Final Project

Radiation of Gluon Jets

Although we have discussed QED radiative corrections at length in the last two chapters, so far we have made no attempt to compute a full radiatively corrected cross section. The reason is of course that such calculations are quite lengthy. Nevertheless it would be dishonest to pretend that one understands radiative corrections after computing only isolated effects as we have done. This "final project" is an attempt to remedy this situation. The project is the computation of one of the simplest, but most important, radiatively corrected cross sections. You should finish Chapter 6 before starting this project, but you need not have read Chapter 7.

Strongly interacting particles—pions, kaons, and protons—are produced in e^+e^- annihilation when the virtual photon creates a pair of quarks. If one ignores the effects of the strong interactions, it is easy to calculate the total cross section for quark pair production. In this final project, we will analyze the first corrections to this formula due to the strong interactions.

Let us represent the strong interactions by the following simple model: Introduce a new massless vector particle, the *gluon*, which couples universally to quarks:

$$\Delta H = \int d^3x\, g\bar{\psi}_{fi}\gamma^\mu\psi_{fi}B_\mu.$$

Here f labels the type ("flavor") of the quark (u, d, s, c, etc.) and $i = 1, 2, 3$ labels the color. The strong coupling constant g is independent of flavor and color. The electromagnetic coupling of quarks depends on the flavor, since the u and c quarks have charge $Q_f = +2/3$ while the d and s quarks have charge $Q_f = -1/3$. By analogy to α, let us define

$$\alpha_g = \frac{g^2}{4\pi}.$$

In this exercise, we will compute the radiative corrections to quark pair production proportional to α_g.

This model of the strong interactions of quarks does not quite agree with the currently accepted theory of the strong interactions, quantum chromodynamics (QCD). However, all of the results that we will derive here are also correct in QCD with the replacement

$$\alpha_g \longrightarrow \frac{4}{3}\alpha_s.$$

We will verify this claim in Chapter 17.

Throughout this exercise, you may ignore the masses of quarks. You may also ignore the mass of the electron, and average over electron and positron polarizations. To control infrared divergences, it will be necessary to assume that the gluons have a small nonzero mass μ, which can be taken to zero only at the end of the calculation. However (as we discussed in Problem 5.5), it is consistent to sum over polarization states of this massive boson by the replacement:

$$\sum \epsilon^{\mu}\epsilon^{\nu *} \rightarrow -g^{\mu\nu};$$

this also implies that we may use the propagator

$$\overline{B^{\mu} B^{\nu}} = \frac{-ig^{\mu\nu}}{k^2 - \mu^2 + i\epsilon}.$$

(a) Recall from Section 5.1 that, to lowest order in α and neglecting the effects of gluons, the total cross section for production of a pair of quarks of flavor f is

$$\sigma(e^+e^- \rightarrow \bar{q}q) = \frac{4\pi\alpha^2}{3s} \cdot 3Q_f^2.$$

Compute the diagram contributing to $e^+e^- \rightarrow \bar{q}q$ involving one virtual gluon. Reduce this expression to an integral over Feynman parameters, and renormalize it by subtraction at $q^2 = 0$, following the prescription used in Eq. (6.55). Notice that the resulting expression can be considered as a correction to $F_1(q^2)$ for the quark. Argue that, for massless quarks, to all orders in α_g, the total cross section for production of a quark pair unaccompanied by gluons is

$$\sigma(e^+e^- \rightarrow \bar{q}q) = \frac{4\pi\alpha^2}{3s} \cdot 3\left|F_1(q^2 = s)\right|^2,$$

with $F_1(q^2 = 0) = Q_f$.

(b) Before we attempt to evaluate the Feynman parameter integrals in part (a), let us put this contribution aside and study the process $e^+e^- \rightarrow \bar{q}qg$, quark pair production with an additional gluon emitted. Before we compute the cross section, it will be useful to work out some kinematics. Let q be the total 4-momentum of the reaction, let k_1 and k_2 be the 4-momenta of the final quark and antiquark, and let k_3 be the 4-momentum of the gluon. Define

$$x_i = \frac{2k_i \cdot q}{q^2}, \qquad i = 1, 2, 3;$$

this is the ratio of the center-of-mass energy of particle i to the maximum available energy. Then show (i) $\sum x_i = 2$, (ii) all other Lorentz scalars involving only the final-state momenta can be computed in terms of the x_i and the particle masses, and (iii) the complete integral over 3-body phase space can be written as

$$\int d\Pi_3 = \prod_i \int \frac{d^3k_i}{(2\pi)^3}\frac{1}{2E_i}\,(2\pi)^4\delta^{(4)}(q - \sum_i k_i) = \frac{q^2}{128\pi^3}\int dx_1\,dx_2.$$

Find the region of integration for x_1 and x_2 if the quark and antiquark are massless but the gluon has mass μ.

