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Electron-positron annihilation energy pattern in quantum chromodynamics: Asymptotically free perturbation theory

C. Louis Basham, Lowell S. Brown, S. D. Ellis, and S. T. Love
Physics Department, University of Washington, Seattle, Washington 98195
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High-energy cross sections of quantum chromodynamics can be computed entirely in terms of the renormalization-group running coupling constant if these cross sections have no infrared mass singularities. Since the theory is asymptotically free, at high energies the running coupling constant vanishes and such cross sections can be computed perturbatively. Thus the theory of quantum chromodynamics may be rigorously tested. We compute the angular distribution of the hadronic energy produced in high-energy electron-positron annihilation to second order and find no mass singularities. Our result can be interpreted in terms of a jet opening angle which vanishes logarithmically as the energy increases. We compute phenomenologically the corrections to the energy pattern resulting from nonperturbative confinement effects. They become small at the energies of the colliding-beam machines now under construction.

Quantum chromodynamics[1] is a very promising candidate for the underlying fundamental field theory of hadronic physics. This theory is asymptotically free at short distances,[2] but it appears to be a very strong coupling theory at long distances. Thus, on the one hand, the theory exhibits the scaling behavior observed at high energies (up to small logarithmic corrections), while on the other it may give very strong long-range forces which confine the quarks. Clearly, it is of great importance to devise methods to test the theory of quantum chromodynamics in a quantitative and unambiguous fashion.

In general, conventional high-energy scattering processes are not immediately suitable for a precise test of the theory. Such reactions contain hadrons in the initial state, and they necessarily involve details of the confinement mechanism, details which are nonperturbative and intractable at present and which spoil any attempt at a precise perturbative test.[2a] High-energy electron-positron annihilation into hadronic final states does not suffer from this defect. These reactions are suitable for testing the theory. We shall be concerned only with them in this paper. The total cross section can be computed unambiguously by exploiting the asymptotic freedom of the theory.[3] The leading term essentially counts the sum of the squares of the quark charges of different types ("flavors"), and the first-order logarithmic correction can also be computed exactly in terms of one free parameter which sets the energy scale in the logarithm. This rigorous computation uses the renormalization-group method to evaluate the high-energy behavior of the photon propagator. The total cross section is identified with the absorptive part of the propagation function. The same result can be obtained in a simpler manner. It is obtained by calculating the total cross section for the production of massless quarks and massless gluons in renormalized perturbation theory and then replacing the fixed renormalized coupling constant by the running energy-dependent coupling constant used in the renormalization-group analysis. Since the running coupling constant vanishes asymptotically, only the first few terms of the perturbation series need be calculated. The validity of this alternative procedure suggests that the total cross section may be only the most elementary of a whole hierarchy of more finely defined partial cross sections which can be calculated perturbatively in quantum chromodynamics, using the asymptotically vanishing running coupling constant. This is the method, recently advocated by Sterman and Weinberg,[4] which we follow.[5]

Let us, for the sake of clarity, review the basic ideas of the method. We consider electron-positron annihilation into a virtual photon of mass W. The virtual photon produces a system of quarks and gluons which eventually combine to form the hadronic final state. We limit our considerations to final-state measurements which do not entail the properties of specific hadrons. Thus we shall not consider, say, the probability for the production of some number of pions, but rather, for example, the amount of energy deposited in a certain solid angle. It is well known empirically that hadrons are produced with limited transverse momentum[6] $\langle p_\perp \rangle$. Thus we shall assume that the restricted measurements which concern us here can be described by the basic partial cross section for the production of the intermediate quarks and gluons. This approximation should incur an error of relative order $\langle p_\perp \rangle / W$

which is much smaller at high energies than are the asymptotically free perturbative corrections which vanish only logarithmically.

The functional form of the basic partial cross section for quark and gluon production is displayed by writing

$$\Delta\sigma = \frac{1}{W^2} F(W, m, \mu, g_\mu, x). \quad (1)$$

A factor of W^{-2} has been extracted to make the function F dimensionless. The quantity m represents a quark or gluon renormalized mass, while x stands for dimensionless variables such as energy ratios or angles. The renormalized coupling constant g_μ is defined by the value of the quark-quark-gluon vertex at the Euclidean-momentum reference point μ. This coupling constant is taken to be that of the theory with *massless* quarks and gluons. We assume that the high-energy limit $W \to \infty$ is equivalent to the massless limit $m \to 0$. Thus at high energies the partial cross section can be written in terms of a dimensionless function of dimensionless parameters,

$$\Delta\sigma = \frac{1}{W^2} f\left(\frac{\mu}{W}, g_\mu, x\right). \quad (2)$$

We should note that this limit is valid only for energies much higher than the quark threshold energies.

It should be emphasized that an arbitrary partial cross section $\Delta\sigma$ will not generally have a finite massless limit; in perturbation theory it will contain factors involving $\ln(W/m)$. Such mass singularities will arise if the cross section is defined in a way that is sensitive to soft-gluon emission or to the branching of a quark into collinear quarks and gluons. If the partial cross section refers to a specific particle type, then it will generally have mass singularities. Thus, if we try to ask questions having to do with the quarklike character of an event, the theory will produce a mass singularity showing that this question cannot be answered. Indeed, such questions should not have an answer in this asymptotic framework which applies only to measurements where the details of quark confinement are not relevant. However, if we ask questions which are "physically sensible[4]" in a massless theory, then the corresponding cross sections should be free of infrared mass singularities.[7] We shall adopt a pragmatic attitude where questions are asked which one would intuitively believe to be "physically sensible" and then calculate in perturbation theory to see if the relevant cross sections are, in fact, free of infrared mass divergences. It should be stressed that we lack a general proof that mass singularities will be absent to all orders—we can only verify the self-consistency of the method in low orders of perturbation theory.

