II. Field Theory Background

This section reviews a number of relevant facts about QCD as a field theory, primarily its Lagrange density and Feynman rules, amplitudes and their renormalization, and the concepts of asymptotic freedom and infrared safety. We assume here a general familiarity with elementary methods in field theory. More detailed discussions of field theory topics may be found in text-books. Asymptotic freedom, infrared safety and the renormalization group applied to QCD are also covered in a number of useful reviews (Muta, 1987; Mueller, 1989; Sterman, 1991; Dok-shitzer *et al.*, 1991).

A. Lagrangian

The flurry of fields, indices, and labels in the telegraphic formulas that follow in this subsection are probably accessible only after the benefit of a pedagogical introduction that must be found elsewhere. We anticipate, however, that some number of readers may find these formulas a useful refresher of memory. Others will be satisfied by the summary of perturbation theory rules in Fig. 1, and will wish to skip to subsection B., which begins a review of quantum theoretic concepts much less dependent on the technical content of QCD, but which, toward the end of this section, explain what is special about QCD.

Quantum Chromodynamics is defined as a field theory by its Lagrange density,

$$\mathcal{L}_{\rm eff}^{\rm QCD}\left[\psi_f(x),\bar\psi_f(x),A(x),c(x),\bar c(x);g,m_f\right] = \mathcal{L}_{\rm invar}+\mathcal{L}_{\rm gauge}+\mathcal{L}_{\rm ghost}\ , \tag{2.1}$$

which is a function of fields [ψ_f (quark), A (gluon), and c (ghost)] and parameters g and m_f, where f labels distinct quark fields. $\mathcal{L}_{\rm invar}$ is the classical density, invariant under local $SU(N_c)$ gauge transformations, with $N_c = 3$ for QCD. $\mathcal{L}_{\rm invar}$ is of the form that was originally written down by Yang and Mills (Yang and Mills, 1954),

$$\begin{aligned}\mathcal{L}_{\rm invar} &= \sum_f \bar\psi_f\left(i\not{D}[A]-m_f\right)\psi_f-\frac{1}{4}F^2[A]\\ &= \sum_{f=1}^{n_f}\sum_{\alpha,\beta=1}^{4}\sum_{i,j=1}^{N_c}\bar\psi_{f,\beta,j}\left(i(\gamma)^\mu_{\beta\alpha}D_{\mu,ji}[A]-m_f\delta_{\beta\alpha}\delta_{ji}\right)\psi_{f,\alpha,i}\\ &\quad -\frac{1}{4}\sum_{\mu,\nu=0}^{3}\sum_{a=1}^{N_c^2-1}F_{\mu\nu,a}[A]F^{\mu\nu}{}_a[A]\ .\end{aligned} \tag{2.2}$$

In the second expression, we have written out all indices explicitly, using the notations

$$D_{\mu,ij}[A]\equiv\partial_\mu\delta_{ij}+igA_{\mu a}(T_a{}^{(F)})_{ij}\ , \tag{2.3}$$

and

$$F_{\mu\nu,a}[A]\equiv\partial_\mu A_{\nu a}-\partial_\nu A_{\mu a}-gC_{abc}A_{\mu b}A_{\nu c}\ . \tag{2.4}$$

Let us describe what these formulas represent, working backwards from Eq. (2.4).

$F_{\mu\nu,a}$ is the nonabelian *field strength* defined in terms of the *gluon* vector field A^μ_b, with $N_c^2 - 1$ group components b. g is the QCD ("strong") coupling and the $C_{abc}, a, b, c = 1 \ldots N_c^2 - 1$, are real numbers, called the structure constants of $SU(N_c)$, which define its *Lie algebra*. As mentioned above, for QCD (Fritzsch *et al.*, 1973; Gross and Wilczek, 1973b; Weinberg, 1973), $N_c = 3$, but for many purposes it is useful to exhibit the N_c-dependence explicitly. N_c is often called the "number of colors".

