APPENDIX A.- SU(N)

Let T_a (a= 1,2,..., N^2-1) be the SU(N) generators, which close a Lie algebra

$$[T_a , T_b] = i f_{abc} T_c$$

$$\mathrm{Tr}\,[T_a] = 0 \tag{A.1}$$

f_{abc} being real and totally antisymmetric, normalized in such a way that

$$f_{abc} f_{dbc} = N \delta_{ad} \tag{A.2}$$

The gluon fields transform as its adjoint (regular) representation and then

$$(T_a)_{bc} = -i f_{abc} \tag{A.3}$$

The quark fields transform under SU(N) as the fundamental (lowest-dimensional cogradient) representation and in this case

$$T_a = \frac{1}{2} \lambda_a \tag{A.4}$$

where λ_a are hermitian, traceless, NxN matrices generalizing the Gell-Mann matrices of SU(3) and which satisfy

$$[\lambda_a, \lambda_b] = i\, 2 f_{abc} \lambda_c$$

$$\{\lambda_a, \lambda_b\} = \frac{4}{N} \delta_{ab} 1 + 2 d_{abc} \lambda_c \tag{A.5}$$

where d_{abc} are real and totally symmetric. Hence

$$\lambda_a \lambda_b = \frac{2}{N} \delta_{ab} I + d_{abc} \lambda_c + i f_{abc} \lambda_c \tag{A.6}$$

By simple algebraic manipulations it is easy to prove the following useful relations

$$d_{abb} = 0 \tag{A.7}$$

$$d_{abc} d_{dbc} = \left(N - \frac{4}{N} \right) \delta_{ad} \tag{A.8}$$

$$f_{abr} f_{cdr} = \frac{2}{N} (\delta_{ac} \delta_{bd} - \delta_{ad} \delta_{bc}) + d_{acr} d_{dbr} - d_{adr} d_{bcr} \tag{A.9}$$

$$f_{abr} d_{car} + f_{acr} d_{abr} + f_{adr} d_{bcr} = 0 \tag{A.10}$$

$$f_{abr} f_{cdr} + f_{acr} f_{dbr} + f_{adr} f_{bcr} = 0 \tag{A.11}$$

Then from (A.6) and (A.9)

$$\mathrm{Tr} [\lambda_a \lambda_b] = 2 \delta_{ab} \equiv 4 T(R) \delta_{ab} \tag{A.12}$$

$$\mathrm{Tr} [\lambda_a \lambda_b \lambda_c] = 2 (d_{abc} + i f_{abc}) \tag{A.13}$$

$$\begin{aligned} \mathrm{Tr} [\lambda_a \lambda_b \lambda_c \lambda_d] = {} & \frac{4}{N} (\delta_{ab} \delta_{cd} - \delta_{ac} \delta_{bd} + \delta_{ad} \delta_{bc}) \\ & + 2 (d_{abr} d_{cdr} - d_{acr} d_{dbr} + d_{adr} d_{bcr}) \\ & + 2i (d_{abr} f_{cdr} - d_{acr} f_{dbr} + d_{adr} f_{bcr}) \end{aligned} \tag{A.14}$$

In order to compute the needed traces in the adjoint representation we must remember [CV 76]

$$(\lambda_a)_{\alpha\beta}(\lambda_a)_{\gamma\delta} = 2\left[\delta_{\alpha\delta}\delta_{\beta\gamma} - \frac{1}{N}\delta_{\alpha\beta}\delta_{\gamma\delta}\right] \tag{A.15}$$

and using (A.3) a straightforward calculation leads to

$$\mathrm{Tr}_{ad}\,[T_a T_b] = N\,\delta_{ab} \tag{A.16}$$

$$\mathrm{Tr}_{ad}\,[T_a T_b T_c] = i\,\frac{N}{2}\,f_{abc} \tag{A.17}$$

$$\mathrm{Tr}_{ad}\,[T_a T_b T_c T_d] = \delta_{ab}\delta_{cd} + \delta_{ad}\delta_{bc} +$$

$$+\frac{N}{4}\left(d_{abr}d_{cdr} - d_{acr}d_{dbr} + d_{adr}d_{bcr}\right) \tag{A.18}$$

Some other useful relations that can be easily derived are

$$(\lambda_a)_{\alpha\beta}(\lambda_a)_{\beta\gamma} = 2\,\frac{N^2-1}{N}\,\delta_{\alpha\gamma} \equiv 4\,C_2(R)\,\delta_{\alpha\gamma} \tag{A.19}$$

$$(T_a)_{bc}(T_a)_{cd} = N\,\delta_{bd} \equiv C_2(G)\,\delta_{bd} \tag{A.20}$$

$$(\lambda_b\lambda_a\lambda_b)_{\alpha\beta} = -\frac{2}{N}(\lambda_a)_{\alpha\beta} \tag{A.21}$$

$$(\lambda_a\lambda_b)_{\alpha\beta}(T_b)_{ca} = N\,(\lambda_c)_{\alpha\beta} \tag{A.22}$$