Assuming that the limit (2) does exist, then it must be independent of the renormalization point μ,

$$\frac{d}{d\mu} f\left(\frac{\mu}{W}, g_\mu, x\right) = 0, \quad (3)$$

and the method of the renormalization group can be applied. Let us briefly review this method.[8] Using the variation of g_μ with μ given by the renormalization-group equation

$$\mu \frac{dg_\mu}{d\mu} = \beta(g_\mu), \quad (4)$$

Eq. (3) can be cast into

$$W \frac{\partial f}{\partial W} = \beta(g_\mu) \frac{\partial f}{\partial g_\mu}. \quad (5)$$

The energy-dependent running coupling constant $\bar{g}_W$ is defined implicitly by

$$\int_{g_\mu}^{\bar{g}_W} \frac{dg}{\beta(g)} = \ln(W/\mu). \quad (6)$$

The variation of this definition yields

$$W \frac{\partial \bar{g}_W}{\partial W} = \beta(\bar{g}_W) \quad (7a)$$

and

$$\frac{\partial \bar{g}_W}{\partial W} = \beta(\bar{g}_W)/\beta(g_\mu). \quad (7b)$$

Hence the solution of Eq. (5) is obtained by transferring the energy dependence in the argument μ/W into the energy dependence in the running coupling constant, by setting $\mu/W = 1$, and by replacing g_μ with $\bar{g}_W$. That is, the solution of Eq. (5) gives

$$\Delta\sigma = \frac{1}{W^2} f\left(\frac{\mu}{W}, g_\mu, x\right) = \frac{1}{W^2} f(1, \bar{g}_W, x). \quad (8)$$

In an asymptotically free theory, the β function is negative for small g, and Eq. (6) forces the running coupling constant $\bar{g}_W$ to vanish as $W \to \infty$. Thus the high-energy limit can be computed perturbatively. The β function in asymptotically free quantum chromodynamics has the lowest-order form[2]

$$\beta(g) = -\frac{1}{(4\pi)^2}(11 - \tfrac{2}{3}N_f)g^3, \quad (9)$$

where N_f is the number of flavors. Although this lowest-order form (9) gives only the leading asymptotic value of the running coupling constant $\bar{g}_W$, it is instructive to use Eq. (9) for all values of g. Then the integral in Eq. (6) gives

$$\bar{g}_W^{\,2} \sim \frac{g_\mu^{\,2}}{1 + (g_\mu^{\,2}/8\pi^2)[11 - (2/3)N_f]\ln(W/\mu)}. \quad (10)$$

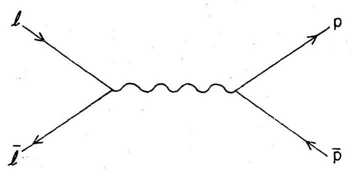

FIG. 1. Lowest-order Feynman graph for $e^+e^- \to \gamma \to q\bar{q}$.

This form illustrates how the powers of $\ln(W/\mu)$ generated in the perturbative expansion of $f(\mu/W, g_\mu, x)$ with the fixed coupling constant g_μ are transferred into the running coupling constant in the function $f(1, \bar{g}_W, x)$. Of course, Eq. (10) is accurate only in the high-energy limit, and one should use only the limiting form

$$\bar{g}_W^{\,2} = \frac{8\pi^2}{[11-(2/3)N_f]\ln(W/\mu)}, \qquad (11)$$

which is independent of $g_\mu^{\,2}$. Present estimates indicate that the renormalization point should be chosen to have the value[9] $\mu \approx 500$ MeV in order to minimize the $(\ln W/\mu)^{-2}$ corrections to the leading form (11).

We turn now to apply this method, the method of asymptotically free perturbation theory, to the simplest measurement beyond that of the total cross section. This is the measurement of the angular pattern of the hadronic energy radiated in electron-positron annihilation. We define the differential energy cross section $d\Sigma/d\Omega$ to be the power radiated into the solid angle $d\Omega$ divided by the energy flux in the incident e^+e^- colliding beams. This quantity is the "antenna pattern" for the annihilation process. It is normalized so that its integral over all solid angle gives the standard total cross section. The energy cross section is not sensitive to the emission of soft gluons or to the branching of a quark into collinear quarks and gluons. Thus it should be calculable by the asymptotically free perturbation method that we have just described. We shall find that it is indeed free of infrared mass singularities in second order.

