = (0, -\sqrt{Q^2}, 0, 0)$$

Following Feynman, each parton carries a fraction of the nucleon momentum, ξP. Neglecting the quark mass compared to p, we have for the scattered parton

$$(q+\xi P)^2 \approx -Q^2 + 2\xi(P\cdot q) = 0$$

and

$$\xi = x \quad p = \sqrt{Q^2}/2x \quad q+\xi P = (xp, -\sqrt{Q^2}, 0, 0)$$

The struck parton scatters with momentum $q+xP$, in the direction of the virtual photon.

Factorization

Assume that in their rest frame, the nucleon is a sphere of radius R. In the Breit frame, due to the large Lorentz contraction, the sphere is viewed as a flat *pancake* with tranverse diameter $2R$ but with longitudinal one, $4RxM/Q \ll 2R$. The trasverse size of the photon is $\sim 1/Q \ll 2R$. The photon therefore interact with a tiny fraction of a thin disk and for sufficiently dilute partons, the photon effectively collides with a single free quark.

We can derive an explicit formula for the cross section. To do that, we introduce the parton distribution functions, pdfs, $f_q(\xi)$ such that $f_q(\xi)\, d\xi$ represents the probability that a parton carries a fraction of the nucleon momentum between ξ and $\xi + d\xi$. Provided that these partons are pointlike $r^2 \ll 1/Q^2$ and dilute $f_q \ll Q^2R^2$ the photons will scatter incoherently off individual partons. The cross section can then be factorized as the convolution of the pdfs with the cross section for parton scattering

$$\frac{d^2\sigma(l+N(P))}{dx\,dy} = \sum_q \int_0^1 d\xi\, f_q(\xi)\, \frac{d^2\hat{\sigma}(l+q(\xi P))}{dx\,dy}$$

Factorization again

To understand why the partons can be viewed as free, consider again the Breit frame in which the nucleon is moving very fast. Suppose the typical interaction time-scale in the nucleon is τ, in the moving frame, the interaction time becomes $\tau\gamma$, where $\gamma = 1/\sqrt{1-v^2}$ is the Lorentz dilation factor. When the speed of the proton approaches the velocity of light $v \approx 1$, the interaction time in the nucleon is so long that the nucleon configurations can be considered esentially frozen.

Alternatively, in the rest frame of the nucleon, the photon interaction time is of order $1/Q$, which is much shorter than the typical hadronic interaction time which is order $1/\Lambda_{QCD}$. Therefore the physics of scattering can be separated from the bound state physics, and the partons can be considered as essentially free during scattering. This is called factorization in QCD.

Hadronization

The central assumption on which the QPM is based is the separation of a collision in two distinct stages

1. **The hard scattering** of leptons on individual partons. During this scattering the influence of other partons is neglected and the cross-section of lepton-parton is calculated as if the partons were as real as leptons.
2. **Hadronization**. The outgoing quark and the remnant diquark cannot separate at infinity. Due to the force acting between colored quarks, which starts to be very strong when the separation is about 1 fm, convert them into observable hadrons.
   In the so-called independent fragmentation model, one uses the so-called fragmentation functions $D_q^h(\xi, p_T)$ to describe hadronization. It describes the probability that a parton q produces (*fragments into*) a hadron h carrying the fraction z of the parton energy and p_T transverse momentum.

Elastic electron-parton scattering

The fundamental process of the QPM is

$$e^-(k) + q(\xi P) \to e^-(k') + q(\xi P + q)$$

We need its matrix element. One possibility is to obtain it by crossing symmetry from the one for $e^+ e^- \to q\bar{q}$.

But it is easier to obtain the contribution of a single unpolarised quark to the hadronic tensor from $L^{\mu\nu}$ simply making the replacements $k \to \xi P$, $k' \to P'$ and $q \to -q$, assuming a parton of charge one.
$H^{\mu\nu}$ has an aditional factor of integration over the final particle phase space. Thus we have

$$\begin{aligned} H^{\mu\nu} &= \frac{1}{4\pi} Q^2 \int \frac{d^3 P'}{(2\pi)^3\, 2E_{P'}} \frac{1}{\xi} (2\pi)^4 \delta^4(\xi P + q - P') \\ &\times \quad 2\left[\xi P^\mu P'^\nu + \xi P^\nu P'^\mu - g^{\mu\nu} \xi (P \cdot P')\right] \end{aligned}$$

Elastic electron-parton scattering

Using that

$$\begin{aligned}
\int \frac{d^3P'}{(2\pi)^3\, 2E_{P'}} &= \int \frac{d^4P'}{(2\pi)^4}\,(2\pi)\,\delta((\xi P + q - P')^2) \\
&= \int \frac{d^4P'}{(2\pi)^4}\,\frac{2\pi}{2(P\cdot q)}\,\delta\left(\xi + \frac{q^2}{2(P\cdot q)}\right)
\end{aligned}$$

we get

$$\begin{aligned}
H^{\mu\nu} &= \frac{1}{2\xi\,(P\cdot q)}\left[\xi P^\mu P'^\nu + \xi P^\nu P'^\mu - g^{\mu\nu}\,\xi\,(P\cdot P')\right]\delta(\xi - x) \\
&= \frac{1}{2\xi\,(P\cdot q)}\left[2\xi P^\mu P^\nu - g^{\mu\nu}\,\xi\,(P\cdot q)\right]\delta(\xi - x)
\end{aligned}$$

where P' has been replaced by $\xi P + q$ and the terms q^μ and q^ν have been dropped.

Structure functions in the QPM

Comparing with the decomposition of the hadronic tensor in terms of the structure functions, we obtain

$$F_1^{(q)} = \frac{1}{2} Q_q^2 \delta(\xi - x) \qquad F_2^{(q)} = \xi Q_q^2 \delta(\xi - x)$$

The antiquark contribution can be computed similarly and is identical to the quark contribution.

Finally, we have to incoherently sum the partonic cross sections. To do that we introduce the distributions functions $q(x)$ and $\bar{q}(x)$, that represent the probability to find a quark (antiquark) in the nucleon with momentum fraction ξ. Integrating, we get

$$\begin{aligned} F_1(x) &= \sum_{q,\bar{q}} \int_0^1 d\xi\, q(\xi)\, x Q_q^2 \delta(x - \xi) = \sum_{q,\bar{q}} Q_q^2\, x\, q(x) \\ F_2(x) &= \sum_{q,\bar{q}} \frac{1}{2} Q_q^2\, q(x) \end{aligned}$$

Bjorken scaling in the Parton model

The main result of the QPM is

$$
\begin{aligned}
F_1(x) &= \sum_{q,\bar{q}} Q_q^2\, x\, q(x) \\
F_2(x) &= \sum_{q,\bar{q}} \frac{1}{2} Q_q^2\, q(x)
\end{aligned}
$$

In addition, they satisfy the celebrated Callan-Gross relation

$$F_2(x) = 2x F_1(x)$$

which is a consequence of the assumption of spin-$\frac{1}{2}$ partons. It is easy to show that por spin-0 partons, $F_1(x) = 0$.

The important conclusion is that $F_{1,2}$ are independent of Q^2 and depend only on x. Therefore the parton model explains the scaling naturally. Notice that in the derivation above no assumptions are made about the quark interactions. Therefore, this model does not explain the internal dynamics of the nucleon.

Scaling violation

Bjorken scaling is satisfied pretty well by the data but systematic deviation from scaling is also clearly seen in the data: with increasing Q^2, the structure function F_2 increases/decreases at small/large values of x.

This was already known at the beginning of the seventies, before the invention of QCD.

Scaling violation is due to the radiation of hard gluons generating transverse momenta for the quarks. When Q^2 increases, more gluons are radiated, leading to logarithmic scaaling violations in the structure functions and to scale dependent parton distribution functions $q_i(x, \mu^2)$. More in the renormalization section.

Asymptotic freedom

As we said before, the success of the QPM is based on the separation of time scales in the deep ineslastic limit: the typical time for parton-parton interaction is much larger than the time for the photon-parton interaction. The correct theory of strong interactions must be such that it exhibits strong binding forces over time or distance scales of the order of 1 fm but is weak, as if the partons were free, over time or distance scales much shorter.

This necessary property of the strong interactons is exhibited by QCD: the forces it mediates grows weaker with shorter interaction times.

Parton distributions

How can we obtain information on the individual parton densities?
We can apply some constraints as flavour conservation of the strong interactions. In fact, as the proton contains two up quarks, one down quark and no strange quark, it follows that

$$\int_0^1 d\xi\, [u(\xi) - \bar{u}(\xi)] = 2 \quad \int_0^1 d\xi\, \left[d(\xi) - \bar{d}(\xi)\right] = 1 \quad \int_0^1 d\xi\, [s(\xi) - \bar{s}(\xi)] = 0$$

Since the longitudinal momentum of the partobns must add up to the longitudinal moemntum of the proton, we have a momentum sum rule

$$\int_0^1 d\xi\, \xi \left[u(\xi) + d(\xi) + s(\xi) + \bar{u}(\xi) + \bar{d}(\xi) + \bar{s}(\xi)\right] = 1$$

Isospin allows us to relate the parton densities of proton and neutron

$$u^n(\xi) = d^p(\xi) \;\; d^n(\xi) = u^p(\xi) \;\; \bar{u}^n(\xi) = \bar{d}^p(\xi) \;\; \bar{d}^n(\xi) = \bar{u}^p(\xi) \;\; s^n(\xi) = s^p(\xi)$$

and similar relations for antiproton parton densities.

Gauge Invariance (I)

Gauge invariance is the main ingredient of QCD (also of the SM)

Abelian case: QED

We start with the Dirac fermion Lagrangian

$$\mathcal{L} = \bar{\psi}(x)\, i\partial\!\!\!/\, \psi(x) - m\,\bar{\psi}(x)\,\psi(x)$$

Both the lagranfian and the equation of motion are invariant under global phase transformations ($U(1)$)

$$\psi(x) \longrightarrow \psi'(x) = e^{-iQ\theta}\,\psi(x)$$

where θ is an arbitrary space-time independent real constant. If the phase depends on the space-time point, things change. The mass term remains invariant, as is easily checked, but the kinetic term transforms as

$$\partial_\mu\psi(x) \to e^{-iQ\theta}\,(\partial_\mu - iQ\partial_\mu\theta(x))\,\psi(x)$$

The procedure for enforcing local invariance is to enlarge the theory by introducing a spin-1 vector field A_μ.

Gauge Invariance (II)

The A_μ transforms in precisely the right way to cancel the piece proportional to $\partial_\mu \theta(x)$. Therefore one replace the ordinary derivative ∂_μ by the covariant one D_μ,

$$D_\mu \psi(x) \to (\partial_\mu + iQA_\mu)\,\psi(x)$$

with

$$A_\mu(x) \to A'_\mu(x) = A_\mu(x) + \partial_\mu \theta(x)$$

The covariant derivative transforms covariantly

$$D_\mu \psi(x) \to (D_\mu \psi)'(x) = e^{-iQ\theta(x)} D_\mu \psi(x)$$

and the Lagrangian is now invariant under local $U(1)$ transformations,

$$\mathcal{L} = \bar{\psi}(x)\left(i\not{D} - m\right)\psi(x) = \mathcal{L}_{free} - QA_\mu \bar{\psi}(x)\gamma^\mu \psi(x)$$

To sum: the requirement of local gauge invariance has generated an interaction term between the fermion field ψ and the gauge field A_μ.