The Lie algebra is defined by the commutation relations of the $N_c^2 - 1, N_c \times N_c$ matrices $(T_a{}^{(F)})_{ij}$ that appear in the definition of $D_{\mu,ij}$, Eq. (2.3),

$$[T_a{}^{(F)}, T_b{}^{(F)}] = iC_{abc} T_c{}^{(F)} \,. \tag{2.5}$$

These commutation relations define the algebra. Here we have taken the $T_a{}^{(F)}$ to be hermitian, which makes QCD look a lot like QED. Some useful facts about the algebra of generators are listed in Appendix A:.

$D^\mu_{ij}[A]$ is the *covariant derivative* in the N_c-dimensional representation of $SU(N_c)$, which acts on the spinor *quark* fields in Eq. (2.2), with color indices $i = 1 \ldots N_c$. There are n_f independent quark fields ($n_f = 6$ in the standard model), labeled by *flavor* $f (= u, d, c, s, t, b)$. In the QCD Lagrangian, they are distinguished only by their masses.

The quark fields all transform as

$$\psi'_{f,\alpha,j}(x) = U_{ji}(x) \psi_{f,\alpha,i}(x) \,, \tag{2.6}$$

under local gauge transformations, where

$$U_{ji}(x) = \left[\exp\left\{ i \sum_{a=1}^{N_c^2-1} \beta_a(x) T_a{}^{(F)} \right\} \right]_{ji,} \tag{2.7}$$

with $\beta_a(x)$ real. Defined this way, $U_{ij}(x)$ for each x is an element of the group $SU(N_c)$, which is the local invariance that has been built into the theory. The corresponding transformation for the gluon field is most easily expressed in terms of an $N_c \times N_c$ matrix, $A_\mu(x)$,

$$[A_\mu(x)]_{ij} \equiv \sum_{a=1}^{N_c^2-1} A_{\mu a}(x) (T_a{}^{(F)})_{ij} \,, \tag{2.8}$$

which is the form that occurs in the covariant derivative. The gluonic field is then defined to transform as

$$A'_\mu(x) \ = U(x) A_\mu(x) U^{-1}(x) + \frac{i}{g} [\partial_\mu U(x)] U^{-1}(x). \tag{2.9}$$

With these transformation rules, the gauge invariance of $\mathcal{L}_{\text{invar}}$ is not difficult to check.

The gauge invariance of $\mathcal{L}_{\rm invar}$ actually makes it somewhat difficult to quantize. This problem is solved by adding to $\mathcal{L}_{\rm invar}$ *gauge-fixing* and *ghost* densities, $\mathcal{L}_{gauge}$ and $\mathcal{L}_{ghost}$, as in Eq. (2.1). The former may be chosen almost freely; the two most common choices being

$$\begin{aligned}\mathcal{L}_{gauge} &= -\frac{\lambda}{2}\sum_{a=1}^{N_c^2-1}\left(\partial_\mu A_a^\mu\right)^2 \quad 1<\lambda<\infty,\\ \mathcal{L}_{gauge} &= -\frac{\lambda}{2}\sum_{a=1}^{N_c^2-1}(n\cdot A_a)^2 \quad \lambda\to\infty,\end{aligned} \tag{2.10}$$

where n^μ is a fixed vector. The first defines the set of "covariant" gauges, the most familiar having $\lambda = 1$, the *Feynman gauge*. The second defines the "axial" or "physical" gauges (Leibbrandt, 1987), since taking λ to infinity eliminates the need for ghost fields. Here, picking n^μ light-like, $n^2 = 0$, defines the *light-cone gauge*. For $\lambda \to \infty$, a nonzero value of $n \cdot A$ leads to infinite action, and for this reason the physical gauges are often called "$n \cdot A = 0$" gauges.

Finally, in the covariant gauges we must add a ghost Lagrangian (Feynman, 1963; DeWitt, 1967; Faddeev and Popov, 1967; 't Hooft and Veltman, 1972)

$$\mathcal{L}_{ghost} = (\partial_\mu \bar{c}_a)(\partial^\mu \delta_{ad} - gC_{abd}A_b^\mu)c_d, \tag{2.11}$$

where $c_a(x)$ and $\bar{c}_a(x)$ are scalar ghost and antighost fields. In the quantization procedure, ghost fields anticommute, despite their spin. In an $SU(N_c)$ theory, the ghost fields ensure that the gauge fixing does not spoil the unitarity of the "physical" S-matrix that governs the scattering of quarks and gluons in perturbation theory.