In order to describe our results clearly, we shall first consider the idealized case in which the initial electrons and positrons are completely polarized along a direction perpendicular to their common beam axis. (We shall, of course, always work in the laboratory frame where the electrons and positrons have equal but oppositely directed momenta.) The lowest-order process $e^+ + e^- \to \gamma \to q + \bar{q}$ is depicted in Fig. 1. Here the differential energy cross section is identical to the ordinary differential cross section since the quark (q) or antiquark ($\bar{q}$) carries away precisely one-half of the total incident energy W. A simple calculation gives

$$\frac{d\Sigma}{d\Omega}^{(0)} = \frac{d\sigma}{d\Omega}^{(0)} = (\alpha^2/2W^2)\sin^2\psi \sum_f 3Q_f^{\,2}. \qquad (12)$$

Here $\alpha \sim \frac{1}{137}$ is the fine-structure constant, ψ is the angle at which the radiated energy is detected relative to the direction of the beam polarization (the magnetic-field direction), Q_f is the value of the fractional quark charge of flavor f, and the factor of 3 accounts for the sum over the three colors. According to the asymptotically free perturbation theory, the lowest-order cross section (12) should be valid in the infinite energy limit. Its structure reflects the spin $\frac{1}{2}$ carried by the quarks, and its magnitude measures the sum of the squares of the quark charges.

To obtain the first correction as the energy is lowered, the order-g^2 terms in the perturbation theory must be computed. These contributions are displayed in Fig. 2. Some regularization scheme must be introduced to deal with the infrared mass singularities that appear in the individual diagrams. The simplest procedure here is to introduce a finite gluon mass λ while keeping the quarks massless.[4] The energy cross section $d\Sigma/d\Omega$ is computed by modifying the ordinary cross section for the production of a particle a into the solid angle $d\Omega$ by inserting the factor E_a/W into the phase-space integrand, where E_a is the energy of the particle, and then summing over all particle types a. We shall first calculate the contributions to the cross section when the energy is carried by the quark or antiquark, $d\Sigma_q/d\Omega$. The lowest-order production process of Fig. 1 is altered by its interference with the vertex correction shown in Fig. 2(a) (including a wave-function renormalization not displayed). The calculation of this interference contribution is a standard one, and we simply quote the result:

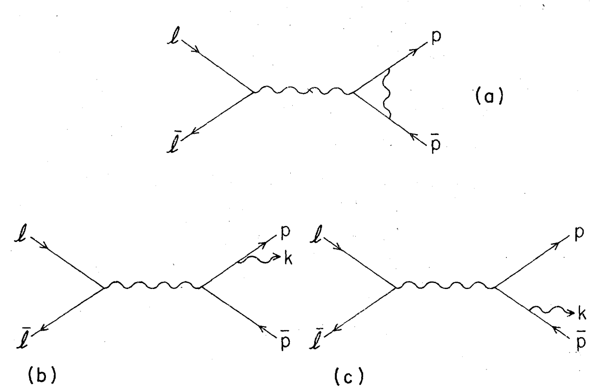

FIG. 2. (a) Vertex modification. (b) and (c) are the lowest-order Feynman graphs for gluon production.

$$\frac{d\Sigma_q}{d\Omega}(\text{virtual}) = \frac{\alpha^2}{2W^2}\sin^2\psi \sum_f 3Q_f^2 \frac{g^2}{6\pi^2}\left(-\ln^2\frac{W^2}{\lambda^2} + 3\ln\frac{W^2}{\lambda^2} + \frac{\pi^2}{3} - \frac{7}{2}\right). \tag{13}$$

The calculation of the contribution of the real-gluon emission process displayed in Figs. 2(b) and 2(c) is outlined in Appendix A. We find that

$$\frac{d\Sigma_q}{d\Omega}(\text{real}) = \frac{\alpha^2}{2W^2}\sum_f 3Q_f^2\frac{g^2}{6\pi^2}\left[\frac{1}{2}(3\cos^2\psi - 1) + \sin^2\psi\left(\ln^2\frac{W^2}{\lambda^2} - \frac{13}{3}\ln\frac{W^2}{\lambda^2} - \frac{\pi^2}{3} + \frac{80}{9}\right)\right]. \tag{14}$$

Thus, to second order, the total contribution to the energy cross section, with the energy being carried by the quarks, is given by

$$\frac{d\Sigma_q}{d\Omega} = \frac{d\Sigma_q}{d\Omega}(\text{virtual}) + \frac{d\Sigma_q}{d\Omega}(\text{real})$$
$$= \frac{\alpha^2}{2W^2}\sum_f 3Q_f^2\frac{g^2}{6\pi^2}\left[\frac{1}{2}(3\cos^2\psi - 1) + \sin^2\psi\left(-\frac{4}{3}\ln\frac{W^2}{\lambda^2} + \frac{97}{18}\right)\right]. \tag{15}$$

Although the leading $\ln^2 W^2/\lambda^2$ mass singularities have canceled between the real and virtual gluon emission contributions, this cross section still has a $\ln W^2/\lambda^2$ mass singularity. We conclude that questions related to energy-weighted measurements of quarklike (flavor-dependent) properties cannot be answered within the context of the asymptotically free perturbation theory. Such properties are inextricably tied up with the confinement mechanism which cannot be treated perturbatively. It is worthwhile to give another example to clarify this point. Naively, one might expect to be able to reveal the underlying presence of fractionally charged quarks by computing, say, the energy-weighted, average squared charge to be detected in some solid angle $d\Omega$. The energy weighting might be expected to remove the ambiguity caused by the transfer of charge from one jet to the other by soft quarks.[10] However, the cross section for this measurement is obtained by replacing Q_f^2 with Q_f^4 in Eq. (15), and we see that it is *not* free of infrared mass singularities.[11] On the other hand, we should note that the ordinary cross section for finding a quark in the solid angle $d\Omega$ is finite to order g^2. As is discussed in Appendix A, the differential cross section is given by

$$\frac{d\sigma_q}{d\Omega} = \frac{\alpha^2}{2W^2}\sum_f 3Q_f^2\left[\left(1 + \frac{g^2}{4\pi^2}\right)\sin^2\psi + \frac{g^2}{6\pi^2}(3\cos^2\psi - 1)\right]. \tag{16}$$

This cross section must have mass singularities in higher order if the method which we are following is to be consistent.