Gauge Invariance (III)

In order to have a propagating gauge field, one adds a kinetic term for the gauge field

$$\mathcal{L} = -\frac{1}{4} F^{\mu\nu} F_{\mu\nu} + \bar{\psi}(x) \left(i \not{D} - m \right) \psi(x)$$

where the field strength tensor is defined as

$$F_{\mu\nu} = -[D_\mu, D_\nu] = \partial_\mu A_\nu - \partial_\nu A_\mu$$

and is automatically gauge invariant.

Finally, setting $Q = -e$, the electron charge, we obtain the QED lagrangian from the free Dirac lagrangian. Notice that a mass term for the gauge field of the form $M^2 A_\mu A^\mu$ is not allowed because it violates gauge invariance. Therefore, the gauge field (photon) is stricly massless.

Gauge Invariance (IV)

Non Abelian case: QCD

Quarks come in three colours with the same mass. Therefore the free quark Lagrangian for a given flavour can be writen as

$$\mathcal{L}_0 = \sum_{i=1}^{3} \bar{q}_i(x) \left(i\not{\partial} - m_q \right) q_i(x)$$

The largest symmetry of this Lagrangian is a unitary transformation of the colour indices

$$q_i \longrightarrow q'_i = U_{ij}\, q_j \qquad U\, U^\dagger = U^\dagger\, U = I$$

Extracting a common phase $U(3) = U(1) \times SU(3)$, we have that the symmetry group is $SU(3)$, the special unitary group of the three-dimensional unitary matrices with unit determinant.

It is easy to see that $SU(3)$ has 8 independent transformations corresponding to 8 parameters of the Lie group.

Gauge Invariance (V)

Therefore, in this case we have more room to impose local gauge invariance. Quarks are in the 3 representation and antiquarks in the 3^*. In this way, both $\bar{q}q$ and qqq contain colour singlets, i.e. hadrons,

$$\bar{\mathbf{3}} \otimes \mathbf{3} = \mathbf{1} \oplus \mathbf{8} \qquad \mathbf{3} \otimes \mathbf{3} \otimes \mathbf{3} = \mathbf{1} \oplus \mathbf{8} \oplus \mathbf{8} \oplus \mathbf{10}$$

Therefore, we must gauge all $SU(3)$ transformations.
Every three-dimensional unitary matrix with unit determinant can be written as

$$U(\theta_1, \cdots, \theta_8) = \exp\left[-i \sum_{a=1}^{8} \theta_a \frac{\lambda_a}{2}\right]$$

where θ_a are eight real parameters and λ_a are the eight traceless hermitian Gell-Mann matrices. The $t_a = \lambda_a/2$ are the generators of the fundamental representation of $SU(3)$ and hence satisfy the Lie algebra commutation relations

$$[t_a, t_b] = i f_{abc} t_c$$

with real, totally antisymmetric structure constants f_{abc}.

Gauge Invariance (VI)

We are ready to impose local gauge invariance of the theory for arbitrary space-time θ_a. As for the abelian case, we have to introduce gauge fields that compensate the extra term in the derivatives. We need 8 gauge fields, $G^\mu_a(x)$ and a covariant derivative

$$(D^\mu q)_i = \left(\partial^\mu \delta_{ij} + i\, g_s \sum_{a=1}^{3} G^\mu_a\, (t^a)_{ij} \right) q_j$$

It is convenient to use a matrix notation

$$G_\mu \equiv G^\mu_a \frac{\lambda^a}{2} = G^\mu_a t^a$$

As before the covariant derivative transforms covariantly (as the quark fields)

$$(D^\mu q)_i \to U_{ij}(\theta_a(x))\, (D^\mu q)'_i$$

Gauge Invariance (VII)

provided the gauge fields transform as

$$G_\mu \to G'_\mu = U(\theta_a(x))\, G_\mu\, U^\dagger(\theta_a(x)) + \frac{i}{g_s}\left(\partial_\mu(\theta_a(x))\right) U^\dagger(\theta_a(x))$$

For infinitesimal values of the parameters, the transformation law of the fields components is

$$G_a^\mu \to G_a^{\mu\prime} = G_a^\mu + \frac{1}{g_s}\partial^\mu\theta_a + f_{abc}\,\theta_b\, G_c^\mu + O(\theta^2)$$

which is similar to the electromagnetic case but now, and this is very important, there is a term proportional to the structure constant. This relation is a generalization and reduces to the abelian case because $f_{abc} = 0$ for an abelian gauge group.

To have propagating gauge fields, we need a field strength tensor, $G^{\mu\nu}$ defined as

$$G_{\mu\nu} = -\frac{i}{g_s}\left[D_\mu, D_\nu\right] = \partial_\mu G_\nu - \partial_\nu G_\mu + i g_s \left[G_\mu, G_\nu\right]$$

QCD Lagrangian

It transforms covariantly

$$G_{\mu\nu} \to G'_{\mu\nu} = U(\theta_a(x))\, G_{\mu\nu}\, U^\dagger(\theta_a(x))$$

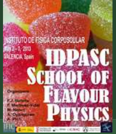
Using this result, it is easy to construct a term quadratic in the gauge field and invariant under gauge transformations

$$\mathrm{Tr}\,(\,G_{\mu\nu}\, G^{\mu\nu}) = \frac{1}{2}\, G_a^{\mu\nu}\, G^a_{\mu\nu}$$

where the trace is over the colour.
Therefore, the $SU(3)_c$ invariant lagrangian for N_F quark flavours is

$$\mathcal{L}_{QCD} = -\frac{1}{2}\,\mathrm{Tr}\,(\,G_{\mu\nu}\, G^{\mu\nu}) + \sum_{f=1}^{N_F} \bar{q}_f\, \left(i\not{D} - m_f\, I\right)\, q_f$$

with massless gauge fields, called gluons.



Feynman rules

Following the book by Greiner, Schramm and Stein, the Feynman rules for each interaction term in the Lagrangian are obtained by varying the corresponding action integral in momentum space. For example, for the quark-gluon vertex, the relevant part of the Lagrangian is

$$\mathcal{L} = \bar{\psi}\,\frac{\lambda^a}{2}\,\gamma_\mu\,G^{a\mu}\,\psi$$

Therefore, we have to calculate

$$\frac{\delta^3}{\delta\psi^i_\gamma(p)\,\delta\bar{\psi}^j_\beta(p')\,\delta G^{b\nu}(k)}\left[\int \bar{\psi}^l_\alpha(p_1)\,g_s\left(\frac{\lambda^a}{2}\right)_{lnm}(\gamma_\mu)_{\alpha\sigma}\right.$$
$$\left.\times\; G^{a\mu}(p_2)\,\psi^m_\sigma(p_3)\,(2\pi)^4\,\delta^4(p_1+p_2-p_3)\,d^4p_1\,d^4p_2\,d^4p_3\right]$$

Using

$$\frac{\delta\psi^i_\gamma(p)}{\delta\psi^j_\beta(p')} = \delta^4(p-p')\,\delta_{ij}\,\delta_{\gamma\beta}$$

Feynman rules (II)

the quark-gluon vertex turns out to be

$$g_s \left(\frac{\lambda^b}{2} \right)_{ji} (\gamma_\nu)_{\beta\gamma}$$

The three-gluon and four-gluon vertices, with momentum routing $\delta^4(k_1 + k_2 + k_3)$, can be evaluated in an analogous way,

$$i\, g_s f_{rst} \left[(k_1^\tau - k_2^\tau)\, g^{\rho\sigma} + \left(k_2^\rho - k_3^\rho\right) g^{\sigma\tau} + (k_3^\sigma - k_1^\sigma)\, g^{\tau\rho} \right]$$

and

$$\begin{aligned} -g_s^2 \,[\quad & f_{eab} f_{ecd} \left(g^{\alpha\gamma} g^{\beta\delta} - g^{\alpha\delta} g^{\beta\gamma} \right) \\ + \quad & f_{eac} f_{edb} \left(g^{\alpha\delta} g^{\beta\gamma} - g^{\alpha\beta} g^{\gamma\delta} \right) \\ + \quad & f_{ead} f_{ebc} \left(g^{\alpha\beta} g^{\gamma\delta} - g^{\alpha\gamma} g^{\beta\delta} \right)] \end{aligned}$$

Feynman rules: propagators

The quark propagator is the same as in QED,

$$\frac{1}{\not{p} - m + i\varepsilon}$$

The gluon propagator is much more subtle because the equation of motion of the free gluon field cannot be inverted to give the Green function of the field equation. This problem appears also in Abelian theories as QED. Consider the free photon Lagrangian

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} = \frac{1}{2} A_\nu \left(g^{\mu\nu} \partial^2 - \partial^\mu \partial^\nu \right) A_\mu$$

where we have integrate by parts and discarded a surface term. The photon propagator is, by definition, the inverse of the operator between brackets,

$$\left(g^{\mu\nu} \partial^2 - \partial^\mu \partial^\nu \right) D_{\nu\lambda}(x-y) = \delta^\mu_\lambda \, \delta(x-y)$$

Feynman rules: Gluon propagator

Contracting with ∂_μ, we get

$$0 \cdot D_{\nu\lambda}(x-y) = \partial_\lambda \delta(x-y) \;\rightarrow\; D_{\nu\lambda} \text{ is ill-defined !!}$$

In other words, the inverse of $P_{\nu\lambda} = \left(g^{\mu\nu}\partial^2 - \partial^\mu\partial^\nu\right)$ does not exist. This is very easy to understand because $P_{\nu\lambda}$ is a projector onto transverse modes and satisfies $P^2 = P$. As all projectors, P does not have inverse because it has a zero eigenvalue corresponding to zero modes as, for example, $\partial_\mu\Lambda(x)$ for an arbitrary Λ. The physical reason is that D propagates physical and unphysical degrees of freedom because fields that are related only by gauge transformations $A_\mu \rightarrow A_\mu + \partial_\mu\Lambda$ are propagated as well, giving an infinit contributions.

The solution is to **fix the gauge**; to choose a particular gauge. At the end of the day, physical observables as the S-matrix will be gauge independent. As a first possibility, we impose the Lorentz condition,

$$\partial_\mu A^\mu = 0$$

Feynman rules: Gauge fixing

This gauge condition can be included in the Lagrangian by using a Lagrange multiplier as

$$\mathcal{L}_{fix} = \mathcal{L} - \frac{1}{2\lambda}\left(\partial_\mu A^\mu\right)^2 = \mathcal{L} + \frac{1}{2\lambda} A^\mu g_{\mu\nu} \partial^2 A^\nu$$

where λ is some arbitrary gauge parameter. The quadratic term in A_μ is now,

$$\frac{1}{2} A^\mu \left(g_{\mu\nu} \partial^2 - (1 - \lambda^{-1}) \partial_\mu \partial_\nu \right) A^\nu$$

that can be inverted to give the photon propagator

$$D_{\mu\nu}(k) = -\frac{1}{k^2}\left(g_{\mu\nu} - (1-\lambda)\frac{k_\mu k_\nu}{k^2} \right)$$

The Feynman gauge corresponds to $\lambda = 1$ whereas in the Landau gauge $\lambda = 0$.

