Generalized $\mu-\tau$ reflection symmetry and leptonic CP violation

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We propose a generalized $\mu-\tau$ reflection symmetry to constrain the lepton flavor mixing parameters. We obtain a new correlation between the atmospheric mixing angle θ_{23} and the "Dirac" CP violation phase δ_{CP}. Only in a specific limit our proposed CP transformation reduces to standard $\mu-\tau$ reflection, for which θ_{23} and δ_{CP} are both maximal. The "Majorana" phases are predicted to lie at their CP-conserving values with important implications for the neutrinoless double beta decay rates. We also study the phenomenological implications of our scheme for present and future neutrino oscillation experiments including T2K, NOνA and DUNE.

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I. INTRODUCTION

The understanding of flavor mixing and CP violation is a long-standing open question in particle physics. In order to shed light upon the structure of fermion mixing various types of flavor symmetry-based approaches have been invoked [1–5]. Non-Abelian flavor symmetries provide a specially attractive framework. These are typically broken spontaneously down to two distinct residual subgroups in the neutrino and charged lepton sectors, the mismatch between the two leading to specific lepton mixing patterns. A complete classification of lepton mixing matrices from finite residual flavor symmetries has been recently given in [6]. The precise measurement of a non-zero reactor angle [7–10] excludes several flavor symmetry groups and encourages future searches for CP violation in neutrino oscillations. It is interesting to notice that a nearly maximal CP-violating phase $\delta_{\mathrm{CP}} \simeq 3\pi/2$ has been reported by the T2K [11], NOνA [12] and Super-Kamiokande experiments [13], although the statistical significance of all these experimental results is below 3σ level. Moreover, such hints of a nonzero δ_{CP} were already present in global analyses of neutrino oscillation data, such as the one in Ref. [14].

Generic lepton mass matrices may admit both remnant CP symmetries as well as remnant flavor symmetries. Moreover remnant flavor symmetries can be generated by remnant CP transformations [15, 16]. As a result it is an interesting idea to constrain the lepton flavor mixing matrix from CP symmetries rather than flavor symmetries. In particular, the maximal Dirac CP-violating phase can be explained by the so-called $\mu-\tau$ reflection symmetry under

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$(\nu_e, \nu_\mu, \nu_\tau) \xrightarrow{\mu-\tau \text{ reflection}} (\nu_e^*, \nu_\tau^*, \nu_\mu^*)$; $S = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$ Interchange operator ; $SMS = M^*$

which a muon (tau) neutrino is transformed into a tau (muon) antineutrino [17–19]. Here we obtain a generalized $\mu - \tau$ reflection symmetry in the context of models based on remnant CP symmetries.

The plan of the paper is as follows. The general form of lepton mixing is reviewed in Sec. II. Based on the residual CP transformation approach we derive in Sec. III a master formula for the lepton mixing matrix. With this we generalize the $\mu - \tau$ reflection, and show explicitly how the CP phase can be constrained by the experimental measurement of the atmospheric mixing angle. In Sec. IV we investigate the phenomenological implications of our scheme for current and upcoming neutrinoless double beta decay as well as neutrino oscillation experiments.

II. GENERAL FORM OF LEPTON MIXING

We start with the fully "symmetrical" presentation of the most general unitary lepton mixing matrix, as originally proposed in Refs. [20, 21], given as:

$$\mathbf{U}_{\text{Sym}} = \begin{pmatrix} c_{12}c_{13} & s_{12}c_{13}e^{-i\phi_{12}} & s_{13}e^{-i\phi_{13}} \\ -s_{12}c_{23}e^{i\phi_{12}} - c_{12}s_{13}s_{23}e^{-i(\phi_{23}-\phi_{13})} & c_{12}c_{23} - s_{12}s_{13}s_{23}e^{-i(\phi_{23}+\phi_{12}-\phi_{13})} & c_{13}s_{23}e^{-i\phi_{23}} \\ s_{12}s_{23}e^{i(\phi_{23}+\phi_{12})} - c_{12}s_{13}c_{23}e^{i\phi_{13}} & -c_{12}s_{23}e^{i\phi_{23}} - s_{12}s_{13}c_{23}e^{-i(\phi_{12}-\phi_{13})} & c_{13}c_{23} \end{pmatrix}, \tag{1}$$

where $c_{ij} = \cos\theta_{ij}$ and $s_{ij} = \sin\theta_{ij}$. In this parametrization the relation between flavor mixing angles and the magnitudes of the entries of the leptonic mixing matrix is

$$\sin^2\theta_{13} = |U_{e3}|^2\,, \quad \sin^2\theta_{12} = \frac{|U_{e2}|^2}{1-|U_{e3}|^2} \quad \text{and} \quad \sin^2\theta_{23} = \frac{|U_{\mu 3}|^2}{1-|U_{e3}|^2}\,. \tag{2}$$

The Particle Data Group presents this parametrization of the mixing matrix in a non symmetrical form [22], in which the two "Majorana" phases appear in the diagonal (there are in principle three ways of doing this). The resulting presentation is motivated by the simple description of neutrino oscillation that results, in which the "Majorana" phases manifestly drop out, as they should [1]. It is very simple to relate both presentations through a similarity transformation involving a diagonal phase matrix (the reader can verify this by using Eq. (2.5) in [20]).

First notice that the above expressions in Eq. (2) also hold when using the PDG form. Therefore, the difference between both parameterizations appears only in the way of writing the CP invariants. We start with the usual Jarlskog invariant describing CP violation in conventional neutrino oscillations. This is defined as

$$J_{\text{CP}} = \mathcal{I}m\left\{U_{e1}^* U_{\mu 3}^* U_{e3} U_{\mu 1}\right\},$$

and takes the following form in the symmetric parametrization

$$J_{\text{CP}} = \frac{1}{8}\sin 2\theta_{12}\,\sin 2\theta_{23}\,\sin 2\theta_{13}\,\cos\theta_{13}\,\sin(\phi_{13}-\phi_{23}-\phi_{12})\,. \tag{3}$$

This invariant is the leptonic analogue of that which characterizes the quark CKM mixing matrix. It is clear that, as expected, in the symmetrical parametrization J_{CP} depends, apart from the three mixing angles, on the rephasing

[1] Of course the Majorana phases also drop out when writing in the symmetric form, but in a less obvious way.

invariant phase combination $\phi_{13} - \phi_{23} - \phi_{12}$. This gives a very transparent interpretation of the "Dirac" leptonic CP invariant. On the other hand, concerning the remaining two invariants

$$I_1 = \mathcal{I}m\left\{U_{e2}^2 U_{e1}^{*2}\right\} \quad \text{and} \quad I_2 = \mathcal{I}m\left\{U_{e3}^2 U_{e1}^{*2}\right\} ,$$

associated with the "Majorana" phases [23–25] they take the form

$$I_1 = \tfrac{1}{4}\sin^2 2\theta_{12}\cos^4\theta_{13}\sin(-2\phi_{12}) \quad \text{and} \quad I_2 = \tfrac{1}{4}\sin^2 2\theta_{13}\cos^2\theta_{12}\sin(-2\phi_{13})\,. \tag{4}$$

These invariants appear in lepton number violating processes such as neutrinoless double beta decay which do not depend, as expected, on the "Dirac" invariant $J_{\rm CP}$. Indeed, one can easily check that this is so. In contrast, however, when written in the PDG form, the amplitude for neutrinoless double beta decay involves all three CP phases. Pulling out an overall phase is, of course, possible but would bring in an ambiguity in the extraction of the phases. For all the reasons explained in this section, we prefer the fully symmetric parametrization to the equivalent PDG form.