To complete the calculation of the energy cross section, we need also the contribution when the energy is carried by the gluon. As is discussed in Appendix A, this contribution is given by

$$\frac{d\Sigma_g}{d\Omega} = \frac{\alpha^2}{2W^2}\frac{g^2}{6\pi^2}\sum_f 3Q_f^2\left[(3\cos^2\psi - 1) + \sin^2\psi\left(\frac{4}{3}\ln\frac{W^2}{\lambda^2} - \frac{35}{9}\right)\right]. \tag{17}$$

The appearance here of the infrared mass singularity shows that questions such as: "What is the fraction of the energy carried off by the gluons?" cannot be answered within the context of the asymptotically free perturbation theory. The complete energy differential cross section is, as was expected, finite. We add together Eqs. (12), (15), and (17), and replace g^2 with the running coupling $\bar{g}_W^2$ to secure

$$\frac{d\Sigma}{d\Omega} = \frac{\alpha^2}{2W^2}\sum_f 3Q_f^2\left(\sin^2\psi + \frac{\bar{g}_W^2}{2\pi^2}\cos^2\psi\right). \tag{18}$$

Inserting the value (11) of the running coupling constant gives

$$\frac{d\Sigma}{d\Omega} = \frac{\alpha^2}{2W^2}\sum_f 3Q_f^2\left(\sin^2\psi + \frac{4\cos^2\psi}{[11 - (2/3)N_f]\ln W/\mu}\right). \tag{19}$$

We recall that ψ is the angle between the direction of the detected energy and the direction of the (complete) polarization of the incident clashing beams. We see that the valley of the $\sin^2\psi$ distribution is filled in at lower energies by the $\cos^2\psi$ perturbative correction.

The fragmentation of the quarks into the observed hadrons modifies our result. We can estimate this effect with a simple phenomenological model. We suppose that a quark, produced with momentum $\vec{p}$ making an angle χ relative to the polarization axis, fragments into hadrons with one hadron of momentum $\vec{h}$ coming out with an opening angle η relative to the quark direction $\vec{p}$. This configuration is illustrated in Fig. 3. The hadron will deposit energy at the angle ψ with respect to the polarization direction. We assume that the quark is produced with the lowest-

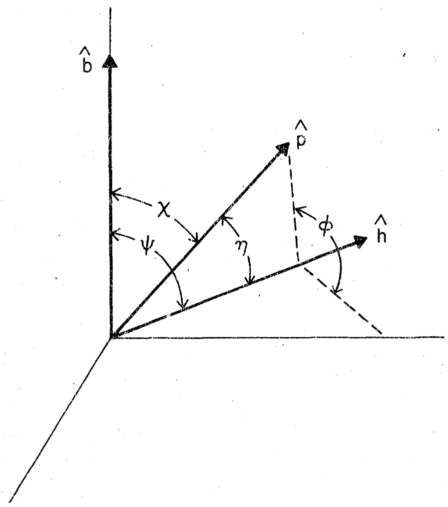

FIG. 3. Geometry for the fragmentation of a quark of momentum $\vec{p}$ into a hadron of momentum $\vec{h}$. The beam-polarization direction (the direction of the magnetic field) is denoted by $\hat{b}$.

order $\sin^2\chi$ distribution. Using the law of cosines, the branching process makes the replacement

$$\sin^2\chi \to \sin^2\psi + \tfrac{1}{2}\sin^2\eta(3\cos^2\psi - 1)\,, \qquad (20)$$

where an average over the azimuthal angle ϕ has been performed. Thus the quark fragmentation modifies the basic cross section to read

$$\frac{d\Sigma}{d\Omega}(\text{frag}) = \frac{\alpha^2}{2W^2}\sum_f 3Q_f{}^2[\sin^2\psi + \tfrac{1}{2}\langle\sin^2\eta\rangle(3\cos^2\psi - 1)]\,, \qquad (21)$$

where $\langle\sin^2\eta\rangle$ involves an energy-weighted average opening angle. This averaged quantity is computed in Appendix B using an energy-scaling quark fragmentation function $f(z, h_\perp)$, where $z = 2h_\parallel/W$ with $h_\parallel$ and $h_\perp$ the components of the hadronic momentum $\vec{h}$ that are parallel and perpendicular to the quark momentum $\vec{p}$. By employing an energy sum rule for $f(z,h_\perp)$, and the integral relating $f(0,h_\perp)$ to the coefficient C of the logarithmic rise of the total hadronic multiplicity $\langle n\rangle$,

$$\langle n\rangle = C\ln W + \text{const}\,, \qquad (22)$$

we find that

$$\langle\sin^2\eta\rangle = \frac{\pi C\langle h_\perp\rangle}{2W}\,, \qquad (23)$$

where $\langle h_\perp\rangle$ is the average transverse momentum.