Feynman rules: Ghosts

In QED, the gauge fixing term in the Lagrangian does not affect the physics of the theory because the field $\chi = \partial_\mu A^\mu$ does not interact with physical degrees of freedom. It obeys a free field equation $\partial^2\chi = 0$ and therefore does not mix with the transverse part of the photon field. In a non-Abelian theory, there is a complication because the field $\chi^a = \partial^\mu G^a_\mu$ satisfies a non-free wave equation

$$\partial^2\chi^a + g_s f^{abc} G^{b\mu}\partial_\mu\chi^c = 0$$

Therefore, the unphysical particles χ^a interact with the physical components of G^a_μ, contributing to gluon loops. Therefore, their effects must be subtracted.

The subtraction is done by the introduction of unphysical ghost fields η^a that exactly cancel the χ^a fields.

First done by De Witt, Fadeev and Popov, the so-called Fadeev-Popov ghost fields obey the same equation as the scalar χ^a fields but they are quantized as fermion field.

Feynman rules: Ghosts

Why must they obey a fermion statistic? Remember that, according to Fermi-Dirac statistics, each closed loop of ghost fields generates a (-1) factor. It follows that if for each gluon loop one includes one ghost loop, the longitudinal part of the gluons is cancelled exactly. The complete gauge-fixing term is

$$\mathcal{L}_{fix} = \mathcal{L} - \frac{1}{2\lambda}\left(\partial_\mu A^\mu\right)^2 + \partial_\mu \eta^{a\,\dagger}\left(\partial^\mu \eta^a + g_s f^{abc} G^{b,\mu} \eta^c\right)$$

From this Lagrangian the Feynman rules for ghosts propagators and vertices can be derived. The ghost Lagrangian repairs the damage done by gauge fixing. The Green functions are gauge dependent but the observable S-matrix elements are gauge invariant and therefore independent of λ.

Experimental group theory

The form of the generators of $SU(3)$, t_a depend on the irreducible representation. For example, in the fundamental (associated to the quarks) and adjoint (associated to the gluons) representations, they are

$$(t_a^F)_{ij} = \frac{(\lambda_a)_{ij}}{2} \qquad (t_a^A)_{bc} = -i f_{abc}$$

It is easy to verify that both of them satisfy the commutation relation

$$[t_a, t_b] = i f_{abc}\, t_c$$

We know that the vertices are determined by the generators of the symmetry Lie group in the representations for quarks and gluons, t_a^F and t_a^A, respectively. In the measurable quantities, like S-matrix elements, the generators appear in two combinations

$$\mathrm{Tr}\left(t_a^R\, t_b^R \right) = T(R)\,\delta_{ab} \qquad \sum_a \left(t_a^R\right)_{ij} \left(t_a^R\right)_{jk} = C(R)\,\delta_{ik} \qquad R = F, A$$

Experimental group theory

The $T(R)$ is called the Dynkin index for the representation R and $C(R)$ is the quadratic Casimir for R. It is easy to redive from the definitions the following relation for a d_R-dimensional irreducible representation

$$d_R\, C(R) = n_G\, T(R)$$

where n_G is the number of independent parameters of the group G. For $SU(N)$, we have $n_G = N^2 - 1$ and

$R = A$ (adjoint representation) $d_A = n_G$ and thus $C(A) = T_A = N$

$R = F$ (fundamental representation) $d_F = N, T_F = \frac{1}{2}$ and thus $C(F) = \frac{N^2-1}{2N}$

For $SU(3)$, $C(F) = \frac{4}{3}$, $C(A) = T_A = N = 3$. We can determining experimentaly $C(F)$ and $C(A)$ from a combined jet analisis in e^+e^- annihilation at LEP

$$C(F) = 1.30 \pm 0.10 \qquad C(A) = 2.89 \pm 0.23$$

in good agreement with $SU(3)$

Quark-quark scattering

Consider the elastic scattering a tree-level of two quarks with momenta, flavors and colors as indicated in Fig, averaged over spins (λ) and colors (c). Two diagrams contribute to lowest order. Working in the Feyman gauge, the invariant amplitudes for the two diagrams are

$$M_t = i\frac{g_s^2}{t}\left(t^a_{ji}\,t^a_{lk}\right)\left[\bar{u}_\alpha(p_4)\gamma^\mu u_\alpha(p_1)\right]\left[\bar{u}_\beta(p_3)\gamma_\mu u_\beta(p_2)\right]$$

$$M_u = i\frac{g_s^2}{u}\left(t^a_{li}\,t^a_{jk}\right)\left[\bar{u}_\beta(p_3)\gamma^\mu u_\alpha(p_1)\right]\left[\bar{u}_\alpha(p_4)\gamma_\mu u_\beta(p_2)\right]$$

in terms of the Mandelstam variables s, t and u, defined as

$$s = (p_1 + p_2)^2 \qquad t = (p_1 - p_4)^2 \qquad u = (p_1 - p_3)^2$$

Consider first the averaged square of M_t,

$$\left\langle |M_t|^2 \right\rangle \equiv \frac{1}{9}\frac{1}{4}\sum_{\lambda,c}|M_t|^2 = \frac{1}{9}\frac{1}{4}\frac{g_s^4}{t^2}\sum_{i,j,k,l}\left[t^a_{ji}\,t^a_{lk}\,t^{b*}_{ji}\,t^{b*}_{lk}\right]\mathrm{Tr}\left[\not{p}_4\gamma_\mu\not{p}_1\gamma_\nu\right]$$

$$\times \quad \mathrm{Tr}\left[\not{p}_3\gamma^\mu\not{p}_2\gamma^\nu\right]$$

Quark-quark scattering

The factor $\frac{1}{9}$ comes from the color and $\frac{1}{4}$ from spin averaging in the initial state. The color trace has the form

$$\sum_{i,j,k,l} \left[t^a_{ji}\, t^a_{lk}\, t^{b*}_{ji}\, t^{b*}_{lk} \right] = \mathrm{Tr}(t^a\, t^b)\, \mathrm{Tr}(t^a\, t^b) = \frac{1}{4} \left(\delta^{ab}\right)^2 = 2$$

while the spinor traces read

$$\begin{aligned} \mathrm{Tr}\left[\not{p}_4\, \gamma_\mu \not{p}_1\, \gamma_\nu\right] \mathrm{Tr}\left[\not{p}_3\, \gamma^\mu \not{p}_2\, \gamma^\nu\right] &= 32\left((p_1p_2)(p_3p_4) + (p_1p_3)(p_2p_4)\right) \\ &= 8\left(u^2 + s^2\right) \end{aligned}$$

Putting all together yields

$$\left\langle |M_t|^2 \right\rangle = \left(\frac{2}{9}\right)_c \frac{g_s^4}{t^2}\, 2\left(u^2 + s^2\right)$$

Quark-quark scattering

The other terms can be evaluated similarly. Finally, we get

$$
\begin{aligned}
\left\langle |M|^2 \right\rangle &= \frac{1}{9}\frac{1}{4}\sum_{\lambda,c}\left(|M_t|^2 + |M_u|^2 + 2M_t^* M_u\right) \\
&= \left(\frac{2}{9}\right)_c g_s^4 \left[\frac{2\left(u^2+s^2\right)}{t^2} + \delta_{\alpha\beta}\frac{2\left(t^2+s^2\right)}{u^2}\right. \\
&+ \left.\delta_{\alpha\beta}\left(-\frac{1}{3}\right)_c \frac{4s^2}{tu}\right]
\end{aligned}
$$

Quark-gluon scattering

This is the simplest process where the gluon selfinteraction vertex appear. The three Feynman diagrams contributing to lowest order are shown in Fig. The first two are, except for the color degree of freedom, exactly the same as in QED.

$$M_s = \frac{-i\, g_s^2}{s} \left(t^a_{li} t^b_{jl}\right) \left[\bar{u}(p_2)\, \not\epsilon_2 \left(\not p_1 + \not q_1\right) \not\epsilon_1\, u(p_1)\right]$$

$$M_u = \frac{-i\, g_s^2}{u} \left(t^a_{jk} t^b_{ki}\right) \left[\bar{u}(p_2)\, \not\epsilon_1 \left(\not p_1 - \not q_2\right) \not\epsilon_2\, u(p_1)\right]$$

where $\varepsilon_{1,2}$ are the polarisation vectors of the of the initial and final state gluons. The last and new diagram contains the 3-gluon vertex. Its contribution is

$$M_t = \frac{g_s^2}{t} \left[f_{abc} \left(t^c\right)_{ij}\right] S^{\lambda\mu\nu}(q_1, -q_2, -q_1, q_2)\, \varepsilon_{1\mu} \varepsilon_{2\nu} \left[\bar{u}(p_2) \gamma_\lambda u(p_1)\right]$$

where $S^{\lambda\mu\nu}$ denotes the combination of momenta in the Feynman rule for the quartic gluon vertex.

Quark-gluon scattering

The evaluation of the Dirac traces is the sames as in QED. What is new are the color factors. For example, the color trace in the color average of $|M_s|^2$,

$$\frac{1}{3}\,\frac{1}{8}\,t^a_{li}\,t^b_{jl}\,t^{b*}_{jk}\,t^{a*}_{ki} = \frac{1}{24}\,\mathrm{Tr}(t^a\,t^b\,t^b\,t^a) = \frac{2}{9}$$

The same color factor appears in $|M_u|^2$. The last diagram, however, involves the color trace

$$\frac{1}{3}\,\frac{1}{8} f_{abc}\,t^c_{ji} f_{abd}\,t^{d*}_{ji} = \frac{1}{24} f_{abc} f_{abd}\,\mathrm{Tr}[t^c\,t^d] = \frac{1}{24}\,3\delta_{cd}\,\frac{1}{2}\,\delta_{cd} = \frac{1}{2}$$

Quark-gluon scattering (gauge invariance)

To sum over the initial and final polarisations of the gluons we need the expression

$$\sum_{\lambda} \varepsilon_\mu(\lambda, q)\, \varepsilon^*_\nu(\lambda, q) = -g_{\mu\nu}$$

valid in the Feynman gauge only. In fact, the polarization vectors depend on the gauge and so do the results for each individual contributions of squares and interference terms from the diagrams. The full result is, however, gauge independent.

Results for quark-gluon processes

Spin and color averaged invariant amplitudes $\left\langle |M|^2 \right\rangle$ normalized such that $\frac{d\sigma}{dt} = \frac{1}{16\pi s^2} \left\langle |M|^2 \right\rangle$.