B. Feynman Rules and Green Functions

The perturbation theory (Feynman) rules for QCD are summarized in Fig. 1. With our choice of (hermitian) generators ${T_a}^{(F)}$, the quark–gluon coupling is just like the QED fermion-photon vertex, except for the extra matrix factor ${T_a}^{(F)}$. The remaining rules for vertices are not difficult to derive in detail, but their essential structure is already revealed by the correspondence $(\partial_\rho\phi) \to -iq_\rho$, where q_ρ is the momentum flowing into the vertex at any field ϕ.

As for the propagators, we pause only to notice some special features of physical gauges. In the $n \cdot A = 0$ gauge, we have, from the propagator in Fig. 1,

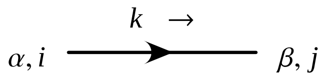
$$k^\mu {G_\mu}^\nu(k,n) = i\left(\frac{n^\nu}{n\cdot k} - \frac{n^2 k^\nu}{(n\cdot k)^2}\right). \tag{2.12}$$

Note the lack of a pole at $k^2 = 0$ on the right-hand side of this relation. This means that the unphysical gluon polarization that is proportional to its momentum does not propagate as a particle in these gauges. The lack of a pole for the gluon scalar polarization is the essential reason why ghosts are not necessary in physical gauges. This simplification also makes these gauges useful for

(a) Propagators: Gluon, quark, and ghost lines of momentum k

$i\frac{\delta_{ba}}{k^2+i\epsilon}[-g^{\mu\nu} + \left(1 - \frac{1}{\lambda}\right)\frac{k^\mu k^\nu}{k^2+i\epsilon}]$ covariant gauge

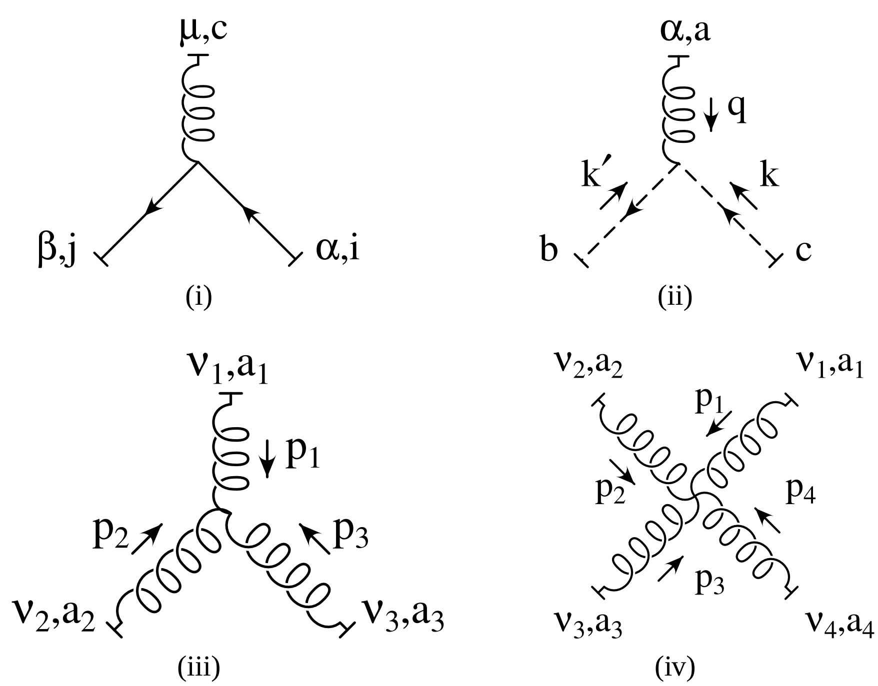
$i\frac{\delta_{ba}}{k^2+i\epsilon}[-g^{\mu\nu} + \frac{k^\mu n^\nu + n^\mu k^\nu}{n\cdot k} - n^2\frac{k^\mu k^\nu}{(n\cdot k)^2}]$ physical gauge