We see that the spreading of the energy distribution by the fragmentation effect vanishes as $1/W$ as the energy increases, while the asymptotically free perturbation corrections vanish much more slowly as $1/\ln W$. Thus at sufficiently high energies only the asymptotically free perturbation corrections are significant. We can get a rough picture of the variation of the energy pattern with energy by simply adding the two effects and, at the same time, putting our results in a suggestive form. We add the fragmentation correction exhibited in Eq. (21) to the perturbative result (18) and write the sum in the form

$$\frac{d\Sigma}{d\Omega} = \frac{\alpha^2}{2W^2}\sum_f 3Q_f{}^2\left(1 + \frac{\bar{g}_W{}^2}{4\pi^2}\right) \times[\sin^2\psi + \tfrac{1}{2}\langle\sin^2\eta\rangle_{\text{total}}(3\cos^2\psi - 1)]\,. \qquad (24)$$

Here we identify $\langle\sin^2\eta\rangle_{\text{tot}}$ in terms of a full jet opening angle. It is given by

$$\langle\sin^2\eta\rangle_{\text{tot}} = \frac{4}{[11 - (2/3)N_f]\ln W/\mu} + \frac{\pi C\langle h_\perp\rangle}{2W}\,. \qquad (25)$$

To assess the size of this total opening angle, we take[12] $N_f = 4$, $\mu = 500$ MeV, $C = 2.5$, and $\langle h_\perp\rangle = 300$ MeV. Then, with W in GeV units,

$$\langle\sin^2\eta\rangle_{\text{tot}} = \frac{0.48}{\ln 2W} + \frac{1.2}{W}\,, \qquad (26)$$

giving, for example,

$$W = 10\text{ GeV}: \ \langle\sin^2\eta\rangle_{\text{tot}} = 0.16 + 0.12\,, \qquad (27a)$$

$$W = 30\text{ GeV}: \ \langle\sin^2\eta\rangle_{\text{tot}} = 0.12 + 0.04\,. \qquad (27b)$$

The discussion thus far has dealt with the idealized case of perfectly polarized colliding beams. In practice, the electron-positron clashing beams will be partially polarized along the direction of the magnetic field $\hat{b}$, a direction which is perpendicular to the beam axis $\hat{l}$. Denoting the degree of polarization by P, the annihilation of an electron-positron pair produces a virtual photon with a photon-spin density matrix

$$L_{kl} = (1 - P^2)(\delta_{kl} - \hat{l}_k\hat{l}_l) + 2P^2\hat{b}_k\hat{b}_l\,. \qquad (28)$$

Thus, in addition to the angle ψ between the direction of the detected energy and the direction of the magnetic field

$$\cos\psi = \hat{h}\cdot\hat{b}\,, \qquad (29)$$

we must now introduce the angle θ between the detected energy direction and the beam axis

$$\cos\theta = \hat{h}\cdot\hat{l}\,. \qquad (30)$$

The previous result (24) holds for the $P = 1$ limit of the tensor (28). The general result is obtained by obvious substitutions $\hat{b}_k \to \hat{l}_k$, $\hat{b}_k\hat{b}_l \to \delta_{kl}$, and we find that

$$\frac{d\Sigma}{d\Omega} = \frac{\alpha^2}{4W^2}\sum_f 3Q_f^{\,2}\left(1+\frac{\bar{g}_W^{\,2}}{4\pi^2}\right)\{(1-P^2)[(1+\cos^2\theta)+\tfrac{1}{2}\langle\sin^2\eta\rangle_{\rm tot}(1-3\cos^2\theta)] + 2P^2[\sin^2\psi+\tfrac{1}{2}\langle\sin^2\eta\rangle_{\rm tot}(3\cos^2\psi-1)]\}. \tag{31}$$

In summary, the method of asymptotically free perturbation theory offers the promise of precise, unambiguous tests of an underlying, fundamental field theory of hadronic physics, the theory of quantum chromodynamics. We have worked out the details of such a test, the energy dependence of the hadronic energy pattern produced in electron-positron annihilation.

Note added in proof. It has been pointed out to us by T. Burnett that an experimental evaluation of the jet opening angle ($\langle\sin^2\eta\rangle_{\rm total}$) can be obtained from the present data on the inclusive production of charged particles. This evaluation requires *only* the assumption that the angular distribution of the energy carried by neutrals is identical to that carried by the charged particles. In terms of the parameter $\alpha(x)$ defined by the angular distribution of the charged-particle inclusive cross section,

$$\frac{d\sigma}{dx d\Omega}\sim[1+\alpha(x)\cos^2\theta],$$

and the $x=2h/W$ dependence of the cross section, $d\sigma/dx$, we get, on comparing with Eq. (31) (for $P=0$),

$$\tfrac{1}{2}\langle\sin^2\eta\rangle_{\rm total}=\int_0^1 x\,dx\frac{d\sigma}{dx}\frac{1-\alpha(x)}{3+\alpha(x)}\Big/\int_0^1 x\,dx\frac{d\sigma}{dx}.$$