Process	$\left\langle \lvert M\rvert^2 \right\rangle / g^4$
$q_\alpha q_\beta \to q_\alpha q_\beta$	$\frac{2}{9}\left[\frac{2(s^2+u^2)}{t^2} + \delta_{\alpha\beta}\left(\frac{2(t^2+s^2)}{u^2} - \frac{1}{3}\frac{4s^2}{ut}\right)\right]$
$q_\alpha \bar{q}_\beta \to q_\alpha \bar{q}_\beta$	$\frac{2}{9}\left[\frac{2(s^2+u^2)}{t^2} + \delta_{\alpha\beta}\left(\frac{2(t^2+u^2)}{s^2} - \frac{1}{3}\frac{4u^2}{st}\right)\right]$
$q\,g \to q\,g$	$\left(1 - \frac{us}{t^2}\right) - \frac{4}{9}\left(\frac{s}{u} + \frac{u}{s}\right) - 1$
$g\,g \to q\,\bar{q}$	$\frac{1}{6}\left(\frac{u}{t} + \frac{t}{u}\right) - \frac{3}{4}\left(1 - \frac{ut}{s^2}\right) + \frac{3}{8}$
$q\,\bar{q} \to g\,g$	$\frac{64}{9} M(gg \to q\bar{q})$
$g\,g \to g\,g$	$\frac{8}{9}\left[-\frac{33}{4} - 4\left(\frac{us}{t^2} + \frac{ut}{s^2} + \frac{st}{u^2}\right)\right] - \frac{9}{16}\left[45 - \left(\frac{s^2}{ut} + \frac{t^2}{us} + \frac{u^2}{ts}\right)\right]$

The cross-section for $ud \to ud$ is just like in $e\mu \to e\mu$ but replacing $\alpha \to \alpha_s$ and including a factor of $2/9$. The $u\bar{u} \to d\bar{d}$ annihilation cross-section is obtained from $ud \to ud$ by crossing $t \leftrightarrow s$ in the amplitude.

Hadron production in e^+e^- collisions

One of the most fundamental quantities of QCD. Consider all diagrams in which a virtual photon with momentum q and Lorentz index ν produces a set of n hadrons with momenta $\{p_1, \cdots, p_n\}$. Let $T_\nu(n, q, \{p_i\})$ be the amplitud for the sum of all these diagrams. The matrix element for the full process is,

$$M = [\bar{v}(q_2)\, e\gamma_\mu\, u(q_1)]\, \frac{-g^{\mu\nu}}{q^2}\, T_\nu(n, q, \{p_i\})$$

Summing over all types of hadrons in the final state and integrating the phase-space, we get the total cross-section

$$\sigma = \frac{1}{2s}\,\frac{1}{4}\,\frac{e^2}{s^2}\,\mathrm{Tr}(\not{q}_2\,\gamma^\mu\,\not{q}_1\,\gamma^\nu)\,H_{\mu\nu}(q)$$

where we have define the so-called hadronic tensor

$$H_{\mu\nu}(q) = \sum_n \int dPS_n\, T_\mu(n, q, \{p_i\})\, T^*_\nu(n, q, \{p_i\})$$

Hadron production in e^+e^- collisions

To construct $H_{\mu\nu}$ there are only two Lorentz covariant two-index tensors $g_{\mu\nu}$ and $q_\mu q_\nu$ with coefficients which depend on q^2 and the masses of the final state hadrons. For energies well above all hadron masses, we can write

$$H_{\mu\nu}(q) = A(q^2)\, g_{\mu\nu} + B(q^2)\, q_\mu q_\nu$$

Gauge invariance implies that $H_{\mu\nu}$ must satisfy $q^\mu H_{\mu\nu} = q^\nu H_{\mu\nu} = 0$, giving a relation between both form factors

$$A(q^2) = -q^2 B(q^2) \rightarrow H_{\mu\nu}(q) = \left(q_\mu q_\nu - q^2 g_{\mu\nu}\right) B(q^2)$$

It is easy to show that B has to be dimensionless, and hence a constant. We arrive at the conclusion that the cross-section to produce any number of hadrons is proportional to that to produce a muon-antimuon pair,

$$R_{had} \equiv \frac{\sigma(e^+e^- \rightarrow q\bar{q})}{\sigma(e^+e^- \rightarrow \mu^+\mu^-)} = \text{constant}$$

Review $e^+e^- \to \mu^+\mu^-$

The amplitude for $e^-(q_1)\,e^+(q_2) \to \mu^-(p_1)\,\mu^+(p_2)$ can be calculated applying the QED Feynman rules to the only possible diagram a tree-level,

$$M = \bar{v}(q_2)\,ie\gamma^\mu\,u(q_1)\,\frac{-g_{\mu\nu}}{(q_1+q_2)^2}\,\bar{u}(p_1)\,ie\gamma^\nu\,u(p_2)$$

Summing over unmeasured polarisations, the square of the matrix element is ($s^2 = (q_1+q_2)^2$)

$$\sum |M|^2 = \frac{(4\pi\alpha)^2}{s^2}\,\mathrm{Tr}\left[\not{q}_2\gamma^\mu\not{q}_1\gamma^\nu\right]\,\mathrm{Tr}\left[\not{p}_1\gamma_\mu\not{p}_2\gamma_\nu\right]$$

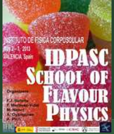
Finally, integrating over the two-body pahse space, we get

$$\sigma = \frac{1}{2s}\,\frac{1}{4}\,\frac{e^2}{s^2}\,\mathrm{Tr}(\not{q}_2\gamma^\mu\not{q}_1\gamma^\nu)\,H^{(\mu)}_{\mu\nu}(q)$$

with

$$H^{(\mu)}_{\mu\nu}(q) = \int dPS_2\,\mathrm{Tr}\left[\not{p}_1\gamma_\mu\not{p}_2\gamma_\nu\right] = \left(q_\mu q_\nu - q^2 g_{\mu\nu}\right)B^{(\mu)}$$

In the last step, we have used the same symmetry arguments than before for the hadronic tensor. Thus, $B^{(\mu)}$ is a constant.



Hadron production in e^+e^- collisions

To calculate the constant we need a model of the production of hadrons.

The quark parton model, provides us with it. The handwaving argument is as follows. The photon is highly virtual and therefore it is produced and decays to quarks in a small space-time volume. The time scale is $t \sim 1/\sqrt{s}$. On the other hand, the wavefunction of a hadron with mass m_h has a spatial extent of order $\sim 1/m_h$ and this is the order of magnitud of the time that the confinement of a quark pair into the hadron takes $t \sim 1/m_h$. Thus, for high energies the confinement time is much larger than the pair production time, and hence cannot affect the annihilation cross-section which should be

$$\begin{aligned} \sigma(e^+e^- \to \text{hadrons}) &\approx \sigma(e^+e^- \to quarks) \\ &= \sigma(e^+e^- \to q\bar{q}) + \sigma(e^+e^- \to q\bar{q}g) + \cdots \end{aligned}$$

and it is calculable using perturbative QCD.

$e^+e^- \rightarrow q\bar{q}$

The calculation is very similar to that for $e^+e^- \rightarrow \mu^+\mu^-$. The only difference is the color structure. As the photon is color blind, the couple of a photon to a quark contains a trivial color matrix, δ^{ij}. Summing over colors and averaging over incoming colors (1 in this case since electrons are not coloured), we obtain

$$\begin{aligned}\sigma(e^+e^- \rightarrow q\bar{q}) &= \sigma(e^+e^- \rightarrow \mu^+\mu^-) \times Q_q^2 \times \sum_{i,j} \delta^{ij}\delta^{ji} \\ &= \sigma(e^+e^- \rightarrow \mu^+\mu^-)\left(Q_q^2 N\right)\end{aligned}$$

Summing over the final state quarks, we get the ratio

$$R_{had} \equiv \frac{\sigma(e^+e^- \rightarrow q\bar{q})}{\sigma(e^+e^- \rightarrow \mu^+\mu^-)} = N\sum_q Q_q^2$$

Hadron production in e^+e^- collisions

Therefore

$$R_{had} = N \sum_q Q_q^2 \left(1 + O(\frac{m_h}{\sqrt{s}}) \right)$$

where the sum is over all quark flavours that are kinematically allowed, $\sqrt{s} > 2m_q$.

Ignoring effects close to threshold, such as formation of bound states, we can expect a plot of R_{had} over $\sqrt{s}$ to present a series of steps at twice the quark masses and flat in between. We can, in principle, read off the number of colors, the quark masses and their charges from this plot.

Ratio R

To lowest order,

$$R = N \sum_q Q_q^2$$

The sum extends to all the quarks which are kinematically allowed. Let us discuss this result.

Below charm threshold, $E \sim 3$ GeV

$$R = N\left(Q_u^2 + Q_d^2 + Q_s^2\right) = 3\left(\tfrac{4}{9} + \tfrac{1}{9} + \tfrac{1}{9}\right) = 2$$

Between charm and beauty threshold, $4 < E < 10$ GeV

$$R = N\left(Q_u^2 + Q_d^2 + Q_s^2 + Q_c^2\right) = 3\left(\tfrac{4}{9} + \tfrac{1}{9} + \tfrac{1}{9} + \tfrac{4}{9}\right) = 3.33$$

Above beauty threshold but well bellow the Z peak, $11 < E < 80$ GeV

$$R = N\left(Q_u^2 + Q_d^2 + Q_s^2 + Q_c^2 + Q_b^2\right) = 3\left(\tfrac{4}{9} + \tfrac{1}{9} + \tfrac{1}{9} + \tfrac{4}{9} + \tfrac{1}{9}\right) = 3.67$$

The ratio R performs a counting of the degrees of freedom involved at a given energy.

Ratio R

This prediction is valid far from the thresholds of quark creation, i.e. far from $c\bar{c}$ and $b\bar{b}$ resonances, because hadronisation effects are important in these regions and affect substantially the recombination of partons. In the resonance region, R has peaks and valleys and our estimate must be compare with an average over an energy interval of say 1 GeV.
In the low energy region the prediction is not accurate. Two reasons explain this results

1. hadron production is still affected by light resonances
2. perturbative corrections are large
3. R depends strongly on (unknown) higher order corrections

As a reference value, consider $E = 34$ GeV. At this energy, $R_{exp} = 3.9$, a 6.3% higher than the theoretical prediction, $R_{th} = 3.67$. Possible reasons:

1. the effect of the Z resonance has not been included
2. radiative corrections (at least to $O(\alpha_s)$) have not been considered

Ratio R at the Z peak

Not only a virtual photon can mediate the annihilation of the $e^+ - e^-$ pair, but a Z^0 contributes at high energies.

$$e^+ + e^- \to \gamma^*, Z^* \to q + \bar{q}$$

It can be shown that at the Z peak ($E = M_Z$), the total cross section is

$$\sigma = \frac{4\pi\alpha^2 k^2}{3\Gamma_Z^2}\left(a_e^2 + v_e^2\right)\left(a_q^2 + v_q^2\right)$$

where Γ_Z is the width of the Z, $k = \sqrt{2}G_F M_Z^2/(4\pi\alpha) = 1.40$ and a_f and v_f are the vector and axial vector couplings of the Z to the fermion f

$$a_f = I_{3f} \qquad v_f = I_{3f} - 2\,Q_f \sin^2\theta_W$$

with θ_W the Weinberg angle and I_3 the third component of the weak isospin: $I_3 = 1/2$ for neutrinos and u-type quarks with $Q = 2/3$, and $I_3 = -1/2$ for charged leptons and d-type quarks with $Q = -1/3$.