(c) Draw the Feynman diagrams for the process $e^+e^- \to \bar{q}qg$, to leading order in α and α_g, and compute the differential cross section. You may throw away the information concerning the correlation between the initial beam axis and the directions of the final particles. This is conveniently done as follows: The usual trace tricks for evaluating the square of the matrix element give for this process a result of the structure

$$\int d\Pi_3\, \frac{1}{4}\sum |\mathcal{M}|^2 = L_{\mu\nu}\int d\Pi_3\; H^{\mu\nu},$$

where $L_{\mu\nu}$ represents the electron trace and $H^{\mu\nu}$ represents the quark trace. If we integrate over all parameters of the final state except x_1 and x_2, which are scalars, the only preferred 4-vector characterizing the final state is q^μ. On the other hand, $H_{\mu\nu}$ satisfies

$$q^\mu H_{\mu\nu} = H_{\mu\nu}q^\nu = 0.$$

Why is this true? (There is an argument based on general principles; however, you might find it a useful check on your calculation to verify this property explicitly.) Since, after integrating over final-state vectors, $\int H^{\mu\nu}$ depends only on q^μ and scalars, it can only have the form

$$\int d\Pi_3\, H^{\mu\nu} = \left(g^{\mu\nu} - \frac{q^\mu q^\nu}{q^2}\right)\cdot H,$$

where H is a scalar. With this information, show that

$$L_{\mu\nu}\int d\Pi_3\, H^{\mu\nu} = \frac{1}{3}\left(g^{\mu\nu}L_{\mu\nu}\right)\cdot\int d\Pi_3\,\left(g^{\rho\sigma}H_{\rho\sigma}\right).$$

Using this trick, derive the differential cross section

$$\frac{d\sigma}{dx_1 dx_2}(e^+e^- \to \bar{q}qg) = \frac{4\pi\alpha^2}{3s}\cdot 3Q_f^2\cdot\frac{\alpha_g}{2\pi}\frac{x_1^2 + x_2^2}{(1-x_1)(1-x_2)}$$

in the limit $\mu \to 0$. If we assume that each original final-state particle is realized physically as a jet of strongly interacting particles, this formula gives the probability for observing three-jet events in e^+e^- annihilation and the kinematic distribution of these events. The form of the distribution in the x_i is an absolute prediction, and it agrees with experiment. The normalization of this distribution is a measure of the strong-interaction coupling constant.

(d) Now replace $\mu \neq 0$ in the formula of part (c) for the differential cross section, and carefully integrate over the region found in part (b). You may assume $\mu^2 \ll q^2$. In this limit, you will find infrared-divergent terms of order $\log(q^2/\mu^2)$ and also $\log^2(q^2/\mu^2)$, finite terms of order 1, and terms explicitly suppressed by powers of (μ^2/q^2). You may drop terms of the last type throughout this calculation. For the moment, collect and evaluate only the infrared-divergent terms.

(e) Now analyze the Feynman parameter integral obtained in part (a), again working in the limit $\mu^2 \ll q^2$. Note that this integral has singularities in the region of integration. These should be controlled by evaluating the integral for q spacelike and then analytically continuing into the physical region. That is, write $Q^2 = -q^2$, evaluate the integral for $Q^2 > 0$, and then carefully analytically continue the result to $Q^2 = -q^2 - i\epsilon$. Combine the result with the answer from part (d) to form the total cross section for $e^+e^- \to$ strongly interacting particles, to order α_g. Show that all infrared-divergent logarithms cancel out of this quantity, so that this total cross section is well-defined in the limit $\mu \to 0$.

(f) Finally, collect the terms of order 1 from the integrations of parts (d) and (e) and combine them. To evaluate certain of these terms, you may find the following formula useful:

$$\int_0^1 dx \, \frac{\log(1-x)}{x} = -\frac{\pi^2}{6}.$$

(It is not hard to prove this.) Show that the total cross section is given, to this order in α_g, by

$$\sigma(e^+e^- \to \bar{q}q \text{ or } \bar{q}qg) = \frac{4\pi\alpha^2}{3s} \cdot 3Q_f^2 \cdot \left(1 + \frac{3\alpha_g}{4\pi}\right).$$

This formula gives a second way of measuring the strong-interaction coupling constant. The experimental results agree (within the current experimental errors) with the results obtained by the method of part (c). We will discuss the measurement of α_s more fully in Section 17.6.