Here we have neglected the mass of the produced hadrons which is permissible since they are mostly pions. We have evaluated this formula at $W=7.4$ GeV using the values of $\alpha(x)$ given in Ref. 6 and the recent analysis of $d\sigma/dx$ performed by G. Hanson (private communication). We find

$$\langle\sin^2\eta\rangle_{\rm total}=0.34\pm0.06\ (W=7.4\ \text{GeV}).$$

This result is in remarkable agreement with the sum of our perturbative result and the phenomenological confinement estimate; Eq. (26) evaluated at $W=7.4$ GeV gives $\langle\sin^2\eta\rangle_{\rm total}=0.34$. One should be troubled by this close agreement since a jet model yields a good description of the data.[6] Why then does our phenomenological jet term not give the full contribution to $\langle\sin^2\eta\rangle$? Part of the explanation lies in the fact that we have determined the coefficient C of the logarithmic rise in the multiplicity from the figures for the total multiplicity presented in the literature.[6] However, these data apparently contain a contamination in the two-prong channel at higher energies that comes from heavy-lepton production. Thus the average multiplicity determined in this way is smaller than that found from the single-particle inclusive data where this contamination has been removed. It should be emphasized, however, that the estimate of confinement effects made in the text was intended only as a semiquantitative measure of these effects. Furthermore, as noted in the text, in order for the theory to rigorously apply, the energy must be well away from any threshold. The simple calculations in this note show that present measurements of angular distributions are already of sufficient accuracy to be sensitive to the perturbative contribution at 7.4 GeV. Hence measurement of the angular distribution with comparable accuracy at higher energies (beyond the 9.5-GeV resonances) will provide a good test of the theory of quantum chromodynamics.

We wish to thank G. Hanson for several useful conversations.

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APPENDIX A

In this appendix, we shall sketch the method used to calculate the second-order effects discussed in the text. We begin by displaying a general formula for the energy cross section. We consider electron-positron annihilation producing N particles in the final state as indicated by the energy-momentum balance

$$p_1+p_2+\cdots+p_N=l+\bar{l}. \tag{A1}$$

According to the definition of the energy cross section, we insert a factor E_a/W for each particle, $a=1,2,\ldots N$, into the usual expression for the differential cross section and integrate over all variables except for the solid angle of the detected energy. Thus

$$\frac{d\Sigma}{d\Omega}=\sum_{a=1}^{N}\int\frac{p_a^{\,2}dp_a}{(2\pi)^3 2W}\prod_{\substack{b=1\\ b\neq a}}^{N}\int\frac{(d^3p_b)}{(2\pi)^3 2E_b}(2\pi)^4 \times\delta(\Sigma_c p_c-l-\bar{l})\,|T|^2\frac{1}{2W^2}. \tag{A2}$$

In general, one must in addition sum over the total number of final particles N.

We need to apply this general formula to the second-order gluon-emission process depicted in Figs. 2(b) and 2(c). We shall only briefly discuss our calculation since similar computations already exist in the published literature.[13] However, some of the expressions that we shall write down are in a particularly simple form that does not exist in the literature. The squared matrix element for the gluon-emission process has the form

$$|T|^2 = \frac{e^4}{2W^2} L_{\mu\nu} H^{\mu\nu}. \tag{A3}$$

Here $L_{\mu\nu}$ is the virtual-photon spin-density tensor which is produced by the spin sum of the square of the leptonic current $(\sqrt{2}/W)\bar{u}(l)\gamma_\mu v(\bar{l})$. It has no time components and its spatial components are identical with those of the tensor L_{kl} given in Eq. (28) in the text. We denote the momentum of the virtual photon by q,

$$q = l + \bar{l}. \tag{A4}$$

The conservation of the leptonic current is made explicit from the fact that $L_{\mu\nu}$ obeys

$$q^\mu L_{\mu\nu} = 0 = L_{\mu\nu} q^\nu. \tag{A5}$$

We denote the momenta of the produced quark and antiquark by p and $\bar{p}$, respectively, and the momentum of the emitted gluon by k. We recall that we are in a limit where the quarks are massless, but the gluon is massive. The normalization of the coupling constant is specified by the interaction Lagrangian

$$\mathcal{L}_I = g\bar{q}\gamma_\mu \lambda_a q A_a^\mu, \tag{A6}$$

where the color SU(3) generating matrices λ_a have an isospin normalization so that, for example,

$$\lambda_3 = \begin{bmatrix} +\frac{1}{2} & 0 & 0 \\ 0 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{A7}$$

Using these conventions, the hadron tensor $H^{\mu\nu}$ is given by

$$H^{\mu\nu} = \sum_f \mathrm{tr}\left[g\lambda_a \gamma^\sigma \frac{\gamma\cdot(p+k)}{(p+k)^2} Q_f \gamma^\mu - Q_f \gamma^\mu \frac{\gamma\cdot(\bar{p}+k)}{(\bar{p}+k)^2} g\lambda_a \gamma^\sigma \right] \times \gamma\bar{p} \left[Q_f \gamma^\nu \frac{\gamma\cdot(p+k)}{(p+k)^2} g\lambda_a \gamma_\sigma - g\lambda_a \lambda_\sigma \frac{\gamma\cdot(\bar{p}+k)}{(\bar{p}+k)^2} Q_f \gamma^\nu \right] \gamma p, \tag{A8}$$

where we use a metric $g_{\mu\nu}$ with signature $(-+++)$ so that $k^2 = -\lambda^2$, and the Dirac matrices obey $\{\gamma_\mu, \gamma_\nu\} = -2g_{\mu\nu}$.