Ratio R at the Z peak

The ratio R reads

$$R = 3\,\frac{\sum_{q=u,d,s,c,b} a_q^2 + v_q^2}{a_\mu^2 + v_\mu^2} = 20.095$$

A comment is in order here. Strictly speaking, we have to add the real photon emission from the initial $e^+ - e^-$ state, the so-called bremsstrahlung. It is the same for the muonic and hadronic channels and hence cancels in the ratio R.

This theoretical results compared very well with the experimental value $R_{exp} = 20.767 \pm 0.025$. The 4% difference can be reduced including radiative corrections.

Radiative corrections for the ratio R

Remember that R is a completely inclusive quantity and neglecting quark masses, it depends on $\sqrt{s}$ only.

Since the pair $q\bar{q}$ is create by the electromagnetic current, that is conserved, the radiative corrections should be UV finite. This does not mean that in intermediate steps of the calculation, we get UV divergent results but the final sum of all contributions must be finite. For example, the UV divergence in the vertex diagram is cancelled by the one in the external legs.

Consider the emission of a gluon in the e^+e^- annihilation. There are two diagrams corresponding to the emission by the quark or the antiquark legs. Remember that the quark-gluon vertex is $i g_s \gamma_\mu t^a$ and the only difference with respect to the Abelian case is the presence of the matrix t^a that produces a factor C_F in the amplitude. Therefore, we can take the result of the photon emission in QED and simply replace α with $C_F\,\alpha_s$ to obtain the QCD result (see Project B of the Peskin and Schroeder).

$e^+e^- \to q\bar{q}g$

We define the energy fractions,

$$x_1 = \frac{E_q}{E_B} \qquad x_2 = \frac{E_{\bar{q}}}{E_B} \qquad x_3 = \frac{E_g}{E_B}$$

where $E_B = \sqrt{s}/2$ is the beam energy. Conservation of energy and momenum gives

$$x_1 + x_2 + x_3 = 2 \qquad 1 - x_i = \frac{1}{2} x_j x_k \left(1 - \cos\theta_{jk}\right) \;\; i \neq j \neq k$$

$$\theta_{12} + \theta_{23} + \theta_{13} = 2\pi$$

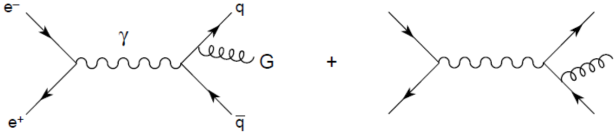
The kinematical region is

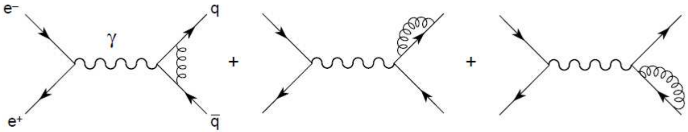
$$0 \leq x_i \leq 1 \qquad x_i + x_j \geq 1 \;\; i \neq j$$

It is easy to see that a parton has the beam energy as the maximum energy when the other two are colinear and a parton pair has the beam energy as the minimum energy and the total energy as the maximum energy.

Feynman diagrams for $e^+e^- \to q\bar{q}g$

$$e^+(q_1)e^-(q_2) \to q(p_1)\overline{q}(p_2)G(p_3)$$





One-loop diagrams for $e^+e^- \to \overline{q}q$.

Three jets differential cross section

The differential cross section for having a quark with an energy fraction x_1 and an antiquark with energy fration x_2 is

$$\frac{1}{\sigma_0}\,\frac{d\sigma}{dx_1\,dx_2} = C_F\,\frac{\alpha_s}{2\pi}\,\frac{x_1^2+x_2^2}{(1-x_1)(1-x_2)} \quad \text{with} \quad \sigma_0 = \frac{4\pi\alpha^2}{s}\sum_q Q_q^2$$

Unfortunately, when in order to obtain the total cross section we integrate in the following region

$$0 \le x_1, x_2 \le 1 \qquad 1 \le x_1 + x_2$$

there are singularities when the quark energy fractions reach their maximum, $x_{1,2} \to 1$, individually (single pole) or simultaneously (double pole). The final result is an infinite total cross section.

Infrared singularities

These infrared divergencies are due to the singular behaviour of the quark propagator that emits the gluon (even for massive quarks)

$$\frac{1}{(p_2+p_3)^2-m_q^2}=\frac{1}{2\,(p_2\cdot p_3)}=\frac{1}{s\,(1-x_1)}$$

There are two singular kinematical configurations:

Soft gluon singularity:
: even for $m_q>0$, $E_g\to 0$, when the gluon is emitted with very small energy, yields $x_1\to 1$.

Collinear singularity:
: when the gluon-(anti)quark angle, $\theta\to 0$; i.e. the gluon is emitted at a very small angle: $x_1=1$ is also possible for $p_3\parallel p_2$.

Regularization of the three-jet total cross section

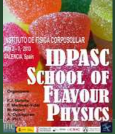
Among the different methods to regularize the cross section (momentum cutoff in the gluon transverse momentum, gluon mass regularization, ...etc), dimensional regularization is the most convenient because it is consistent with all the symmetries of QCD and can be used to regularize both UV and infrared divergencies.

The total cross section can be regularized performing the integration in the kinematical region in $D = 4 - 2\varepsilon$ dimensions. Doing so, the singulaties will appear as double, $1/\varepsilon^2$ and single, $1/\varepsilon$ poles. The result is

$$\sigma(e^+e^- \to q\bar{q}g) = \sigma_0 \frac{C_F \alpha_s}{2\pi} H(\varepsilon) \left[\frac{2}{\varepsilon^2} + \frac{3}{\varepsilon} + \frac{19}{2} - \pi^2 + O(\varepsilon) \right]$$

where

$$H(\varepsilon) = \frac{3(1-\varepsilon)^2}{(3-2\varepsilon)\Gamma(2-2\varepsilon)} \qquad H(0) = 1$$



$e^+e^- \to q\bar{q}$ at one loop

At leading order, in α_s, $q\bar{q}$ is the only process that contributes to $e^+e^- \to$ partons with amplitude M_0. At one-loop, however, there are three additional diagrams that include quark-leg selfenergy and one-loop correction to the quark-photon vertex. Their amplitudes, M_1, are down by a factor α_s with respect to the tree-level diagram. Therefore, the cross-section is two powers of α_s down and hence negligible at the order we are working.

This is, however, incorrect because there is an $O(\alpha_s)$-interference between the one-loop amplitude and the tree-level one. The reason is that since both processes have the same final state, one must add their amplitudes $M = M_0 \alpha_s M_1$ and hence $|M|^2$ contains the interference $\alpha_s \mathrm{Re}(M_0^* M_1)$ that contributes at order α_s.

The one-loop diagrams are individually both infrared and UV divergent. Dimensional regularization can be used to regularize the three diagrams.

Cancellation of infrared divergences

The important results are that the UV divergences cancel when the two self-energy diagrams are added to the vertex correction. Infrared divergences, however, do not cancel and one obtains

$$\sigma(e^+e^- \to q\bar{q}) = \sigma_0 \frac{C_F \alpha_s}{2\pi} H(\varepsilon) \left[-\frac{2}{\varepsilon^2} - \frac{3}{\varepsilon} - 8 + \pi^2 + O(\varepsilon) \right]$$

If we recall the expression for three-parton cross-section,

$$\sigma(e^+e^- \to q\bar{q}g) = \sigma_0 \frac{C_F \alpha_s}{2\pi} H(\varepsilon) \left[\quad \frac{2}{\varepsilon^2} + \frac{3}{\varepsilon} + \frac{19}{2} - \pi^2 + O(\varepsilon) \right]$$

and taking into account that the total cross section is the sum of the two, we get

$$\sigma(e^+e^- \to \text{hadrons}) = \sigma_0 \left[1 + C_F \frac{\alpha_s}{2\pi} \frac{3}{2} \right] = \sigma_0 \left(1 + \frac{\alpha_s}{\pi} \right)$$

Moreover, it can be shown that this result is scheme independent.

Ratio R at NNNLO

The QCD corrections to R are know up to order α_s^3. The renormalization procedure introduces a renormalization-scale dependence into α_s and the coefficients functions beyond the first one

$$R_{e^+e^-}(s) = R^{(0)}_{e^+e^-}\left[1 + c_1\,\frac{\bar{\alpha}_s(\sqrt{s})}{\pi} + c_2\left(\frac{\bar{\alpha}_s(\sqrt{s})}{\pi}\right)^2 + c_3\left(\frac{\bar{\alpha}_s(\sqrt{s})}{\pi}\right)^3\right]$$

where $R^{(0)}_{e^+e^-} = N\sum_q Q_q^2$. In the $\overline{MS}$ scheme, the coefficients are

$$\begin{aligned} c_1 &= 1 \\ c_2 &= 1.986 - 0.115\,N_f \\ c_3 &= -6.637 - 1.2\,N_f - 0.005\,N_f^2 - 1.240\,\eta \end{aligned}$$

with $\eta = 0.303$ for $N_f = 5$.

Scale dependence of R at NNNLO

As we know, the knowledge of the running coupling constant at a given energy scale μ is sufficient to know it at any other scale,

$$\bar{\alpha}_s(s) = \frac{\alpha_s(\mu)}{1 + \beta_0 \frac{\alpha(\mu)}{4\pi} \log(s/\mu^2)}$$

Inserting this relation in the expression for the ratio R at order α_s^2, we find

$$R_{e^+e^-}(s) = R^{(0)}_{e^+e^-} \left[1 + c_1 \frac{\alpha_s(\mu)}{\pi} + \left(c_2 - c_1 \frac{\beta_0}{4} \log(\frac{s}{\mu^2}) \right) \left(\frac{\alpha_s(\mu)}{\pi} \right)^2 + \cdots \right]$$

In principle R is independent of the arbitrary scale μ but the unavoidable truncation of the perturbative series introduces a scale dependence. There is no unique prescription for the optimal value of μ. The obvious choice to avoid large logarithms is $\mu = \sqrt{s}$. This choice turns out to be very reasonable and sufficient to understand phenomenology.

Measurements of α_s

The experimental measurement of $R_{e^+e^-}$ gives one of the best measurements of α_s. For example, the LEP combined value of the ratio R at the peak of the Z is

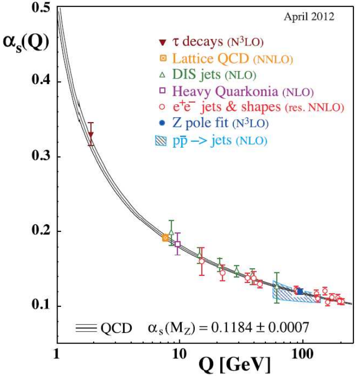
$$R_{e^+e^-}(M_Z) = 20.767(25) \text{ to be compared with } R^{(0)}_{e^+e^-}(M_Z) = 19.984$$

Simply using the one-loop order result with $\mu = M_Z$, we obtain $\alpha_s(M_Z) = 0.124(4)$, close to the four-loop result, $\alpha_s(M_Z) = 0.119(3)$.
Another test to QCD is to perform measurements at other scales and evolve them to a single scale using the scale dependence of $\bar{\alpha}_s$. For example, PETRA measurement of R at 34 GeV is

$$R_{e^+e^-}(34\text{ GeV}) = 3.88(3) \text{ to be compared with } \mathrm{R}^{(0)}_{\mathrm{e^+e^-}}(34\text{ GeV}) = 3.69$$

Using the leading order result, we obtain $\alpha_s(34\text{GeV}) = 0.162(26)$.
Finally using the one-loop RGE, we convert this result into $\alpha_s(M_Z) = 0.134(18)$, in good agreement with the LEP value.