III. GENERALIZED $\mu - \tau$ REFLECTION

We now turn to the method of residual CP symmetry transformations proposed in Ref. [15]. This will allow us to obtain CP-violating extensions systematically. Moreover it will, in principle, allow us to make CP predictions, starting from the general CP-conserving form of the lepton mixing matrix. Without loss of generality, we adopt the charged lepton diagonal basis, i.e. $\mathbf{m}_l \equiv \text{diag}\,(m_e, m_\mu, m_\tau)$. Then the neutrino mass matrix $\mathbf{m}_\nu$ can be expressed via the mixing matrix $\mathbf{U}$ as $\mathbf{m}_\nu = \mathbf{U}^*\text{diag}\,(m_1, m_2, m_3)\,\mathbf{U}^\dagger$ under the assumption of Majorana neutrinos. The invariance of the neutrino mass matrix under the action of a CP transformation $\mathbf{X}$ implies [15]

$$\mathbf{X}^\top \mathbf{m}_\nu \mathbf{X} = \mathbf{m}_\nu^*\,, \tag{5}$$

where $\mathbf{X}$ should be a symmetric unitary matrix to avoid degenerate neutrino masses. As a result we find a master formula for the lepton mixing matrix [15]

$$\mathbf{U} = \mathbf{\Sigma}\,\mathbf{O}_{3\times3}\,\mathbf{Q}_\nu\,, \tag{6}$$

where $\mathbf{\Sigma}$ is the Takagi factorization matrix of $\mathbf{X}$ fulfilling $\mathbf{X} = \mathbf{\Sigma}\mathbf{\Sigma}^T$, $\mathbf{Q}_\nu$ is a diagonal phase matrix whose form is $\mathbf{Q}_\nu = \text{diag}\left(e^{-ik_1\pi/2},\, e^{-ik_2\pi/2},\, e^{-ik_3\pi/2}\right)$ with the natural numbers $k_i = 0, 1, 2, 3$. Actually, the entries of $\mathbf{Q}_\nu$ are ±1 and $\pm i$ which encode the CP-parity or CP-signs of the neutrino states and it renders the light neutrino mass eigenvalues positive [26]. The matrix $\mathbf{O}_{3\times3} = \mathbf{O}_1\mathbf{O}_2\mathbf{O}_3$ is a generic three dimensional real orthogonal matrix, and it can be parameterized as

$$\mathbf{O}_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta_1 & \sin\theta_1 \\ 0 & -\sin\theta_1 & \cos\theta_1 \end{pmatrix}, \mathbf{O}_2 = \begin{pmatrix} \cos\theta_2 & 0 & \sin\theta_2 \\ 0 & 1 & 0 \\ -\sin\theta_2 & 0 & \cos\theta_2 \end{pmatrix} \text{and}\, \mathbf{O}_3 = \begin{pmatrix} \cos\theta_3 & \sin\theta_3 & 0 \\ -\sin\theta_3 & \cos\theta_3 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{7}$$

A possible overall minus sign of $\mathbf{O}_{3\times3}$ is dropped since it is irrelevant. Therefore the lepton mixing matrix is predicted to depend on three free parameters $\theta_{1,2,3}$ besides the parameters characterizing the residual CP transformation $\mathbf{X}$.

II. CP SYMMETRY AND LEPTON FLAVOR MIXING

Here we adopt a model-independent approach in which we assume neutrinos to be Majorana particles. In the flavor basis, the Lagrangian describing the lepton masses reads

$$\mathcal{L}_{mass} = -\bar{l}_R \mathbf{m}_l l_L + \frac{1}{2}\nu_L^\top \mathbf{C}^{-1}\mathbf{m}_\nu \nu_L + h.c.\,. \tag{1}$$

Here $\mathbf{C}$ is the charge-conjugation matrix, l_L and l_R are vectors of the three left and right-handed charged lepton fields in generation space, ν_L refers to the three left-handed neutrino fields. Without loss of generality, we adopt the charged lepton diagonal basis, i.e. $\mathbf{m}_l \equiv \mathrm{diag}\{m_e, m_\mu, m_\tau\}$. Then the neutrino mass matrix $\mathbf{m}_\nu$ can be expressed via the mixing matrix $\mathbf{U}$ as

$$\mathbf{m}_\nu = \mathbf{U}^*\mathrm{diag}\,\{m_1, m_2, m_3\}\,\mathbf{U}^\dagger\,, \tag{2}$$

where m_1, m_2 and m_3 are the light neutrino masses. We assume that the, otherwise generic, neutrino mass term in Eq. (1) is invariant under both the CP transformations

$$\nu_L(x) \longmapsto i\mathbf{X}_j\gamma^0\mathbf{C}\bar{\nu}_L^T(x_P), \qquad \mathbf{X}_j = \mathbf{U}\,\mathbf{d}_j\,\mathbf{U}^\top, \quad j = 1, \cdots 4 \tag{3}$$

and the flavor transformations

$$\nu_L(x) \longmapsto \mathbf{G}_i\nu_L(x), \qquad \mathbf{G}_i = \mathbf{U}\,\mathbf{d}_i\,\mathbf{U}^\dagger, \quad i = 1, \cdots 4\,, \tag{4}$$

where $x_P = (t, -\vec{x})$ and $\mathbf{d}_1 = \mathrm{diag}\,\{1, -1, -1\}$, $\mathbf{d}_2 = \mathrm{diag}\,\{-1, 1, -1\}$, $\mathbf{d}_3 = \mathrm{diag}\,\{-1, -1, 1\}$ and $\mathbf{d}_4 = \mathrm{diag}\,\{1, 1, 1\}$. As shown previously [15], only three of the four remnant CP transformations are independent. The reason is that any one of the residual CP transformations can be generated by the remaining three via

$$\mathbf{X}_i = \mathbf{X}_j\mathbf{X}_m^*\mathbf{X}_n, \qquad i \neq j \neq m \neq n\,. \tag{5}$$

The remnant flavor symmetry and remnant CP transformations are closely related with each other. The remnant flavor symmetry can be generated by performing two CP transformations as follows,

$$\begin{aligned}
\mathbf{G}_1 &= \mathbf{X}_2\mathbf{X}_3^* = \mathbf{X}_3\mathbf{X}_2^* = \mathbf{X}_4\mathbf{X}_1^* = \mathbf{X}_1\mathbf{X}_4^*,\\
\mathbf{G}_2 &= \mathbf{X}_1\mathbf{X}_3^* = \mathbf{X}_3\mathbf{X}_1^* = \mathbf{X}_4\mathbf{X}_2^* = \mathbf{X}_2\mathbf{X}_4^*,\\
\mathbf{G}_3 &= \mathbf{X}_1\mathbf{X}_2^* = \mathbf{X}_2\mathbf{X}_1^* = \mathbf{X}_4\mathbf{X}_3^* = \mathbf{X}_3\mathbf{X}_4^*,\\
\mathbf{G}_4 &= \mathbf{X}_1\mathbf{X}_1^* = \mathbf{X}_2\mathbf{X}_2^* = \mathbf{X}_3\mathbf{X}_3^* = \mathbf{X}_4\mathbf{X}_4^* = \mathbf{1}\,.
\end{aligned} \tag{6}$$

Furthermore, one can see that $\mathbf{X}_i$ and $\mathbf{G}_j$ must fulfill the following relation $\mathbf{X}_i\mathbf{G}_j^*\mathbf{X}_i^* = \mathbf{U}\,\mathbf{d}_j\,\mathbf{U}^\dagger = \mathbf{G}_j$ for $i, j = 1, \cdots 4$. This means that the residual flavor symmetry and residual CP symmetry should generally commute with each other in the neutrino sector. Given the experimentally measured mixing matrix $\mathbf{U}$, the CP transformation matrix $\mathbf{X}_i$ can be easily fixed by Eq. (3). Conversely, $\mathbf{U}$ can be deduced from any well-defined four CP transformations.