$i\frac{\delta_{ij}}{k^2-m^2+i\epsilon}[\not{k} + m]_{\beta\alpha}$

$i\frac{\delta_{ba}}{k^2+i\epsilon}$

(b) Vertices (all momenta defined to flow in)



(i) $-ig[T_c^{(F)}]_{ji}[\gamma_\mu]_{\beta\alpha}$

(ii) $gC_{abc}k'_\alpha$

(iii) $-gC_{a_1a_2a_3}[g^{\nu_1\nu_2}(p_1 - p_2)^{\nu_3} + g^{\nu_2\nu_3}(p_2 - p_3)^{\nu_1} + g^{\nu_3\nu_1}(p_3 - p_1)^{\nu_2}]$

(iv)
$$\begin{aligned} -ig^2[\quad & C_{ea_1a_2}C_{ea_3a_4}(g^{\nu_1\nu_3}g^{\nu_2\nu_4} - g^{\nu_1\nu_4}g^{\nu_2\nu_3}) \\ + \quad & C_{ea_1a_3}C_{ea_4a_2}(g^{\nu_1\nu_4}g^{\nu_3\nu_2} - g^{\nu_1\nu_2}g^{\nu_3\nu_4}) \\ + \quad & C_{ea_1a_4}C_{ea_2a_3}(g^{\nu_1\nu_2}g^{\nu_4\nu_3} - g^{\nu_1\nu_3}g^{\nu_4\nu_2})] \end{aligned}$$

Figure 1: Perturbation theory rules for QCD.

many all-order arguments in pQCD. The price, however, is the unphysical poles at $n \cdot k = 0$, which are often thought of as principal values,

$$P \frac{1}{(n \cdot k)^\alpha} \equiv \frac{1}{2} \left[\frac{1}{(n \cdot k + i\epsilon)^\alpha} + \frac{1}{(n \cdot k - i\epsilon)^\alpha} \right]. \tag{2.13}$$

This definition, however, is awkward beyond tree level (when loops are present) and other definitions (Mandelstam, 1983; Leibbrandt, 1987) are necessary to carry out loop calculations correctly (Bassetto, Nardelli, and Soldati, 1991; Bassetto *et al.*, 1993). In any case, it is often desirable to back up results derived in physical gauges with calculations or arguments based on covariant gauge reasoning.

The Feynman rules allow us to define *Green functions* in momentum space. These are the vacuum expectation values of time-ordered products of fields,

$$(2\pi)^4 \delta(p_1 + \ldots + p_n) G_{\alpha_1 \cdots \alpha_n}(p_1, \ldots, p_n) = \prod_{i=1}^{n} \int d^4 x_i e^{-ip_i \cdot x_i} \\ \times \langle 0 | \, T[\phi_{\alpha_1}(x_1) \ldots \phi_{\alpha_n}(x_n)] \, | 0 \rangle \,, \tag{2.14}$$

where the α_i represent both space-time and group indices of the fields, collectively denoted by ϕ. At any fixed order in perturbation theory, $G_{\alpha_1 \cdots \alpha_n}$ is given by the sum of all diagrams constructed according to the rules of Fig. 1. Corresponding to each of the fields in the matrix element, every diagram will have an external propagator carrying momentum p_i into the diagram, with free external indices α_i. Essentially all of the physical information of the theory is contained in its Green functions.

C. From Green Functions to Experiment

The route from Feynman rules, through Green functions to experimentally observable quantities is straightforward, but involves a number of steps which it may be useful to outline. In what follows, we shall briefly review the roles of the S-matrix, cross sections, renormalization schemes and regularization.

We do not address yet the issue of whether perturbation theory is of any use for reliable calculations of physical quantities in QCD.