Running of α_s



Summary of measurements of α_s as a function of the respective energy scale Q. The respective degree of QCD perturbation theory used in the extraction of α_s is indicated in brackets (NLO: next-to-leading order; NNLO: next-to-next-to leading order; res. NNLO: NNLO matched with resummed next-to-leading logs; N^{3}LO: next-to-NNLO).

Renormalization

But, what is renormalization?

1. In order to cut the high-momentum modes of the theory, a cut-off Λ is introduced. Thus an amplitude $A(p_i, e_0, m_0; \Lambda)$ is finite and depends not only on the momenta (p_i), the (bare) electric charge (e_0) and the mass (m_0), but also on Λ. The amplitude diverges as $\Lambda \to \infty$, i.e. as the cutoff is removed. This step is called **regularization**.
2. One defines physical parameters $e(\mu)$ and $m(\mu)$ with the help of measurable quantities as cross sections. They depend on an arbitrary renormalization scale μ.
3. One writes e_0 and m_0 in terms of $e(\mu)$ and $m(\mu)$ to a given order of PT. Then, the limit

$$\lim_{\Lambda \to \infty} A(p_i, e_0(e, m, \Lambda), m_0(e, m, \Lambda); \Lambda) = \hat{A}(p_i, e(\mu), m(\mu))$$

is finite.

Divergent Feynman integrals

Perturbative calculations of cross-sections are perfomed in QCD using the Feynman diagram formalism. The higher order corrections can be quite large due to the not-so-small value of α_s, say, ~ 0.1.

Diagrams beyond the leading order and usually divergent. For example, consider quark selfenergy in QCD (but the conclusion is valid also for QED). The integral of the Feynman diagram is

$$\Sigma(p) = g^2 C_F \int \frac{d^4k}{(2\pi)^4} \frac{(-2)(\not{p} - \not{k}) + 4m}{k^2 \left[(p-k)^2 + m^2 \right]}$$

When $k^\mu \to \infty$, the asymtotic behaviour of the integral is

$$\Sigma(p) \sim g^2 C_F \int \frac{dk}{(2\pi)^4} \frac{k^3 \, 2\not{k}}{k^4} \sim g^2 C_F \int_0^\infty dk \sim \lim_{k\to\infty} k$$

This is called a UV (ultraviolet) divergence.

Divergent Feynman integrals

Actually, due to symmetry, it can be shown that

$$\Sigma(p) \sim \int_0^\infty \frac{dk}{k} = \lim_{k\to\infty} \log k$$

To cope with the divergencies, integrals have to be **regularized**, for example, using dimensional regularization. Next, a renormaliztion procedure is applied to absorb all divergencies arising from Feynman diagrams at all orders, into a refefinition of fields, masses and coupling constants.

Regularization of loop integrals

We need to give a definite mathematical meaning to Feynman loop integrals in order to isolate their UV divergent parts and eliminate it through the procedure of renormalization. The procedure of regularization consists in cutting the high-momentum components of the particles in the loop. There are many methods to do that,

1. **Hard cutoff** The range of integration in loop integrals is cut from above by introducing a cutoff.
2. **Lattice** Discretize the space-time by introducing a lattice spacing a. Therefore, $1/a$ autoimatically plays the role of a momentum cutoff. The prize to pay is thatn Lorentz invariaance, but not gauge invaariaance, is lost and the calculations are much more involved.
3. **Dimensional regularization** Extend the dimension of the spacetime to, in general, complex values, $D = 4 + 2\varepsilon$ in such a way that for $\varepsilon \neq 0$, the integral is well defined. THe ultraviolet divergences appear at $\varepsilon = 0$ as poles of the Gamma function.

Dimensional regularization: formal definition

Dimensional regularization is currently the standard procedure for regularizing QFT. The basic observation is that in standard QFT only logarithmic divergences appear and they vanish if the dimension is smaller than 4. Therefore, by defining a generalized integration in noninteger D dimensions, the divergences will appear as poles in $D-4$ which are relatively easy to isolate. Let us define mathematically what we will call a D-dimensional integral,

$$\int d^D k\, f(k)$$

The first step is to perform the so-called Wick rotation,i.e. to continue to imaginary loop momentum and energy: $k^0 \rightarrow i\,k_E^0$ and $\vec{k} \rightarrow \vec{k}_E$,

$$k^2 = (k^0)^2 - (\vec{k}^2) \rightarrow -k_E^2 = -\left((k_E^0)^2 + (\vec{k}_E)^2\right)$$

These Euclidean vectors satisfy that k_E^2 is large as $(k_E)_\mu \rightarrow \infty$ and hence UV divergencies are more easily analized. Minkowsky vectors, however, due to their non Euclidean metric, can have a small $k^2 = (k^0)^2 - (\vec{k})^2$ even when k^0 and $|\vec{k}|$ both become large.

Dimensional regularization: formal definition

Mathematical conditions:

1. Linearity

$$\int d^D k_E \left[\alpha f(k_E^\mu) + \beta g(k_E^\mu)\right] = \alpha \int d^D k_E f(k_E^\mu) + \beta \int d^D k_E\, g(k_E^\mu)$$

2. Invariance under finite p_μ shifts

$$\int d^D k_E f(k_E^\mu) = \int d^D k_E f(k_E^\mu + p_E^\mu)$$

3. Scaling

$$\int d^D k_E f(\lambda k_E^\mu) = \lambda^{-D} \int d^D k_E f(k_E^\mu)$$

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Dimensional regularization: master formula

It can be shown that vector like integrals can always be reduced to scalar integrals because the former are completely specified by their contractions with few linearly independent vectors.

Therefore, we need to analyticaly continue scalar integrals only. The master most important d-dimensional integral is

$$\int_{-\infty}^{\infty} d^D k_E \, f(k_E^2) = \frac{2\pi^{D/2}}{\Gamma(D/2)} \int_0^{\infty} dk_E \, k_E^{D-1} f(k_E^2)$$

Peculiar property:

$$\int d^D k_E \, \left(k_E^2\right)^z = 0 \qquad \forall z$$

This is very easy to demostrate using the scaling axiom but in reality is a definition consistent with the axioms and motivated by the following argument,

$$\int d^D k_E \, \left(\lambda k_E^2\right)^z = \lambda^{2z} \int d^D k_E \, \left(k_E^2\right)^z = \lambda^{-D} \int d^D k_E \, \left(k_E^2\right)^z$$

Dimensional regularization: basic formula

Using the master formula, the following basic integral can be evaluated

$$\int d^D k_E \frac{\left(k_E^2\right)^\alpha}{\left(k_E^2 + m^2\right)^\beta} = \pi^{D/2}\,(m^2)^{\alpha-\beta+D/2}\,\frac{\Gamma(\alpha+\frac{D}{2})\,\Gamma(\beta-\alpha-\frac{D}{2})}{\Gamma(\frac{D}{2})\,\Gamma(\beta)}$$

This is an important formula since any integral appearing in the calculation of a Feynman graph can be brought into this form.

Dimensional regularization: reduction to basic-formula integrals

Steps for reducing any Feynman integral:

1. Feynman parametrization
The integral of a Feynman integral contains products of propagators associated to the internal lines of the loops. By means the Feynman parametrization this product of N momentum dependent factors can be written as a unique propagator, using the identity

$$\prod_{i=1}^{N} \frac{1}{a_i^{n_i}} = \frac{\Gamma(n)}{\prod_{i=1}^{N} \Gamma(n_i)} \int_0^1 \frac{\prod_{i=1}^{N} dx_i\, x_i^{n_i-1}}{\left(\sum_{i=1}^{N} a_i x_i\right)^n} \delta\left(1 - \sum_{i=1}^{N} x_i\right)$$

with a_i arbitrary complex numbers, $n = \sum_{i=1}^{N} n_i$ and x_i the so-called Feynman parameters.

Dimensional regularization: reduction to basic-formula integrals

2. Momentum shift
Shift the momentum variable of the loop, say, k_μ such that the linear term in the denominator vinishes and you get an integral like
$$\int d^D k_E \, \frac{f(k_E^\mu)}{\left(k_E^2 + m^2\right)^\beta}$$

3. Reducing to scalar integrals
Use that pairs of momentum components can be contracted to give $k_\mu k_\nu \to g_{\mu\nu} k^2/D$ and that terms with remaining individual components k_μ vanish when inverted $k_\mu \to -k_\mu$. The resulting expressions are sums of terms of the form of the basic formula. For example, consider
$$\begin{aligned}\int d^D k \, \frac{k_\mu k_\nu k_\lambda}{(k^2 + 2\,(k\cdot p) + m^2)} &= F_1(p^2)\, g_{\mu\nu} p_\lambda + F_2(p^2)\, g_{\mu\lambda} p_\nu \\ &+ F_3(p^2)\, g_{\nu\lambda} p_\mu + F_4\, p_\mu p_\nu p_\lambda\end{aligned}$$

Dimensional regularization: reduction to basic-formula integrals

3. Reducing to scalar integrals
   The "form factors" F_i depend on p^2 only and can be calculated by contracting with $g_{\alpha\beta}$ and/or p_α.

4. Expand the Γ functions
   Taking $D = 4 \pm 2\varepsilon$ and using the useful property $\Gamma(z+1) = z\Gamma(z)$, all divergencies can be related to $\Gamma(\varepsilon)$,

$$\Gamma(\varepsilon) = \frac{1}{\varepsilon} - \gamma + \frac{1}{2}\left[\gamma^2 + \zeta(2)\right]\varepsilon^2 + \cdots$$

   and can be easily isolated. In the expansion above, γ is Euler's constant, $\gamma = 0.5772157$, and $\zeta(s)$ the Riemann's Zeta function, $\zeta(2) = \frac{\pi^2}{6}$.

Power counting

We will determine which diagrams are UV divergent in QCD using power counting.

The UV behaviour of a Feynman diagram is found by the limit of loop momenta to infinity.

Type	Feynman rule	Power of k
Loop integration	$\int \frac{d^D k}{(2\pi)^D} = \int \frac{k^{D-1}\, dk}{(2\pi)^D}\, d\Omega_D$	D
Quark propagator	$\frac{\not{k}}{k^2} \sim \frac{1}{k}$	-1
Gluon propagator	$\frac{1}{k^2}$	-2
Ghost propagator	$\frac{1}{k^2}$	-2
Quark-gluon vertex	$t^a \gamma_\mu$	0
Three-gluon vertex	$\propto k^\mu$	1
Four-gluon vertex	$f_{abc}\, g^{\alpha\beta}$	0
Ghost-gluon vertex	$\propto k^\mu$	1

Superficial degree of divergence

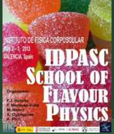
Denoting by n_L, the number of loops, V_3 the number of triple gluon vertices, $P_{q,g,G}$ the number of quark, gluon and ghost propagators, respectively, and V_G the number of ghost-gluon vertices, the superficial degree of divergence is defined by

$$d = Dn_L + V_3 - P_q - 2P_g - P_G + V_G$$

The superficial degree of divergence of a diagram is the difference between the number of momenta in the numerator and the number of momenta in the denominator.