In the following, we discuss some specific interesting cases. For definiteness we take the case where a single [1] remnant CP transformation $\mathbf{X}$ is preserved by the neutrino mass matrix. In this case $\mathbf{X}$ should be a symmetric

[1] Cases with one, two, three or four remnant CP transformations preserved by the neutrino mass matrix have been investigated previously [15, 16].

Here we focus on a generalization of the widely discussed $\mu-\tau$ reflection [17–19]. This interesting CP transformation takes the following form:

$$\mathbf{X}=\begin{pmatrix} e^{i\alpha} & 0 & 0 \\ 0 & e^{i\beta}\cos\Theta & ie^{i\frac{(\beta+\gamma)}{2}}\sin\Theta \\ 0 & ie^{i\frac{(\beta+\gamma)}{2}}\sin\Theta & e^{i\gamma}\cos\Theta \end{pmatrix}, \tag{8}$$

$\beta=\tau=\gamma=0$; $\Theta=\pm\frac{\pi}{2}$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

where the parameters α, β, γ, and Θ are real. The corresponding Takagi factorization matrix is given by

$$\mathbf{\Sigma}=\begin{pmatrix} e^{i\frac{\alpha}{2}} & 0 & 0 \\ 0 & e^{i\frac{\beta}{2}}\cos\frac{\Theta}{2} & ie^{i\frac{\beta}{2}}\sin\frac{\Theta}{2} \\ 0 & ie^{i\frac{\gamma}{2}}\sin\frac{\Theta}{2} & e^{i\frac{\gamma}{2}}\cos\frac{\Theta}{2} \end{pmatrix}. \tag{9}$$

As a result the resulting lepton mixing angles are determined as

$$\sin^2\theta_{13}=\sin^2\theta_2,\quad \sin^2\theta_{12}=\sin^2\theta_3,\quad \sin^2\theta_{23}=\tfrac{1}{2}\left(1-\cos\Theta\cos 2\theta_1\right), \tag{10}$$

while the CP violation parameters are predicted as

$$\begin{aligned} &J_{\rm CP}=\tfrac{1}{4}\sin\Theta\sin\theta_2\sin 2\theta_3\cos^2\theta_2\,,\quad \sin\delta_{\rm CP}=\tfrac{\sin\Theta\,{\rm sign}[\sin\theta_2\sin 2\theta_3]}{\sqrt{1-\cos^2\Theta\cos^2 2\theta_1}}\,,\\ &\tan\delta_{\rm CP}=\tan\Theta\csc 2\theta_1\,,\quad \phi_{12}=\tfrac{k_2-k_1}{2}\pi\,,\quad \phi_{13}=\tfrac{k_3-k_1}{2}\pi\,,\quad \delta_{\rm CP}=\tfrac{k_3-k_2}{2}\pi-\phi_{23}\,. \end{aligned} \tag{11}$$

In general, as we saw in the previous section, the lepton mixing matrix is specified by six parameters, three angles and three phases. In our scenario only four free independent parameters appear: θ_1, θ_2, θ_3 and Θ. Notice also that the parameters α, β and γ in Eq. (8) do not appear in the mixing parameters. It follows that the three mixing angles are not correlated with each other. Hence we have no genuine prediction for mixing angles. In contrast, however, an important prediction concerning CP violation is that the "Majorana" phases ϕ_{12} and ϕ_{13} are restricted to lie at their CP-conserving values, and correspond simply to the CP parities of the neutrino states [26, 27]. Moreover, one sees that the atmospheric angle and the Dirac phase $\delta_{\rm CP}$ are given in terms of two parameters θ_1 and Θ, and they are correlated with each other according to [2]

$$\sin^2\delta_{\rm CP}\sin^2 2\theta_{23}=\sin^2\Theta\,. \tag{12}$$

Taking $\Theta=\pm\frac{\pi}{2}$, both θ_{23} and $\delta_{\rm CP}$ are maximal, since the residual CP transformation $\mathbf{X}$ reduces to the standard $\mu-\tau$ reflection. When $\theta_1=\pm\frac{\pi}{4}$, the atmospheric mixing angle θ_{23} is maximal and $\tan\delta_{\rm CP}=\pm\tan\Theta$. On the other hand, we have maximal $\delta_{\rm CP}$ and $\sin^2\theta_{23}=\sin^2\frac{\Theta}{2}$ for $\theta_1=0,\pi$. Present global fits of neutrino oscillation data indicate the θ_{23} deviates from the maximal value [14]. If non-maximal θ_{23} was confirmed by forthcoming more sensitive experiments, the standard $\mu-\tau$ reflection would be disfavored, while our present CP transformation would provide a good alternative, with the value of Θ determined from the measured values of θ_{23} and $\delta_{\rm CP}$. We display the contour regions for $\sin^2\theta_{23}$ and $|\sin\delta_{\rm CP}|$ in the plane θ_1 versus Θ in Fig. 1 and Fig. 2 respectively.

Given the 3σ range of the atmospheric mixing angle $0.393\leq\sin^2\theta_{23}\leq 0.643$, the correlation in Eq. (12) allows us to predict the range of the Dirac CP violating phase $|\sin\delta_{\rm CP}|$ as a function of the parameter Θ which characterizes

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[2] We note that in the A_4 flavor-symmetry-based model in Ref. [28] we also have a correlation between $\delta_{\rm CP}$ and the atmospheric angle.

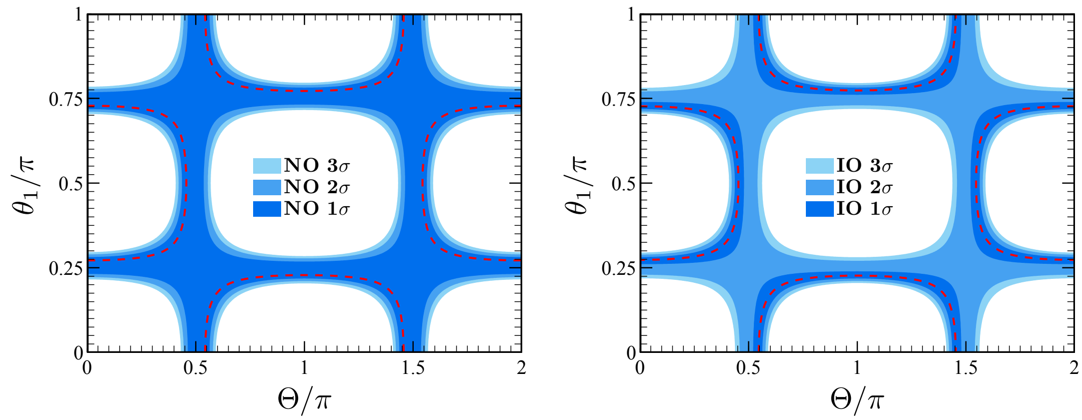

FIG. 1: The contour region of $\sin^2\theta_{23}$ in the plane of θ_1 and Θ for both normal ordering (NO) and inverted ordering (IO) mass spectrum. The different contours correspond to 1σ, 2σ and 3σ. The red solid lines represent the best fitting values.