1. The S-matrix and Cross Sections

By themselves, Green functions are not always direct physical observables. For one thing, their external lines are not necessarily on-mass-shell, and, in a gauge theory, the Green functions are not even gauge invariant. The relation between Green functions and physical quantities like cross sections is, however, quite simple. Let us review the basic steps in a generic situation with fields ϕ_α.

First, a two-point Green function has a pole at $p^2 = m^2$. Near the pole, it has the form of a "free" propagator (Fig. 1) times a scalar constant R_ϕ,

$$G_{\alpha\beta}(p) \to R_\phi G_{\alpha\beta}(p)^{free} + \text{finite} . \tag{2.15}$$

If the particles under discussion are hadrons, then R_ϕ and the physical mass M are not perturbatively calculable. If, nevertheless, we discuss the perturbative S-matrix for quarks and gluons, then R_ϕ and M can be computed as a power series in the coupling

$$\begin{aligned} R_\phi &= 1 + \mathrm{O}(g^2) \\ M &= m + \mathrm{O}(g^2) . \end{aligned} \tag{2.16}$$

The *S-matrix* is simply the amplitude for the scattering of momentum eigenstates into other momentum eigenstates. In particle physics, the most important S-matrix elements describe the scattering of two incoming particles into some set of outgoing particles. The S-matrix is derived from Green functions by "reduction formulas", of the general form

$$\begin{aligned} S\left((p_1, s_1) + (p_2, s_2) \to (p_3, s_3) + \ldots (p_n, s_n)\right) = \prod_i \psi(p_i, s_i)_{\alpha_i} \\ \times \left[\frac{G^{-1}_{\alpha_i\beta_i}(p_i)^{free}}{R_\phi^{1/2}} \right] G_{\beta_1\cdots\beta_n}(p_1, p_2, -p_3, \ldots, -p_n) , \end{aligned} \tag{2.17}$$

where now s_i represents the spin (and other quantum numbers) of particle i. Here $\psi(p_i, s_i)_{\alpha_i}$ represents the wave function of external particle i, given by

$$\begin{aligned} u(p,s) &\quad \text{for an incoming Dirac particle} \\ \bar{u}(p,s) &\quad \text{for an outgoing Dirac particle} \\ \bar{v}(p,s) &\quad \text{for an incoming Dirac antiparticle} \\ v(p,s) &\quad \text{for an outgoing Dirac antiparticle} \\ \epsilon(p,s) &\quad \text{for an incoming vector particle} \\ \epsilon^*(p,s) &\quad \text{for an outgoing vector particle} . \end{aligned} \tag{2.18}$$

Once again, $G_{\alpha_i\beta_i}(p_i)^{free}$ is the free propagator, for field i, but with the correct physical mass of the corresponding particle.

From the S-matrix, it is customary to define the *transition matrix* T by

$$S = I + iT , \tag{2.19}$$

with I the identity matrix in the space of states. For momentum eigenstates, T contains an explicit momentum-conservation delta function, which it is convenient to separate explicitly,

$$\begin{aligned} iT((p_1, s_1) + (p_2, s_2) \to\ & (p_3, s_3) + \ldots (p_n, s_n)) = \\ & (2\pi)^4 \delta^4(p_1 + p_2 - p_3 - \ldots - p_n) \\ & \times \mathcal{M}((p_1, s_1) + (p_2, s_2) \to (p_3, s_3) + \ldots (p_n, s_n)) . \end{aligned} \tag{2.20}$$

It is $\mathcal{M}$-matrix elements that are used to derive cross sections, by integrating the general infinitesimal cross section,

$$\begin{aligned} d\sigma\left((p_1, s_1) + (p_2, s_2) \to (p_3, s_3) + \ldots (p_n, s_n)\right) \\ = \frac{1}{4\sqrt{(p_1.p_2)^2 - m_1^2 m_2^2}} \mathrm{dPS}_n \\ \times \left|\mathcal{M}\left((p_1, s_1) + (p_2, s_2) \to (p_3, s_3) + \ldots (p_n, s_n)\right)\right|^2 \ , \end{aligned} \tag{2.21}$$

over n-particle phase space,

$$\mathrm{dPS}_n = \prod_i \left(\frac{d^3 p_i}{2\omega_i (2\pi)^3}\right) N_i (2\pi)^4 \delta^4 (p_1 + p_2 - \sum_{j=3}^{n} p_j) \ . \tag{2.22}$$