It is worthwhile displaying the result of calculating the traces in Eq. (A8) in a manner that makes the current conservation

$$q_\mu H^{\mu\nu} = 0 = H^{\mu\nu} q_\nu \tag{A9}$$

manifest. This gives us some assurance that the answer is correct. We find that $H^{\mu\nu}$ can be put into the form

$$\begin{aligned} H^{\mu\nu} = 32 g^2 \Sigma_f Q_f^2 \Bigg\{ & \frac{1}{(\bar{p}+k)^2} \frac{1}{(p+k)^2} \{(q^2 g^{\mu\nu} - q^\mu q^\nu) q^2 + [\Delta^\mu \Delta^\nu q^2 - (q^\mu \Delta^\nu + \Delta^\mu q^\nu) q\cdot\Delta + g^{\mu\nu}(q\cdot\Delta)^2] \\ & + [k^\mu k^\nu q^2 - (q^\mu k^\nu + k^\mu q^\nu) q\cdot k + g^{\mu\nu}(q\cdot k)^2]\} \\ & - (g^{\mu\nu} q^2 - q^\mu q^\nu)\left[\frac{1}{(p+k)^2} + \frac{1}{(\bar{p}+k)^2}\right] \\ & + \frac{2\lambda^2}{[(p+k)^2(\bar{p}+k)^2]^2} [(k^\mu q\cdot\Delta - \Delta^\mu q\cdot k)(k^\nu q\cdot\Delta - \Delta^\nu q\cdot k) + (g^{\mu\nu} q^2 - q^\mu q^\nu)(q\cdot\Delta)^2] \Bigg\}, \end{aligned} \tag{A10}$$

where

$$\Delta = \bar{p} - p. \tag{A11}$$

Factors which are proportional to q^μ or q^ν can be deleted. They do not contribute to the squared matrix element (A3) since the lepton current is conserved [Eq. (A5)]. Thus we can, for example, make the replacement

$$\Delta^\mu \Delta^\nu + k^\mu k^\nu \to 2(p^\mu p^\nu + \bar{p}^\mu \bar{p}^\nu). \tag{A12}$$

We also specialize to the laboratory frame and use relations such as

$$(p+k)^2 = (q-\bar{p})^2 = -W(W - 2\bar{E}). \tag{A13}$$

Finally, we neglect terms which give a vanishing contribution to the cross section in the massless limit $\lambda \to 0$. We get

$$H^{\mu\nu}=64\,g^2\sum_f Q_f{}^2\,\frac{1}{W-2E}\,\frac{1}{W-2\bar{E}}$$
$$\times[(E^2+\bar{E}^2)g^{\mu\nu}-p^\mu p^\nu-\bar{p}^\mu\bar{p}^\nu]$$
$$\times\left[1-\frac{1}{2}\frac{\lambda^2}{W^2}\left(\frac{W-2\bar{E}}{W-2E}+\frac{W-2E}{W-2\bar{E}}\right)\right]. \quad \text{(A14)}$$

Some terms involving k^σ in factors multiplying λ^2 have been omitted here since they do not contribute in the $\lambda\to 0$ limit. Moreover, we have used the fact that $E^2g^{\mu\nu}-p^\mu p^\nu$ and $\bar{E}^2g^{\mu\nu}-\bar{p}^\mu\bar{p}^\nu$ are equivalent in the λ^2 term when $\lambda\to 0$. However, the remaining λ^2 terms cannot be discarded for they give finite contributions to the cross sections in the $\lambda\to 0$ limit.

Since the general case of arbitrary polarization is easily obtained from that with pure polarization, we shall compute the squared matrix element for perfectly polarized beams. Putting $P=1$ in Eq. (28) and also inserting Eq. (A14) into Eq. (A3) gives

$$|T|^2=64\,e^4g^2\,\frac{1}{W^2}\sum_f Q_f{}^2\,\frac{1}{W-2E}\,\frac{1}{W-2\bar{E}}$$
$$\times(E^2\sin^2\chi+\bar{E}^2\sin^2\bar{\chi})$$
$$\times\left[1-\frac{1}{2}\frac{\lambda^2}{W^2}\left(\frac{W-2\bar{E}}{W-2E}+\frac{W-2E}{W-2\bar{E}}\right)\right]. \quad \text{(A15)}$$

Here χ and $\bar{\chi}$ are the angles which the momenta of the quark and antiquark form with the polarization direction, $\cos\chi=\hat{p}\cdot\hat{b}$, $\cos\bar{\chi}=\hat{\bar{p}}\cdot\hat{b}$. To evaluate the integrals over the three-body phase space, we use the identity

$$\int\frac{(d^3p)}{E}\,\frac{(d^3\bar{p})}{\bar{E}}\,\frac{(d^3k)}{\omega}\,\delta(p+\bar{p}+k-q)=\int dE\,d\omega\,d\Omega\,d\cos\eta\,d\phi\,\delta\left[\cos\eta-\frac{W^2-2W(E+\omega)+2E\omega+\lambda^2}{2E(\omega^2-\lambda^2)^{1/2}}\right] \quad \text{(A16)}$$

and its permutations. We denote the gluon energy by ω and the element of its solid angle by $d\Omega$. The angles η and ϕ describe the orientation of $\vec{p}$ relative to $\vec{k}$, with $\cos\eta=\hat{p}\cdot\hat{k}$. The limits of integration on the two energy integrals follow by requiring that $-1\le\cos\eta\le+1$. The evaluation of the integrals in this manner yields the results quoted in the text.