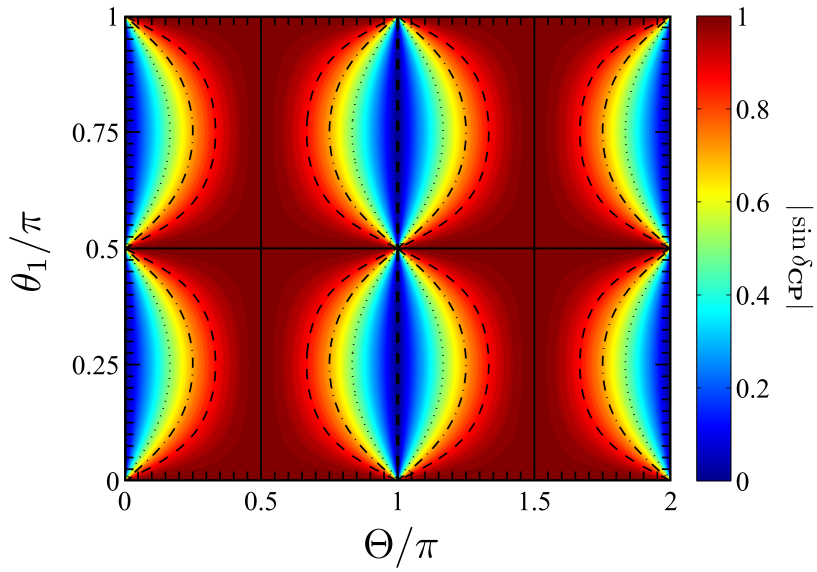

FIG. 2: Contour plot of $|\sin\delta_{CP}|$ defined in Eq. (11). The thick dashed lines, dotted lines, dot-dashed lines, dashed line and thick solid lines refer to $|\sin\delta_{\rm CP}| = 0$, $1/2$, $1/\sqrt{2}$, $\sqrt{3}/2$ and 1 respectively.

the CP transformation $\mathbf{X}$. The result is shown in Fig. 3. It is remarkable that $|\sin\delta_{\rm CP}|$ is predicted to lie in a rather narrow region for a given value of Θ.

On the other hand, as we can see from Eq. (12), the correlation between the atmospheric angle and the CP phase is weighted by the value of the Θ angle. In Fig. 4 we map the allowed ranges of the $\delta_{\rm CP}$ phase versus the atmospheric angle for given values of the Θ parameter determining a given CP scheme. The best fit points (BFP), 1σ and 3σ ranges of θ_{23} reported in [14] are indicated. For the benchmark value of $\Theta = 3\pi/8$, $2\pi/5$ and $5\pi/12$, the range of $|\sin\delta_{\rm CP}|$ allowed by the data of θ_{23} at 3σ level is given in Table I. One sees that the experimentally observed nearly maximal $\delta_{\rm CP}$ can be reproduced.

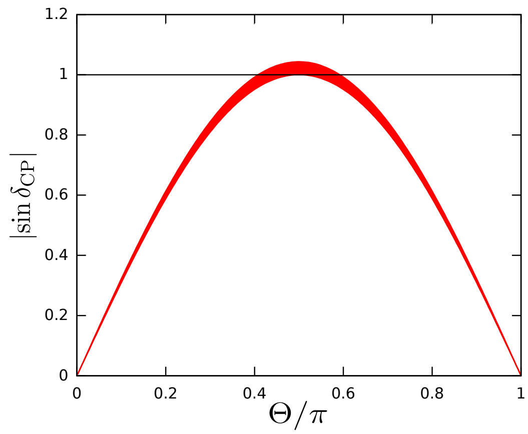

FIG. 3: The regions of $|\sin\delta_{CP}|$ versus Θ, where the atmospheric mixing varies within its experimentally allowed 3σ range [14].

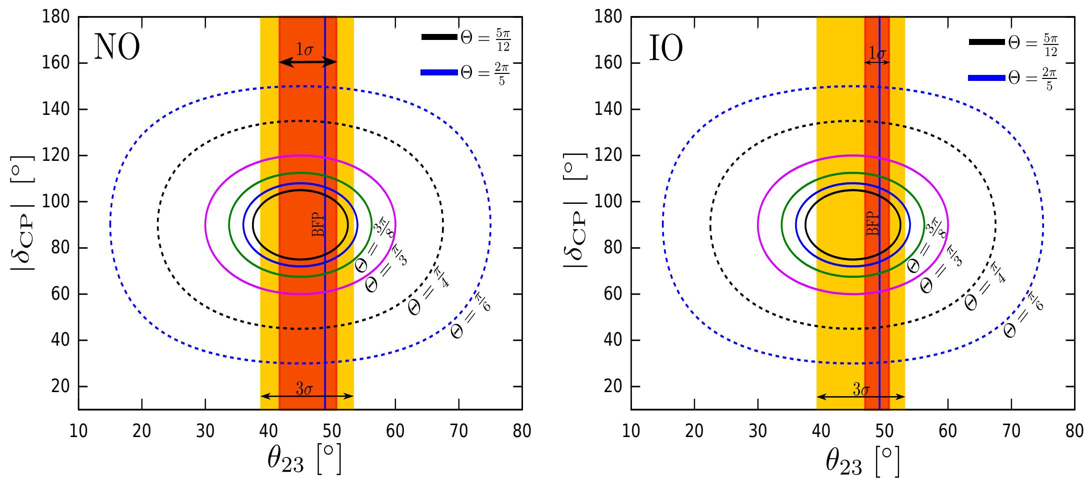

FIG. 4: Predicted range of $|\delta_{\rm CP}|$ phase, for given illustrative values of the Θ parameter characterizing our CP scheme, where Θ is fixed to $\pi/6$, $\pi/4$, $\pi/3$, $3\pi/8$, and $5\pi/12$, $2\pi/5$. The best fits, 1σ and 3σ ranges of the atmospheric mixing angle from [14] are indicated.

IV. PHENOMENOLOGICAL IMPLICATIONS

We have seen that our generalized $\mu-\tau$ reflection symmetry schemes make well-defined predictions for CP violation. In the following, we shall investigate the phenomenological implications of these predictions for lepton number violating processes such as neutrinoless double beta decay ($0\nu\beta\beta$), as well as conventional neutrino oscillations.

Θ	$3\pi/8$	$2\pi/5$	$5\pi/12$
$\lvert\sin\delta_{\rm CP}\rvert$	$[0.92, 0.96]$	$[0.95, 0.99]$	$[0.97, 1]$

TABLE I: Predicted range of $|\sin\delta_{\rm CP}|$ for the benchmark values $\Theta = 3\pi/8$, $2\pi/5$ and $5\pi/12$, allowed by the current 3σ range $38.8^\circ \leq \theta_{23} \leq 53.3^\circ$ given in [14].

A. Neutrinoless double beta decay

The rare decay $(A, Z) \to (A, Z+2) + e^- + e^-$ is the lepton number violating process "par excellence". Its observation would establish the Majorana nature of neutrinos irrespective of their underlying mass generation mechanism [29, 30]. Within the simplest light neutrino exchange mechanism its amplitude is sensitive to the "Majorana phases". Up to nuclear matrix elements [31] and experimental factors [32, 33] the amplitude for the decay is proportional to the effective mass parameter

$$|m_{ee}| = \left| m_1 \cos^2\theta_{12} \cos^2\theta_{13} + m_2 \sin^2\theta_{12} \cos^2\theta_{13} e^{-i2\phi_{12}} + m_3 \sin^2\theta_{13} e^{-i2\phi_{13}} \right| , \tag{13}$$

where we used the symmetric parametrization of the lepton mixing matrix. It is clear that only the two "Majorana phases" appear but not the "Dirac phase" [21].