If $d \geq 0$ the diagram is UV divergent: $d = 0$, logarithmically, $d = 1$, linearly and so on.

If, however, $d < 0$, the diagram is UV finite.



Superficial degree of divergence

Using some topological relations between the number of external and internal lines and vertices, it is not difficult to arrive at the formula for the superficially degre of divergence of a Feynman diagram in QCD,

$$d = 4 - \frac{3}{2} F_E - B_E$$

where $F_E(B_E)$ is the number of external fermionic (bosonic) lines in the Feynman (sub)diagram. Some comments are in order here:

1. Ghost lines count as bosonic lines.
2. Notice that the counting must be applied to every subgraph.
3. The actual behaviour of the corresponding integral might be better due to cancellations coming from symmetries.

Divergent subdiagrams

Using the formula above, it is easy to find the divergent diagrams in QCD and their superficial degree of divergence

Diagram	F_E	B_E	d	d_{eff}
Quark selfenergy	2	0	1	0
Gluon selfenergy	0	2	2	0
Ghost selfenergy	0	2	2	0
Quark vertex	2	1	0	0
Triple gluon vertex	0	3	1	0
Quartic gluon vertex	0	4	0	0
Quartic ghost vertex	0	4	0	< 0
Quartic ghost-gluon vertex	0	4	0	< 0

Therefore, all UV divergences in QCD appear in one of the above 8 diagrams. The idea of the renormalization program is to absorb them in the redefinition of field normalizations, masses and couplings. Only then all amplitudes will be UV finite.

Renormalized Lagrangian

To eliminate all divergent parts in divergent diagrams, we add to the initial Lagrangian of QCD

$$\begin{aligned}\mathcal{L} &= -\frac{1}{4} G^a_{\mu\nu} G^{\mu\nu}_a + \bar{\psi}\left(i\not{D} - m\right)\psi - \frac{1}{2\lambda}\left(\partial_\mu A^\mu\right)^2 \\ &+ \partial_\mu \eta^{a*} \partial^\mu \eta^a - g\,\partial_\mu \eta^{a*} f_{abc} A^{b\mu} \eta^c\end{aligned}$$

where

$$\begin{aligned}G^a_{\mu\nu} &= \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + igf^{abc} A^b_\mu A^c_\nu \\ D_\mu &= \partial_\mu + igA^a_\mu t^a\end{aligned}$$

some counterterms corresponding to each superficially divergent diagram in the theory

$$\mathcal{L}(x) \to \mathcal{L}(x) + \mathcal{L}_{ct}(x)$$

For example, for the quark mass term we have

$$m\bar{\psi}\psi \to m\bar{\psi}\psi + c_4 m\bar{\psi}\psi$$

Renormalized Lagrangian (II)

We choose c_4 in such a way that it cancels exactly the pole in $1/\varepsilon$ of this term in the Lagrangian. This can be extended to each divergent term in the Lagrangian by defining the so-called renormalization constants (RCs), $Z_i = 1 - c_i$. The renormalized lagrangian is

$$
\begin{aligned}
\mathcal{L} \; = \; & -Z_{3YM}\,\frac{1}{4}\,(\partial_\mu A^a_\nu - \partial_\nu A^a_\nu)\,(\partial^\mu A^\nu_a - \partial^\nu A^\nu_a) \\
& - \; Z_6\,\frac{1}{2\lambda}\,(\partial_\mu A^\mu_a)^2 \\
& + \; Z_{2F}\, i\bar{\psi}\partial\!\!\!/\,\psi \\
& - \; Z_4\, m\bar{\psi}\psi \\
& + \; Z_{1F}\, g\bar{\psi}\gamma^\mu t^a\psi A^a_\mu \\
& - \; Z_{1YM}\,\frac{1}{2}\, g f_{abc}\,(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu)\,A^\mu_b A^\nu_c \\
& - \; Z_5\,\frac{1}{4}\, g^2 f_{abc} f_{ade} A^b_\mu A^c_\nu A^\mu_d A^\nu_e \\
& + \; \tilde{Z}_3\,[\partial_\mu \eta^*_a]\,\partial^\mu \eta^a \\
& + \; \tilde{Z}_1\, g f_{abc}\,[\partial_\mu \eta^*_a]\,\eta_b A^\mu_c
\end{aligned}
$$

Renormalization schemes

In DR, all the counterterms are power expansions in α of the form

$$c_i = \sum_{n=1}^{\infty} \sum_{m=1}^{n} C_{i,m}^{(n)} \frac{1}{\varepsilon^m} \left(\frac{\alpha}{\pi} \right)^n$$

The c_i are chosen in such a way that they cancel the $1/\varepsilon$ terms. But finite parts are arbitrary. We have some alternatives:

MS scheme: subtract **exactly** the poles in $1/\varepsilon$; nothing more.

$\overline{MS}$ scheme: the poles always come in the combination

$$\frac{1}{\hat{\varepsilon}} \equiv \frac{1}{\varepsilon} - \log 4\pi + \gamma$$

Therefore, we can subtract $1/\hat{\varepsilon}$ poles instead of $1/\varepsilon$.

MOM scheme: to regularize an amplitude, subtract its value with all momenta taken in the arbitrary Euclidean point $p^2 = -\mu^2$,

$$A_R(p^2;\mu^2) = A(p^2,\varepsilon) - A(p^2 = -\mu^2,\varepsilon)$$

Renormalization constants

A long but straightforward calculation in dimensional regularization yields

$$
\begin{aligned}
Z_{3YM} &= 1 - \frac{\alpha}{\pi}\left[-\frac{T(R)}{3}N_F + \frac{C_2(G)}{4}\left(\frac{13}{6} - \frac{\lambda}{2}\right)\right]\frac{1}{\hat{\varepsilon}} \\
Z_6 &= 1 \\
Z_{2F} &= 1 + \frac{\alpha}{\pi}\lambda\,\frac{C_2(R)}{4}\,\frac{1}{\hat{\varepsilon}} \\
Z_4 &= 1 + \frac{\alpha}{\pi}(3+\lambda)\,\frac{C_2(R)}{4}\,\frac{1}{\hat{\varepsilon}} \\
Z_{1F} &= 1 + \frac{\alpha}{\pi}\frac{1}{4}\left[(3+\lambda)\frac{C_2(G)}{4} + \lambda\, C_2(R)\right]\frac{1}{\hat{\varepsilon}} \\
Z_{1YM} &= 1 - \frac{\alpha}{\pi}\frac{1}{4}\left[-\frac{4T(R)}{3}N_F + \left(\frac{17}{12} - \frac{3\lambda}{4}\right)C_2(G)\right]\frac{1}{\hat{\varepsilon}} \\
Z_5 &= 1 + \frac{\alpha}{\pi}\frac{1}{4}\left[\frac{4T(R)}{3}N_F + \left(-\frac{2}{3} + \lambda\right)C_2(G)\right]\frac{1}{\hat{\varepsilon}}
\end{aligned}
$$

Slavnov-Taylor identities

$$
\begin{aligned}
\tilde{Z}_3 &= 1 - \frac{\alpha}{\pi}\,\frac{C_2(G)}{4}\,\frac{3-\lambda}{4}\,\frac{1}{\hat{\varepsilon}} \\
\tilde{Z}_1 &= 1 + \frac{\alpha}{\pi}\,\frac{1}{4}\,\lambda\,\frac{C_2(G)}{2}\,\frac{1}{\hat{\varepsilon}}
\end{aligned}
$$

It is easy to show that they satisfy the so-called Slavnov-Taylor identities

$$
\frac{Z_{3YM}}{Z_{1YM}} = \frac{\tilde{Z}_3}{\tilde{Z}_1} = \frac{Z_{2F}}{Z_{1F}} = \frac{Z_{1YM}}{Z_5} \qquad Z_6 = 1
$$

Using the so-called BRST symmetry of the QCD Lagrangian, a residual gauge symmetry after the fixing of the gauge, it is shown that the Slavnov-Taylor identities are valid at all orders in Perturbation Theory. These identities are used to demonstrate the renormalizability of QCD because they imply subtle cancellations between divergencies at different orders of PT.

Bare and renormalized fields and couplings

Now, we define the bare, ϕ_0, and renormalized, ϕ_R, fields and couplings in the form

$$\begin{array}{rclcrcl} A^a_{\mu 0} &=& \sqrt{Z_3}\, A^a_{\mu R} & \qquad & g_0 &=& Z_g\, g_R \\ \eta^a_0 &=& \sqrt{\tilde{Z}_3}\, \eta^a_R & & m_0 &=& Z_m\, m_R \\ \psi_0 &=& \sqrt{Z_2}\, \psi_R & & \lambda_0 &=& Z_\lambda\, \lambda_R \end{array}$$

We will drop the renormalized index because the renormalized couplings and fields are the same as the ones in the renormalized Lagrangian. They are finite and the fields give rise to finite amplitudes in $D = 4$. Both bare and renormalized quantities are defined in a D-dimensional space. The lagrangian in terms of bare quantities has the same form as the initial one (before adding the counterterms).

It is easy to find that

$$\begin{aligned} Z_g &= Z_{1YM}\, Z_{3YM}^{-3/2} = \tilde{Z}_1\, \tilde{Z}_3^{-1}\, Z_{3YM}^{-1/2} = Z_{1F}\, Z_{3YM}^{-1/2}\, Z_{2F}^{-1} = Z_5^{1/2}\, Z_{3YM}^{-1} \\ Z_m &= Z_4\, Z_{2F}^{-1} \\ Z_\lambda &= Z_6^{-1}\, Z_{3YM} \end{aligned}$$

The mass scale in dimensional regularization

The great advantage of DR is that all symmetry properties, including local gauge invariance, are preserved, except for dilation invariance because only in $D=4$ dimensions the coupling constant is dimensionless, and chiral invariance, due to the difficulties in the definition of γ_5 in D dimensions.
Taking into account that the action must remain dimensionless, it is easy to find the canonical dimensions of the parameters and fields

$$[\psi(x)] = M^{\frac{3}{2}+\varepsilon} \quad [A_\mu^a] = [\eta^a] = M^{1+\varepsilon} \quad [g] = M^{-\varepsilon} \quad [\lambda] = M^0 \quad [m] = M$$

Therefore, to preserve a dimensionless coupling constant, an arbitrary mass scale parameter μ must be introduced and all expressions in DR should be considered as power series in $g\mu^\varepsilon$ or $\alpha = \frac{(g\mu^\varepsilon)^2}{4\pi}$.