APPENDIX B

The fragmentation of a quark into hadrons may be described phenomenologically by the function $f(z,h_\perp)$, where $z=2h_\parallel/W$. The number dn of hadrons produced in a momentum interval (d^3h) is given by

$$dn=\frac{(d^3h)}{h^0}f(z,h_\perp), \quad \text{(B1)}$$

with the variable z limited to the range $0\le z<1$. In the e^+e^- annihilation, a quark and antiquark are produced, giving a total multiplicity of

$$\langle n\rangle=2\int_0^1 dz\int\frac{(d^2h_\perp)f(z,h_\perp)}{[z^2+(4/W^2)(h_\perp{}^2+m^2)]^{1/2}}, \quad \text{(B2)}$$

where m is the mass of the hadron with momentum h. The $W\to\infty$ limit of this expression gives

$$\langle n\rangle=C\ln W+\text{const}, \quad \text{(B3)}$$

where

$$C=2\int(d^2h_\perp)f(0,h_\perp). \quad \text{(B4)}$$

The amount of energy carried off by the hadrons in the momentum interval (d^3h) is given by h^0dn. Since this energy must add up to $\frac{1}{2}W$, we have the energy sum rule

$$\tfrac{1}{2}W=\int(d^3h)f(z,h_\perp). \quad \text{(B5)}$$

With these results in hand, we can now proceed to the evaluation of the effect of quark fragmentation on the energy cross section.

We assume that the final hadrons are produced from a $q\bar{q}$ pair which has an angular distribution proportional to $\sin^2\chi$, where χ is the angle between the quark momentum $\vec{p}$ and the beam polarization direction $\hat{b}$. With the hadron carrying off a fraction h^0/W of the total energy, the partial energy cross section is given by

$$\Delta\Sigma=\frac{\alpha^2}{W^2}\sum_f 3Q_f{}^2\int_\Delta\frac{(d^3h)}{h^0}\left(\frac{h^0}{W}\right)\int d\Omega_p\sin^2\chi f(z,h_\perp), \quad \text{(B6)}$$

where Δ indicates the phase-space volume of the final hadron and $d\Omega_p$ denotes the solid angle interval of the intermediate quark of momentum $\vec{p}$. The energy sum rule (B5) ensures that this cross section is correctly normalized, giving a total cross section

$$\Sigma=\frac{\alpha^2}{2W^2}\sum_f 3Q_f{}^2\int d\Omega_p\sin^2\chi \quad \text{(B7)}$$

that agrees with Eq. (12). We use the law of cosines to write $\sin^2\chi$ in terms of the jet opening angle η, which is the angle between $\vec{h}$ and $\vec{p}$, and the angle ψ between $\vec{h}$ and $\hat{b}$ (the angles are displayed in Fig. 3). Then, on writing $(d^3h) = d\Omega h^2 dh$ and averaging over an azimuthal angle, we get

$$\frac{d\Sigma}{d\Omega} = \frac{\alpha^2}{W^3} \sum_f 3Q_f^2 \times \int d\Omega_p \int h^2 dh [\sin^2\psi + \tfrac{1}{2}\sin^2\eta(3\cos^2\psi - 1)] \times f(z, h_\perp). \quad \text{(B8)}$$

The integrand here now involves only the direction of the quark momentum $\vec{p}$ relative to the hadronic momentum $\vec{h}$—it involves only $\vec{p}\cdot\vec{h}$. Hence we may consider $d\Omega_p$ to be an element of solid angle of the hadronic momentum $\vec{h}$ rather than that of the quark momentum $\vec{p}$. We may write

$$\frac{d\Sigma}{d\Omega} = \frac{\alpha^2}{W^3} \sum_f 3Q_f^2 \int (d^3h)[\sin^2\psi + \tfrac{1}{2}\sin^2\eta(3\cos^2\psi - 1)] \times f(z, h_\perp), \quad \text{(B9)}$$

or, using the energy sum rule (B5),

$$\frac{d\Sigma}{d\Omega} = \frac{\alpha^2}{2W^2} \sum_f 3Q_f^2 [\sin^2\psi + \tfrac{1}{2}\langle\sin^2\eta\rangle(3\cos^2\psi - 1)]. \quad \text{(B10)}$$

Since

$$\sin^2\eta = \frac{h_\perp^2}{(1/4)z^2W^2 + h_\perp^2}, \quad \text{(B11)}$$

we have

$$\langle\sin^2\eta\rangle = \int_0^1 dz \int (d^2h_\perp) \frac{h_\perp^2}{(1/4)z^2W^2 + h_\perp^2} f(z, h_\perp). \quad \text{(B12)}$$

In the high-energy limit, the z integration produces terms of order $1/W^2$ except for the neighborhood of $z = 0$. Hence we can set $z = 0$ in $f(z, h_\perp)$ and evaluate the resulting elementary integral over z to obtain

$$\langle\sin^2\eta\rangle = \pi \int (d^2h_\perp) \frac{h_\perp}{W} f(0, h_\perp). \quad \text{(B13)}$$

In view of the normalization given in Eq. (B4), we may write this as

$$\langle\sin^2\eta\rangle = \frac{\pi C\langle h_\perp\rangle}{2W}, \quad \text{(B14)}$$

where $\langle h_\perp\rangle$ is the average hadronic transverse momentum in the quark fragmentation. This is the result quoted in Eq. (23) of the text.

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