The crucial prediction of our CP scheme concerns CP violation, in particular, the absence of Majorana CP violation, as seen in Eq. (11). Within our scheme the Majorana phases are predicted as $\phi_{12} = \frac{k_2 - k_1}{2}\pi$ and $\phi_{13} = \frac{k_3 - k_1}{2}\pi$. In other words, these phase factors are predicted to lie at their CP conserving values, which correspond to the CP signs of neutrino states [26, 27]. This implies that the two Majorana phases (ϕ_{12}, ϕ_{13}) can only take the following nine values $(0,0)$, $(0, \pm\pi/2)$, $(\pm\pi/2, 0)$ and $(\pm\pi/2, \pm\pi/2)$.

The effective mass m_{ee} is an even function of the phases ϕ_{12} and ϕ_{13}. Hence, the difference of signs between Majorana phase values is irrelevant, hence the only relevant values for Majorana phases are $(0,0)$, $(0,\pi/2)$, $(\pi/2, 0)$ and $(\pi/2, \pi/2)$. This means that for each possible neutrino mass ordering, there are only four independent regions for the effective mass. Now, inputting the experimentally allowed 3σ ranges of neutrino oscillation parameters [14], the resulting regions of the effective mass $|m_{ee}|$ correlate with the lightest neutrino mass as shown in Fig. 5.

The first comment is that, compared with the generic case, the predictions of our scheme for the neutrino-mass-induced neutrinoless double beta decay amplitude are in some cases rather powerful. Consider, for example, the case of inverted ordering (IO), when the lightest neutrino mass is m_3. In this case the predicted effective mass for $\phi_{13} = 0$ and $\phi_{13} = \pi/2$ almost coincide, as shown in Fig 5. However, the predictions for $\phi_{12} = 0$ and $\phi_{12} = \pi/2$ can be probably be distinguished from each other in the next generation of experiments.

Turning to the case of normal neutrino mass ordering (NO) it is remarkable that one can place a lower bound for the effective mass despite the possibility of destructive interference amongst the three light neutrinos. Indeed no such interference can take place for $(0,0)$ and $(0,\pi/2)$. This situation is analogous to what occurs in a number of flavour symmetry models [34–41].

For completeness we now summarize the above results as tables II and III, for the cases of normal and inverted ordering, respectively. In these tables, the first column gives possible forms of the Q_ν matrix, while the second and third columns show the corresponding (CP conserving) values of the Majorana phases, and the resulting allowed ranges for the effective mass parameter $|m_{ee}|$.

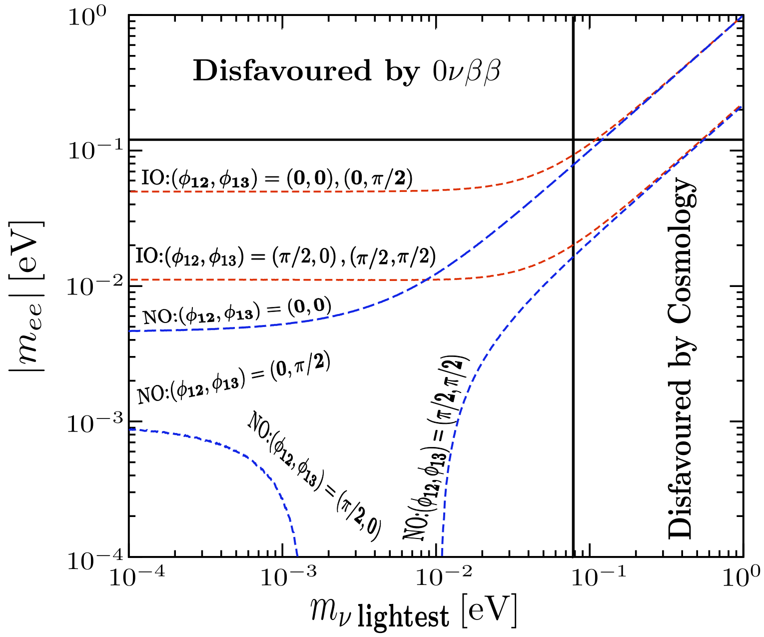

FIG. 5: Effective mass $|m_{ee}|$ describing neutrinoless double beta decay in our scenario where the Majorana phases are predicted at their CP conserving values 0 and $\pm\pi/2$. The red and blue dashed lines indicate the regions currently allowed at 3σ by neutrino oscillation data [14] for inverted and normal neutrino mass ordering, respectively. The allowed values of $|m_{ee}|$ for different values of ϕ_{12} and ϕ_{13} are displayed. For comparison we show the most stringent upper bound $|m_{ee}| < 0.120$eV from EXO-200 [42, 43] in combination with KamLAND-ZEN [44]. The upper limit on the mass of the lightest neutrino is derived from the lastest Planck result $\sum_i m_i < 0.230$eV at 95% level [45].

Normal Ordering		
CP signs Q_ν	$(\phi_{12},\, \phi_{13})$	$\lvert m_{ee}\rvert$ $(10^{-2}$ eV)
diag$(1,1,1)$	$(0,\, 0)$	$[0.32\,, 7.22]$
diag$(1,1,-i)$	$(0,\, \frac{\pi}{2})$	$[9.50\times10^{-2}\,, 6.89]$
diag$(1,-i,1)$	$(\frac{\pi}{2},\, 0)$	$[0\,, 3.31]$
diag$(1,-i,-i)$	$(\frac{\pi}{2},\, \frac{\pi}{2})$	$[0\,, 2.94]$

TABLE II: The allowed ranges for the effective mass in neutrinoless double beta decay for the case of normal ordering. Notice that in our generalized $\mu-\tau$ reflection scenario the Majorana phases can only be 0 and $\pm\pi/2$.

Inverted Ordering		
CP signs Q_ν	$(\phi_{12},\, \phi_{13})$	$\lvert m_{ee}\rvert$ $(10^{-2}$ eV)
diag $(1,1,1)$ diag $(1,1,-i)$	$(0,\, 0)$ $(0,\, \frac{\pi}{2})$	$[4.59\,, 8.20]$
diag $(1,-i,1)$ diag $(1,-i,-i)$	$(\frac{\pi}{2},\, 0)$ $(\frac{\pi}{2},\, \frac{\pi}{2})$	$[1.10\,, 3.45]$

TABLE III: Same as above for the case of inverted ordering.