Let us now consider a vertex with n external gluon lines, then its contribution to the invariant T-matrix element can be writte as $V_{a_1 a_2 \ldots a_n}$, where only the colour indices have been explicited. This can always be written as [PT 80]

$$V_{a_1 a_2 \dots a_m} = \sum_i V^{(i)} S^{(i)}_{a_1 a_2 \dots a_m} \tag{A.23}$$

where the tensors $S^{(i)}_{a_1 a_2 \dots a_n}$ $(i = 1,2,\dots)$ form a basis constructed uniquely from δ_{ab}, f_{abc} and d_{abc}. Let us try to characterize this basis for the lowest values of n. For $n = 1$ the base does not exist. For $n = 2$ the basis contains only one element : δ_{ab}. For $n = 3$ the elements of the basis are f_{abc} and d_{abc}. Finally let us consider the situation for $n = 4$. Let us characterize an irreducible representation of $SU(N)$ by its Young diagram $(\lambda_1, \lambda_2, \dots, \lambda_{N-1})$ where $\lambda_j + \lambda_{j+1} + \dots + \lambda_{N-1}$ denotes the number of boxes in the row j. Then one can prove that the Clebsch-Gordan series for the product of two adjoint representations can be written as

$$N = 3 \qquad (1,1) \otimes (1,1) = (0,0) \oplus (1,1) \oplus (1,1) \oplus (3,0) \oplus (0,3) \oplus (2,2)$$

$$N = 4 \qquad (1,0,1) \otimes (1,0,1) = (0,0,0) \oplus (1,0,1) \oplus (1,0,1) \oplus (0,2,0)$$
$$\oplus (2,1,0) \oplus (0,1,2) \oplus (2,0,2)$$

$$N \geq 5 \qquad (1,0,\dots,0,1) \otimes (1,0,\dots,0,1) = (0,0,\dots,0,0) \oplus (1,0,\dots,0,1)$$
$$\oplus (1,0,\dots,0,1) \oplus (0,1,\dots,1,0) \oplus (2,0,\dots,1,0) \oplus$$
$$+ (0,1,\dots,0,2) \oplus (2,0,\dots,0,2)$$

where the dots denote $(N-5)$ zeroes. If we characterize the irreducible representations by their dimension we can write $(N \geq 3)$

$$(N^2-1) \otimes (N^2-1) = 1 \oplus (N^2-1) \oplus (N^2-1) \oplus \frac{1}{4} N^2 (N+1)(N-3)$$

$$\oplus \frac{1}{4} (N^2-4)(N^2-1) \oplus \overline{\frac{1}{4} (N^2-4)(N^2-1)} \oplus \frac{1}{4} N^2 (N-1)(N+3) \tag{A.24}$$

Clearly for $N = 3$ the fourth term in the r.h.s. disappears. For one external gluon line (n=1) there are no invariant tensors. For n=2 since $(N^2-1) \otimes (N^2-1)$ contains only once the trivial representation, there is only one irreducible tensor: δ_{ab}. For n=3 and since $(N^2-1) \otimes (N^2-1) \otimes (N^2-1)$ contains only twice the trivial representation there are two invariant tensors: d_{abc} and f_{abc}. For n=4 and $N \geq 4$ the Clebsh-Gordon series for $(N^2-1) \otimes (N^2-1) \otimes (N^2-1) \otimes (N^2-1)$ contains nine times the trivial representation, and therefore the independent invariant tensors can be chosen as

$$\begin{array}{lll}
\delta_{ab}\,\delta_{cd} & \delta_{ac}\,\delta_{db} & \delta_{ad}\,\delta_{bc} \\
 & & \\
d_{abr}\,d_{cdr} & d_{acr}\,d_{dbr} & d_{adr}\,d_{bcr} \\
 & & \\
d_{abr}\,f_{cdr} & d_{acr}\,f_{dbr} & d_{adr}\,f_{bcr}
\end{array} \tag{A.25}$$

If N=3 the same reasoning proves that there are only eight invariant tensors and there must therefore exist a relation among these nine ten sors and it is

$$\delta_{ab}\,\delta_{cd} + \delta_{ac}\,\delta_{db} + \delta_{ad}\,\delta_{bc} =$$

$$= 3 \left[d_{abr}\,d_{cdr} + d_{acr}\,d_{dbr} + d_{adr}\,d_{bcr} \right] \tag{A.26}$$

For $N = 3$ an explicit representation of the λ_a is

$$\lambda_1 = \begin{vmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{vmatrix} \qquad \lambda_2 = \begin{vmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{vmatrix} \qquad \lambda_3 = \begin{vmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{vmatrix}$$

$$\lambda_4 = \begin{vmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{vmatrix} \qquad \lambda_5 = \begin{vmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{vmatrix} \qquad \lambda_6 = \begin{vmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{vmatrix} \tag{A.27}$$

$$\lambda_7 = \begin{vmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{vmatrix} \qquad \lambda_8 = \frac{1}{\sqrt{3}} \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{vmatrix}$$

Furthermore

$$\begin{aligned} f_{123} &= +1 \\ f_{147} &= -f_{156} = f_{246} = f_{257} = f_{345} = -f_{367} = +\frac{1}{2} \\ f_{458} &= f_{678} = +\frac{\sqrt{3}}{2} \end{aligned} \tag{A.28}$$

$$\begin{aligned} d_{118} &= d_{228} = d_{338} = -d_{888} = +\frac{1}{\sqrt{3}} \\ d_{146} &= d_{157} = -d_{247} = d_{256} = d_{344} = d_{355} = -d_{366} = -d_{377} = +\frac{1}{2} \\ d_{448} &= d_{558} = d_{668} = d_{778} = -\frac{1}{2\sqrt{3}} \end{aligned} \tag{A.29}$$

The components of these tensors which cannot be obtained by permutations of indices of the above given ones are zero.