Here $N_i = 1$ for vector and scalar particles, as well as for Dirac particles when we normalize their wave functions according to $\bar{u}(p, s)u(p, s) = 2m$. For the other common choice, $\bar{u}(p, s)u(p, s) = 1$, we have $N_i = 2m$ for Dirac fermions. If one integrates a differential cross section over the phase space for n identical particles, then one should include an additional factor of $\mathcal{S}_n = 1/n!$ that compensates for counting the same physical state $n!$ times. When discussing the perturbative expansion of a cross section, it is often useful to work directly with diagrams for $|\mathcal{M}|^2$. The rules for this expansion are almost the same as for the S-matrix, and are summarized in Appendix B:.

2. UV Divergences, Renormalization and Schemes

Green functions, and consequently cross sections, calculated according to the unmodified Feynman rules described above suffer a severe problem when we include diagrams with loops. These are the ultraviolet (UV) divergences, associated with infinite loop momenta. We may think of these divergences as due to virtual states in which energy conservation is violated by an arbitrarily large amount. Let us see how these problems come about, and review how they can be solved in perturbative calculations.

A typical one-loop integral UV divergence is illustrated by the diagram with scalar lines in Fig. 2. For scalar lines the diagram is given, before renormalization, by

$$\begin{aligned} \Gamma^{(un)}(p) &= \int \frac{d^4 k}{(2\pi)^4} \frac{1}{(k^2 - m^2)((p-k)^2 - m^2)} \\ &= \int_0^1 dx \int \frac{d^4 k}{(2\pi)^4} \frac{1}{\left(k^2 - 2xp \cdot k + xp^2 - m^2\right)^2} \\ &= \int_0^1 dx \int \frac{d^4 k'}{(2\pi)^4} \frac{1}{\left(k'^2 + x(1-x)p^2 - m^2\right)^2} \ . \end{aligned} \tag{2.23}$$

A: Color Matrix Identities and Invariants

Only a few identities are necessary for the calculations described in the text. In general, for representation R, $SU(N)$ generators can be picked to satisfy

$$\mathrm{Tr}\,[\,T_a^{(R)}T_b^{(R)}\,] = T(R)\delta_{ab}\ , \tag{1.1}$$

with $T(R)$ a number characteristic of the representation. Also of special interest is the representation-dependent invariant, $C_2(R)$, defined by

$$\sum_{a=1}^{N^2-1} (T_a^{(R)})^2 = C_2(R)I, \tag{1.2}$$

with I the identity matrix.

We encounter only two representations here, the N-dimensional "defining" representation, F, and the N^2-1-dimensional adjoint representation, A. The generators $T_a^{(F)}$ are a complete set of $N \times N$ traceless hermitian matrices, while the generators $T_a^{(A)}$ are defined by the $SU(N)$ structure constants C_{abc} (Eq. (2.5)) as

$$(T_a^{(A)})_{bc} = -iC_{abc}\ . \tag{1.3}$$

For these two representations, the relevant constants are

$$\begin{aligned} T(F) &= \frac{1}{2} \quad C_2(F) = \frac{N^2-1}{2N} \\ T(A) &= N \quad C_2(A) = N\ . \end{aligned} \tag{1.4}$$

Another useful identity, special to the defining representation, enables us to work with simple products of the generators,

$$T_a^{(F)}T_b^{(F)} = \tfrac{1}{2}[iC_{abc}T_c^{(F)} + d_{abc}T_c^{(F)}] + \tfrac{1}{6}\delta_{ab}I\ , \tag{1.5}$$

with I the 3×3 identity, and the d_{abc} real. Unlike the previous equations, this and the following equation apply only to $SU(3)$. A numerical value that occurs in the three-loop correction to the total e^+e^- annihilation cross section is

$$D = \sum_{abc} d_{abc}^2 = \frac{40}{3}\ . \tag{1.6}$$