B. CP violation in conventional neutrino oscillations

The existence of leptonic CP violation would show up as the difference of oscillation probabilities between neutrino and anti-neutrinos in the vacuum [46]:

$$\Delta P_{\alpha\beta} \equiv P\left(\nu_\alpha \to \nu_\beta\right) - P\left(\bar{\nu}_\alpha \to \bar{\nu}_\beta\right) = -16\, J_{\alpha\beta}\, \sin\Delta_{21} \sin\Delta_{23} \sin\Delta_{31},$$

where $\Delta_{kj} = \Delta m^2_{kj} L/(4E)$ with $\Delta m^2_{kj} = m^2_k - m^2_j$, L is the baseline, E is the energy of neutrino, and $J_{\alpha\beta} = \mathcal{I}m\left(U_{\alpha 1} U^*_{\alpha 2} U^*_{\beta 1} U_{\beta 2}\right) = \pm J_{CP}$, whereby it is called Jarlskog-like invariant. The positive (negative) sign for (anti-)cyclic permutation of the flavour indices e, μ and τ. For example for the oscillation between electron and muon neutrinos, the transition probability $\nu_\mu \to \nu_e$ in vacuum has the form $P\left(\nu_\mu \to \nu_e\right) \simeq P_{\rm atm} + 2\sqrt{P_{\rm atm}}\sqrt{P_{\rm sol}} \cos\left(\Delta_{32} + \delta_{\rm CP}\right) + P_{\rm sol}$, where $\sqrt{P_{\rm atm}} = \sin\theta_{23} \sin 2\theta_{13} \sin\Delta_{31}$ and $\sqrt{P_{\rm sol}} = \cos\theta_{23} \cos\theta_{13} \sin 2\theta_{12} \sin\Delta_{21}$ [46]. Hence, the neutrino anti-neutrino asymmetry in the vacuum is

$$A_{\mu e} = \frac{P\left(\nu_\mu \to \nu_e\right) - P\left(\bar{\nu}_\mu \to \bar{\nu}_e\right)}{P\left(\nu_\mu \to \nu_e\right) + P\left(\bar{\nu}_\mu \to \bar{\nu}_e\right)} = \frac{2\sqrt{P_{\rm atm}}\sqrt{P_{\rm sol}} \sin\Delta_{32} \sin\delta_{\rm CP}}{P_{\rm atm} + 2\sqrt{P_{\rm atm}}\sqrt{P_{\rm sol}} \cos\Delta_{32} \cos\delta_{\rm CP} + P_{\rm sol}}\,. \tag{14}$$

In order to describe long baseline neutrino oscillations it is important to include the effect of matter associated to

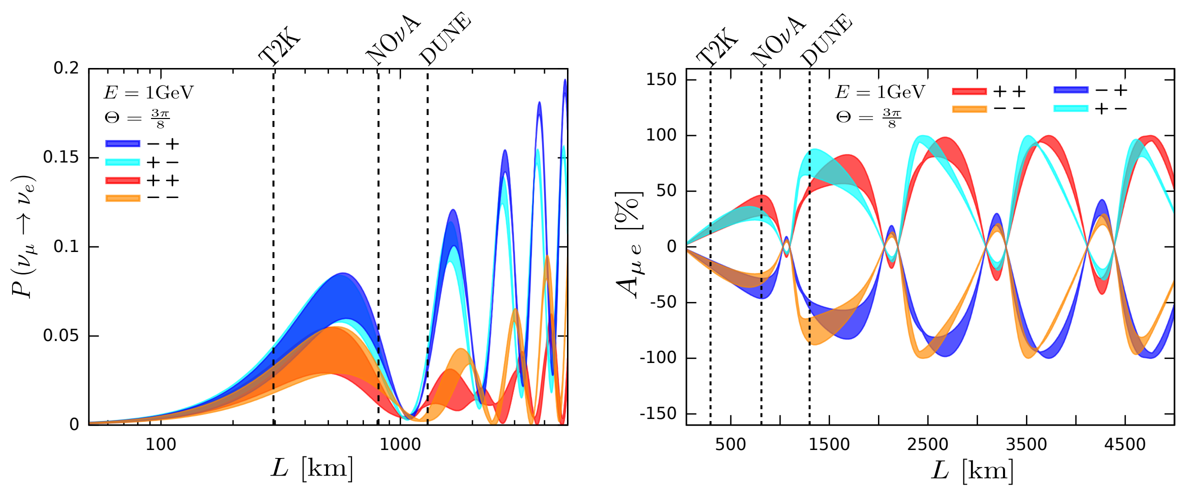

FIG. 6: In the left panel we show the $\nu_\mu \to \nu_e$ transition probability in matter for a neutrino energy of $E = 1$GeV. The right panel shows the neutrino anti-neutrino asymmetry $\mathcal{A}_{\mu e}$ in matter. The mixing angle θ_{23} is taken within the currently allowed 3σ range $0.393 \le \sin^2\theta_{23} \le 0.643$ [14]. The remaining neutrino oscillation parameters are fixed at their best fit values: $\Delta m^2_{21} = 7.60 \times 10^{-5}\text{eV}^2$, $|\Delta m^2_{31}| = 2.48 \times 10^{-3}\text{eV}^2$, $\sin\theta_{12} = 0.323$ and $\sin\theta_{13} = 0.0226$. The Θ parameter is fixed to the value $3\pi/8$. The figure corresponds to the case of normal ordering and the sign combinations refer to Eqs. (16) and (17).

neutrino propagation in the Earth , as it can induce a fake CP violating effect. In this case the expressions for $\sqrt{P_{\rm atm}}$ and $\sqrt{P_{\rm sol}}$ in matter have the form:

$$\sqrt{P_{\rm atm}} = \sin\theta_{23} \sin 2\theta_{13} \tfrac{\sin(\Delta_{31} - aL)}{(\Delta_{31} - aL)}\, \Delta_{31}\,, \quad \sqrt{P_{\rm sol}} = \cos\theta_{23} \sin 2\theta_{12} \tfrac{\sin(aL)}{aL}\, \Delta_{21}\,, \tag{15}$$

where $a = G_F N_e/\sqrt{2}$, G_F is the Fermi constant and N_e is the density of electrons. The approximate value of a is $(3500\text{km})^{-1}$ for $\rho Y_e = 3.0\text{g cm}^{-3}$, where Y_e is the electron fraction [46]. The relative phase $(\Delta_{32} + \delta_{\rm CP})$ between $\sqrt{P_{\rm atm}}$ and $\sqrt{P_{\rm sol}}$ remains unchanged.

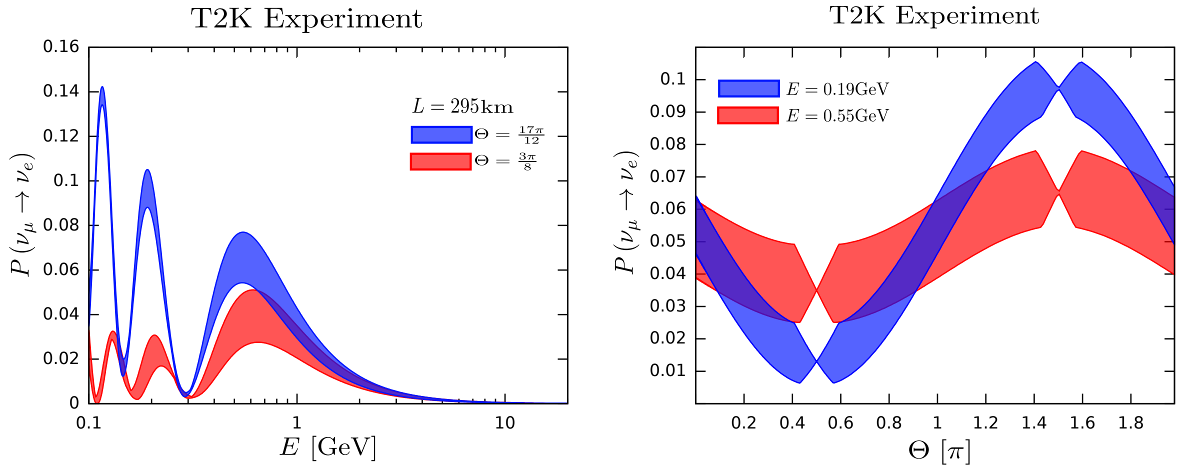

FIG. 7: The transition probability $P(\nu_\mu \to \nu_e)$ at a baseline of 295km which corresponds to the T2K experiment. The mixing angle θ_{23} is taken within its currently allowed 3σ regions $0.393 \leq \sin^2\theta_{23} \leq 0.643$ [14]. Remaining oscillation parameters as in Fig. 6