Renormalization constants of the coupling, mass and gauge parameter

Using the expressions for the renormalization constants, we find the very important results

$$
\begin{aligned}
Z_\alpha &= \tilde{Z}_1^2 \tilde{Z}_3^{-2} Z_{3YM}^{-1} = 1 + \frac{\alpha}{\pi}\left[\frac{11}{12} C_2(G) - \frac{T(R)}{3} N_f\right]\frac{1}{\hat{\varepsilon}} \\
Z_m &= Z_4 Z_{2F}^{-1} = 1 + \frac{\alpha}{\pi}\frac{3}{4} C_2(R)\frac{1}{\hat{\varepsilon}} \\
Z_\lambda &= Z_6^{-1} Z_{3YM} = 1 - \frac{\alpha}{\pi}\left[-\frac{T(R)}{3} N_F + \frac{C_2(G)}{4}\left(\frac{13}{6} - \frac{\lambda}{2}\right)\right]\frac{1}{\hat{\varepsilon}}
\end{aligned}
$$

where $\alpha = (g\mu^\varepsilon)^2/(4\pi)$.

Renormalization Group Equation

Consider a dimensionless physical observable R which depends on a single large energy scale $Q \gg m$, where m is any mass. Taking the limit $m \to 0$, it is clear, by dimensional analysis, that R should be independent of Q.

On QFT this is not true. To calculate R as a perturbative series in the renormalized coupling constant $\alpha(\mu) = g^2(\mu)/4\pi$, we have to renormalize the theory to remove UV divergences. This introduces a second mass scale: the renormalization point, μ. Therefore, R is a function of the ratio Q^2/μ^2 and the renormalized coupling: R is not a constant.

But μ is arbitrary. Any physical observable must be independent of μ. Therefore,

$$\mu^2 \frac{d}{d\mu^2} R\left(\frac{Q^2}{\mu^2}, \alpha(\mu)\right) = \left[\mu^2 \frac{\partial}{\partial\mu^2} + \mu^2 \frac{\partial\alpha}{\partial\mu^2}\frac{\partial}{\partial\alpha}\right] R = 0$$

Beta function

Introducing the beta function

$$\beta(\alpha) = \mu^2 \frac{\partial \alpha}{\partial \mu^2}$$

we get the Renormalization group equation,

$$\left[-\frac{\partial}{\partial t} + \beta(\alpha) \frac{\partial}{\partial \alpha} \right] R(t, \alpha) = 0$$

where $t = \log\left(\frac{Q^2}{\mu^2}\right)$.
This equation can be solved by defining the characteristic function, also called running coupling, $\bar{\alpha}(Q)$,

$$t = \int_{\alpha(\mu)}^{\bar{\alpha}(Q)} \frac{dx}{\beta(x)}$$

Beta function

In fact, it immediately follows that

$$\frac{\partial\bar{\alpha}(Q)}{\partial t} = \beta(\bar{\alpha}(Q)) \qquad \frac{\partial\bar{\alpha}(Q)}{\partial\alpha} = \frac{\beta(\bar{\alpha}(Q))}{\beta(\alpha)}$$

and hence

$$R(\frac{Q^2}{\mu^2}, \alpha(\mu)) = R(1, \bar{\alpha}(Q))$$

all scale dependence in R comes from the running coupling constant.

Calculation of the beta function

We expand the beta function as

$$\beta(\alpha) = -\beta_0 \left(\frac{\alpha^2}{4\pi}\right) - \beta_1 \left(\frac{\alpha^3}{(4\pi)^2}\right) - \cdots$$

Writting the renormalization constant Z_α in the form

$$Z_\alpha = 1 + \sum_{n=1}^{\infty} \frac{Z_{\alpha,n}(\alpha)}{\varepsilon^n}$$

and recalling that $\alpha_0 = Z_\alpha(\alpha)\,\alpha(\mu)\,\mu^{2\varepsilon}$ and the bare coupling constant α_0 is independent of μ, it is not difficult to arrive at the equation we will use to calculate $\beta(\alpha)$,

$$\beta(\alpha) = -\alpha \frac{\partial Z_{\alpha,1}}{\partial \alpha}$$

Therefore, the beta function, in minimal subtraction renormalization schemes, is determined by the residue of the pole $1/\varepsilon$.

Calculation of the beta function

Using our expression for Z_α, we get at one-loop (and two-loops)

$$\begin{aligned}\beta_0 &= \frac{11}{3} C_2(G) - \frac{4}{3} T(R) N_f \\ \beta_1 &= \frac{34}{3} C_2(G)^2 - \frac{20}{3} C_2(G) T(R) N_f - 4 C_2(R) T(R) N_f\end{aligned}$$

Gross and Wilczek, and independently Politzer, compute the one-loop beta function for QCD in 1973 (they received the Nobel Prize in 2004). Previously, in 1971, 't Hooft computed the one-loop β-function but did not publish it. He wrote the formula on the blackboard at a conference and his PhD supervisor, Veltman, told him that it was not interesting.

In 1974, Caswell and Jones, calculated β_1. In 1980, Tarasov, Vladimirov and Zharkov computed the three-loop beta function in QCD. Van Ritbergen, Vermaseren and Larin, after computing more than 50.000 Feynman diagramas, calculated the four-loop beta function in 1997.

Beta function in QCD at four-loops

Using the color factors in $SU(N)$,

$$T(R) = 1/2 \qquad C_2(G) = N \qquad C_2(R) = \frac{N^2-1}{2N}$$

for $N = 3$, we find the following numerical results for the beta function of QCD,

$$\begin{aligned}
\beta_0 &= 11 - 0.66667\, N_f \\
\beta_1 &= 102 - 12.6667\, N_f \\
\beta_2 &= 1428.50 - 279.611\, N_f + 6.01852\, N_f^2 \\
\beta_3 &= 29243.0 - 6946.30\, N_f + 405.089\, N_f^2 + 1.49931\, N_f^3
\end{aligned}$$

Running coupling constant in QCD

The running coupling constant $\bar{\alpha}(Q)$ can be calculated solving the equation

$$\frac{\partial \bar{\alpha}(Q)}{\partial t} = \beta(\bar{\alpha}(Q)) = -\beta_0 \left(\frac{\bar{\alpha}^2(Q)}{4\pi} \right) - \beta_1 \left(\frac{\bar{\alpha}^3(Q)}{(4\pi)^2} \right) + \cdots$$

Neglecting β_1 and higher coefficients, the solution is simply

$$\bar{\alpha}(Q) = \frac{\alpha(\mu)}{1 + \frac{\alpha(\mu)}{4\pi} \beta_0 \log(Q^2/\mu^2)}$$

The behaviour of $\bar{\alpha}(Q)$ for large Q is determined by the sign of β_0. If the number of active flavours is $N_f < 33/2$, $\beta_0 > 0$ and therefore the running coupling constant decreases, $\bar{\alpha}(Q) \to 0$ as $Q \to \infty$. This is asymptotic freedom and we say that QCD is asymptotically free.

Thus for large Q we can safely use PT. The knowledge of $R(1, \bar{\alpha}(Q))$ at a given order allows us to predict the variation of R with Q.

Running coupling constant in QED

From our expressions for QCD, as a subproduct, we can find both the beta function and the running coupling constant for QED. The necessary modifications to make are due to color factors,

$$C_2(G) = N \to 0 \qquad C_2(R) \to 1 \qquad T(R)N_f \to 1$$

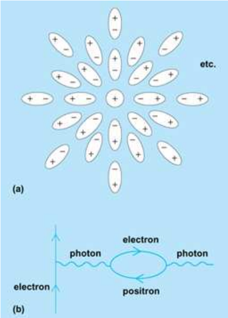
Then the value of β_0 is

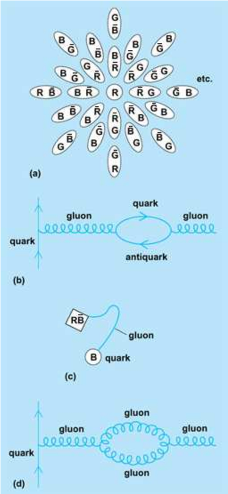
$$\beta_0^{QED} = -\frac{4}{3}$$

that have the opposite sign with respect to QCD.

QED is therefore not asymptotically free because the coupling increases at large Q. This result can be explained using the charge screening concept. In fact, the observed electron charge is distance-dependent due to the vacuum polarisation. At short distances (high momenta) we see more of the bare charge and hence the effective coupling increases. At large distances (low momenta), however, the screening is larger and the effective charge decreases.

Polarisation





Antiscreening

Due to polarisation effects, the QED vacuum behaves as an ordinary medium with electric dipoles and a dielectric constant $\varepsilon > 1$. This produces a screening effect.

In QCD, the vacuum polarisation gives anti-screening due to the non-Abelian character of the gluon field. Due to gluon-gluon couplings (three and four gluon vertices), the test charge is surrounded by gluons of the same charge. The vacuum behaves as a hypothetical medium with dielectric constant $\varepsilon < 1$. The measured color charge will be

$$q^{(c)} = \frac{q_0^{(c)}}{\varepsilon} \to q^{(c)} > q_0^{(c)}$$

This is an anti-screening effect that compensates the screening at short distances (large momenta) producing a decreasing of the effective coupling. And the other side round: the coupling constant increases at large distances. Asymptotic freedom means that the measured charge approaches zero at an infinitesimal distance.

The Λ parameter

Perturbation theory allows us to know the evolution of $\bar{\alpha}(Q)$ with Q, but its absolute value has to be obtained from experiment. One usually choose as the fundamental parameter the value of the coupling at $Q = M_Z$, a reference scale large enough to be in the perturbative region.

It is also useful to express $\bar{\alpha}(Q)$ in terms of a parameter with dimension of energy, Λ, defined by

$$\log\left(\frac{Q^2}{\Lambda^2}\right) = -\int_{\bar{\alpha}(Q)}^{\infty} \frac{dx}{\beta(x)} = \int_{\bar{\alpha}(Q)}^{\infty} \frac{4\pi\, dx}{\beta_0\, x^2\, \left(1 + \beta_1/\beta_0\, x/(4\pi) + \dots\right)}$$

At leading order, i.e keeping only the first coefficient β_0,

$$\bar{\alpha}^{(0)}(Q) = \frac{4\pi}{\beta_0 \log\left(Q^2/\Lambda^2\right)}$$

Notice that, if PT were the whole story, $\bar{\alpha}_s(Q) \to \infty$ as $Q \to \Lambda$. The parameter Λ sets the scale at which $\bar{\alpha}_s(Q)$ becomes large.

The Λ parameter

In the next-to-leading order, i.e. including β_1,

$$\frac{4\pi}{\bar{\alpha}_s(Q)} + \frac{\beta_1}{4\pi}\log\left(\frac{\frac{\beta_1}{4\pi}\frac{\bar{\alpha}_s(Q)}{4\pi}}{1+\frac{\beta_1}{4\pi}\frac{\bar{\alpha}_s(Q)}{4\pi}}\right) = \beta_0 \log\left(\frac{Q^2}{\Lambda^2}\right)$$

An approximate solution to second order in $1/\log(Q^2/\Lambda^2)$ can be obtained (Particle Data Group)

$$\bar{\alpha}(Q) = \frac{4\pi}{\beta_0 \log\left(Q^2/\Lambda^2\right)}\left[1 - \frac{\beta_1}{4\pi\beta_0}\frac{\log\log\left(Q^2/\Lambda^2\right)}{\log\left(Q^2/\Lambda^2\right)}\right]$$

Note that Λ depends on the number of active flavours and on the renormalization scheme: $\Lambda_{\overline{MS}} = 2.66\,\Lambda_{MS}$.