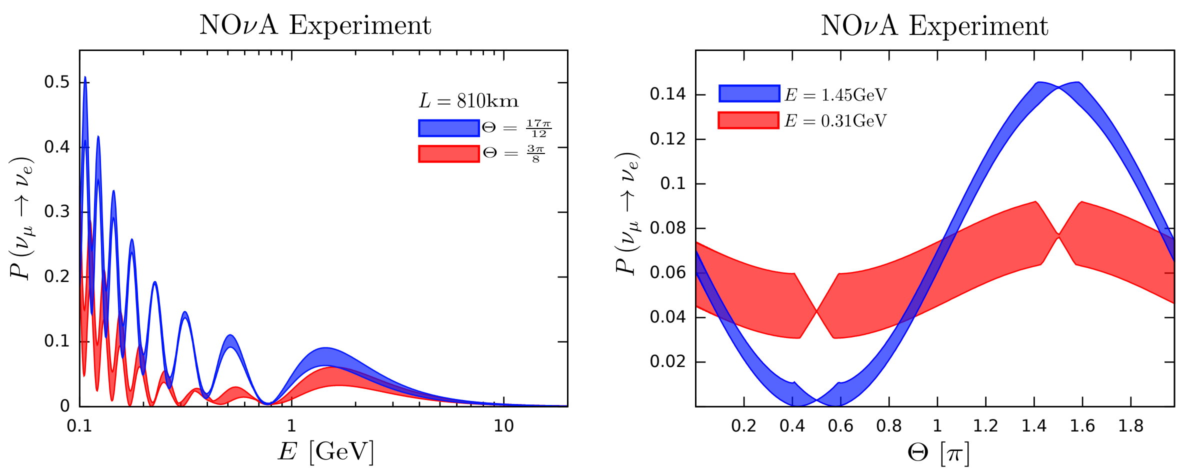

FIG. 8: The transition probability $P(\nu_\mu \to \nu_e)$ at a baseline of 810km which corresponds to the NOνA experiment. The mixing angle θ_{23} is considering into the currently allowed 3σ regions $0.393 \leq \sin^2\theta_{23} \leq 0.643$ [14]. Remaining oscillation parameters as in Fig. 6.

Within the framework of our generalized of $\mu - \tau$ reflection scenario, the transition probability $P(\nu_\mu \to \nu_e)$ in matter has the form

$\mathrm{sen}^2\delta_{cp}\,\mathrm{sen}^2 2\theta_{23} = \mathrm{sen}^2\Theta$

$$P(\nu_\mu \to \nu_e) \simeq P_{\rm atm} + P_{\rm sol} \pm 2\sqrt{P_{\rm atm}}\sqrt{P_{\rm sol}}\cos\left(\Delta_{32} \pm \arcsin\left(\frac{\sin\Theta}{\sin 2\theta_{23}}\right)\right). \tag{16}$$

The neutrino anti-neutrino asymmetry in matter is given by

$$A_{\mu e} = \pm\frac{2\sqrt{P_{\rm atm}}\sqrt{P_{\rm sol}}\sin\Delta_{23}\sin\Theta}{(P_{\rm atm}+P_{\rm sol})\sin 2\theta_{23} \pm 2\sqrt{P_{\rm atm}}\sqrt{P_{\rm sol}}\sqrt{\sin^2 2\theta_{23} - \sin^2\Theta}\,\cos\Delta_{23}}, \tag{17}$$

where $\sqrt{P_{\rm atm}}$ and $\sqrt{P_{\rm sol}}$ are given in Eq. (16).

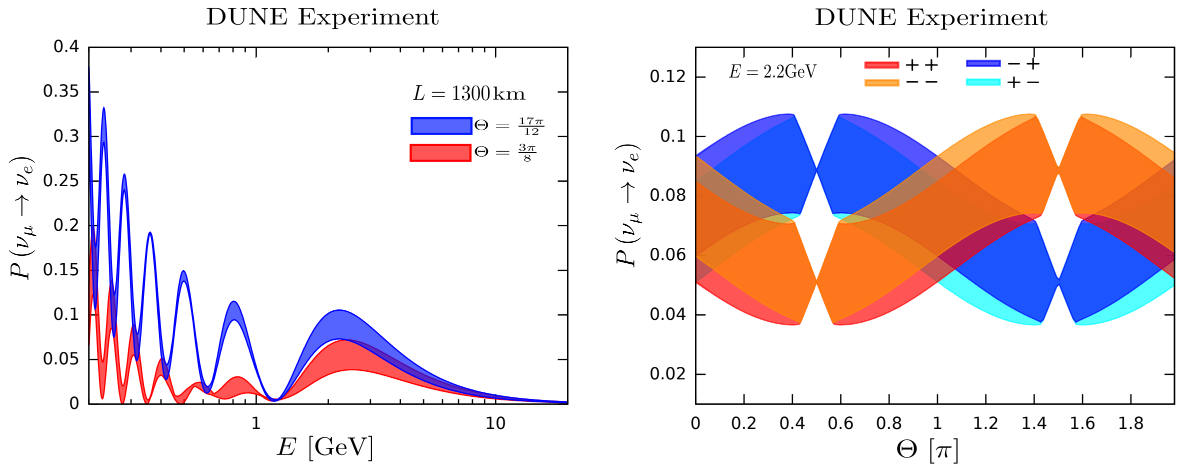

FIG. 9: The transition probability $P(\nu_\mu \to \nu_e)$ at a baseline of 1300km, which corresponds to the DUNE proposal. The mixing angle θ_{23} is taken within the currently allowed 3σ regions $0.393 \le \sin^2\theta_{23} \le 0.643$ [14], while the remaining oscillation parameters are chosen as in Fig. 6.

In Fig. 6 we show the $\nu_\mu \to \nu_e$ transition probability and the neutrino anti- neutrino asymmetry in matter. In this figure we take the atmospheric mixing angle within its currently allowed 3σ region, while for the remaining neutrino oscillation parameters are taken at their best fit values [14]. In Figs. 7, 8 we show the behavior of the transition probability $P(\nu_\mu \to \nu_e)$ in terms of neutrino energy E and the CP parameters Θ describing our approach, for baseline values 295 and 810 km, which correspond to the current T2K and NOνA experiments, respectively.

Note that so far we have discussed the predictions of our scenario for neutrino oscillations at the T2K and NOνA experiments, for a fixed sign combination in Eq. (16), which is $(+,+)$. We now consider the variation of our prediction with respect to the choice of sign conbination. For definiteness we now consider the future DUNE experiment. Fist we display in the left panel of Fig. 9 the behaviour of the $\nu_\mu \to \nu_e$ transition probability with respect to energy for the $(+,+)$ case and two fixed values of the model parameter Θ. In the right panel of Fig. 9 we display the model-dependence of the $\nu_\mu \to \nu_e$ transition probability for different sign combinations.

V. CONCLUSION

CP violation is the least studied aspect of the lepton mixing matrix. Other unknown features in the neutrino sector include the neutrino mass ordering and the octant of the atmospheric mixing parameter θ_{23}, not yet reliably determined by current global oscillation fits. In this letter we have proposed a generalized $\mu - \tau$ reflection scenario for leptonic CP violation and derived the corresponding restrictions on lepton flavor mixing parameters. We found that the "Majorana" phases are predicted to lie at their CP-conserving values with important implications for the neutrinoless double beta decay amplitudes, which we work out in detail. In addition to this prediction concerning the vanishing of the "Majorana-type" CP violation, we have obtained a new correlation between the atmospheric mixing angle θ_{23} and the "Dirac" CP phase $\delta_{\rm CP}$. Only in a very specific limit our CP transformation reduces to standard $\mu - \tau$ reflection, for which θ_{23} and $\delta_{\rm CP}$ become both maximal. We have also analysed the phenomenological implications of our scheme for present as well as upcoming neutrino oscillation experiments T2K, NOνA and DUNE.

Acknowledgments

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unitary matrix with $\mathbf{X}=\mathbf{X}^\top$, $\mathbf{X}\mathbf{X}^*=\mathbf{X}^*\mathbf{X}=1$. So by means of Takagi factorization the $\mathbf{X}$ matrix can be written as $\mathbf{X}=\boldsymbol{\Sigma}\boldsymbol{\Sigma}^\top$, where $\boldsymbol{\Sigma}$ is a unitary matrix. The invariance of the neutrino mass matrix under the action of $\mathbf{X}$ implies

$$\mathbf{X}^\top \mathbf{m}_\nu \mathbf{X}=\mathbf{m}_\nu^*\,, \tag{7}$$

Substituting the neutrino mass matrix $\mathbf{m}_\nu$ in Eq. (2), we find the lepton mixing matrix is constrained to fulfill

$$\mathbf{U}^{-1}\mathbf{X}\mathbf{U}^*=\mathrm{diag}\left(\pm1,\pm1,\pm1\right)\,. \tag{8}$$

As a result we find a master formula for the lepton mixing matrix [15]

$$\mathbf{U}=\boldsymbol{\Sigma}\,\mathbf{O}_{3\times3}\,\mathbf{Q}_\nu=\boldsymbol{\Sigma}'\,\mathbf{O}'_{3\times3}, \tag{9}$$

where $\boldsymbol{\Sigma}'=\boldsymbol{\Sigma}\,\mathbf{Q}_\nu$ with $\mathbf{Q}_\nu$ is a diagonal and unitary matrix whose shape is $\mathbf{Q}_\nu=\mathrm{diag}\left(e^{-i\frac{k_1}{2}\pi},\,e^{-i\frac{k_2}{2}\pi},\,e^{-i\frac{k_3}{2}\pi}\right)$ with the natural numbers $k_\mathtt{i}=0,\cdots,3$ and $\mathtt{i}=1,2,3$. Actually, the entries of ±1 and $\pm i$ $\mathbf{Q}_\nu$ which encode the CP parity or CP signs of the neutrino states and it renders the light neutrino mass eigenvalues positive. The matrix $\mathbf{O}_{3\times3}=\mathbf{O}_1\mathbf{O}_2\mathbf{O}_3$ is a generic three dimensional real orthogonal matrix, and it can be parameterized as

$$\mathbf{O}_1=\begin{pmatrix}1&0&0\\0&\mathrm{C}_{\theta1}&\mathrm{S}_{\theta1}\\0&-\mathrm{S}_{\theta1}&\mathrm{C}_{\theta1}\end{pmatrix},\ \mathbf{O}_2=\begin{pmatrix}\mathrm{C}_{\theta2}&0&\mathrm{S}_{\theta2}\\0&1&0\\-\mathrm{S}_{\theta2}&0&\mathrm{C}_{\theta2}\end{pmatrix}\ \text{and}\ \mathbf{O}_3=\begin{pmatrix}\mathrm{C}_{\theta3}&\mathrm{S}_{\theta3}&0\\-\mathrm{S}_{\theta3}&\mathrm{C}_{\theta3}&0\\0&0&1\end{pmatrix}\,, \tag{10}$$

where $\mathrm{C}_{\theta\mathtt{i}}=\cos\theta_\mathtt{i}$ and $\mathrm{S}_{\theta\mathtt{i}}=\sin\theta_\mathtt{i}$ are real parameters, and a possible overall minus sign of $\mathbf{O}_{3\times3}$ is dropped since it is insignificant. Therefore the PMNS matrix is predicted to depend on three free parameters $\theta_\mathtt{i}$ besides the parameters characterizing the residual CP transformation $\mathbf{X}$. Finally, the last term in Eq. (9) is $\mathbf{O}'_{3\times3}=\mathbf{O}'_1\mathbf{O}'_2\mathbf{O}'_3$ where $\mathbf{O}'_\mathtt{i}=\mathbf{Q}_\nu^\dagger\mathbf{O}_\mathtt{i}\mathbf{Q}_\nu$.

III. GENERAL PREDICTIONS FOR LEPTON MIXING

We now turn to the method of residual CP symmetry transformations proposed in Ref. [15]. Starting from the general CP-conserving form of the lepton mixing matrix it allows us to obtain CP-violating extensions systematically and, in principle, make CP predictions. The "symmetrical" presentation of the lepton mixing matrix originally proposed in Ref. [17, 18] is parametrized as:

$$\mathbf{U}_{Sym}=\begin{pmatrix}\mathrm{C}_{12}\mathrm{C}_{13} & \mathrm{S}_{12}\mathrm{C}_{13}e^{-i\phi_{12}} & \mathrm{S}_{13}e^{-i\phi_{13}}\\ -\mathrm{S}_{12}\mathrm{C}_{23}e^{i\phi_{12}}-\mathrm{C}_{12}\mathrm{S}_{13}\mathrm{S}_{23}e^{-i(\phi_{23}-\phi_{13})} & \mathrm{C}_{12}\mathrm{C}_{23}-\mathrm{S}_{12}\mathrm{S}_{13}\mathrm{S}_{23}e^{i(\phi_{23}+\phi_{12}-\phi_{13})} & \mathrm{C}_{13}\mathrm{S}_{23}e^{-i\phi_{23}}\\ \mathrm{S}_{12}\mathrm{S}_{23}e^{i(\phi_{23}+\phi_{12})}-\mathrm{C}_{12}\mathrm{S}_{13}\mathrm{C}_{23}e^{i\phi_{13}} & -\mathrm{C}_{12}\mathrm{S}_{23}e^{i\phi_{23}}-\mathrm{S}_{12}\mathrm{S}_{13}\mathrm{C}_{23}e^{-i(\phi_{12}-\phi_{13})} & \mathrm{C}_{13}\mathrm{C}_{23}\end{pmatrix}, \tag{11}$$

where $\mathrm{C}_{ij}=\cos\theta_{ij}$ and $\mathrm{S}_{ij}=\sin\theta_{ij}$. In the symmetric parametrization the relation between flavor mixing angles and the magnitudes of entries of leptonic mixing matrix is

$$\sin^2\theta_{13}=\left|(\mathbf{U}_{Sym})_{13}\right|^2\,,\quad \sin^2\theta_{12}=\frac{\left|(\mathbf{U}_{Sym})_{12}\right|^2}{1-\left|(\mathbf{U}_{Sym})_{13}\right|^2}\quad\text{and}\quad \sin^2\theta_{23}=\frac{\left|(\mathbf{U}_{Sym})_{23}\right|^2}{1-\left|(\mathbf{U}_{Sym})_{13}\right|^2}\,. \tag{12}$$

From the comparing the above expressions with those obtained in the Standard parametrization used by PDG [19], we conclude that are exactly the same expressions. Hence, the deference between both parameterizations will