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$SO(10)$ models with flavour symmetries: Classification and examples

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Abstract

We establish the full list of flavour symmetry groups which may be enforced, without producing any further accidental symmetry, on the Yukawa-coupling matrices of an $SO(10)$ Grand Unified Theory with arbitrary numbers of scalar multiplets in the **10**, $\overline{\mathbf{126}}$, and **120** representations of $SO(10)$. For each of the possible discrete non-Abelian symmetry groups, we present examples of minimal models which do not run into obvious contradiction with the phenomenological fermion masses and mixings.

Analysis is not really SO(10)-specific; should be the same with, say, E6

1 Introduction

There is a long history of attempts at explaining the fermion masses and mixings through (discrete) symmetry groups in models beyond the Standard Model (bSM). They started, back in the 1970's, with guesses or hopes that permutation groups might help explain and predict the patterns of the quark masses and mixing [1], and over the course of decades evolved into an elaborate group-theoretic machinery, especially for the lepton sector—for recent reviews see [2]. In the simplest approach, one assumes that several scalars exist which couple to the fermions through Yukawa matrices which inherit symmetries from the model bSM. Consider, for example, the quark sector with the following Yukawa Lagrangian with n_ϕ scalar doublets and n_g generations:

$$\mathcal{L}_Y = -\sum_{a=1}^{n_H} \sum_{i,j=1}^{n_g} \bar{Q}_{Li} \left(\Gamma^a_{ij} \phi_a d_{Rj} + \Delta^a_{ij} \tilde{\phi}_a u_{Rj} \right) + \text{H.c.} \tag{1}$$

If this Lagrangian inherits some symmetry from a high-energy model bSM, then the Yukawa matrices Γ^a_{ij} and Δ^a_{ij} are invariant under a transformation acting simultaneously in the flavour

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spaces of the left-handed quark doublets Q_{Li}, right-handed up-type quark singlets u_{Rj}, down-type quark singlets d_{Rj}, and scalar doublets ϕ_a. When the scalars acquire vacuum expectation values (vevs), those Yukawa matrices produce the mass matrices $M_d = \sum_a \Gamma^a \langle \phi_a^0 \rangle$ and $M_u = \sum_a \Delta^a \langle \phi_a^0 \rangle^*$ and the symmetry may get lost. However, if the symmetry was *ab initio* sufficiently restrictive, then the mass matrices might still have predictive power. In this field of research one wants to use a symmetry group to construct a model that is able to fit the known observables, *viz.* the fermion masses and the mixing parameters, without requiring fine-tuning and that is sufficiently predictive. This activity naturally splits into two parts: firstly, to find which symmetry groups are available for a given model bSM, and secondly, to check which symmetry groups lead to masses and to mixing patterns in agreement with the phenomenology. In this paper we address only the first task.

Grand Unified Theories (GUTs) based on the group $SO(10)$ are particularly attractive in this context, because in those theories all the left- and right-handed fermions of each generation are united in a single irreducible representation (irrep) **16** of $SO(10)$. As a consequence, all the Yukawa couplings take the simple form $f^T \Gamma_a H_a f$, where f stands for the column vector of the three fermionic **16** and the Γ_a are 3×3 Yukawa-coupling matrices in family space. The scalar multiplets H_a may be either **10** or $\overline{\mathbf{126}}$ of $SO(10)$, which couple to the symmetric part of the tensor product $\mathbf{16} \otimes \mathbf{16}$, or **120** of $SO(10)$, which couple to the antisymmetric part of the tensor product; correspondingly, the Γ_a will be either symmetric or antisymmetric, respectively. The (anti)symmetry properties of the Γ_a are preserved in weak-basis changes

$$\Gamma_a \to \sum_b U_{ab} \left(W^T \Gamma_b W \right), \tag{2}$$

where U is a unitary matrix which mixes the various scalar **10** (or $\overline{\mathbf{126}}$, or **120**) and W is a 3×3 unitary matrix which mixes the three fermionic **16**. For an overview of flavour model-building opportunities with $SO(10)$, see the classical review [3] and the more recent summary [4]. In this paper we want to analyze which symmetry groups one may impose on the Yukawa-coupling matrices of an $SO(10)$ GUT. In our search, we do not restrict ourselves in our choice of scalars—we derive results that are valid for arbitrary numbers of scalar multiplets **10**, $\overline{\mathbf{126}}$, and **120** of $SO(10)$. Thus, we go far beyond not only the early $SO(10)$ models, but also, for instance, the very recent study [5]; the examples presented in that paper emerge as specific cases of our general classification.

It might happen that by imposing a symmetry group G one ends up producing a model which is symmetric under a larger group $G' \supset G$. (This is sometimes called an 'accidental symmetry'.) A common instance of this occurs when G is a cyclic group and G' is $U(1)$. In our analysis, we shall always try to identify accidental symmetries which may be present in the Yukawa-coupling matrices that we write down.

In our search for discrete, non-Abelian symmetry groups, we shall use the method of [6]. Namely, we shall firstly derive all the possible Abelian symmetry groups; then, we shall use group-theoretical methods to combine the Abelian symmetry groups in all possible ways into non-Abelian groups. Here, knowing the full list of possible Abelian symmetry groups, *i.e.* knowning that no other Abelian group may be a subgroup of the non-Abelian symmetry group that we are constructing, is a strong factor limiting the possible choices.

It is important to stress that in this paper we only focus on the Yukawa-coupling sector of the $SO(10)$ GUT. We disregard the scalar sector, *viz.* the scalar potential, of the GUT. This sector depends, in particular, on which scalar $SO(10)$ irreps exist beyond the **10**, $\overline{\mathbf{126}}$, and **120**;

on whether those scalar representations constitute basic building blocks of a model or they are just the effective combination of other scalar irreps; on whether the GUT is supersymmetric or not; and on whether the scalar potential is renormalizable or not. Depending on the potential, some symmetries that exist in the Yukawa couplings may or may not be partially broken. So, the accidental symmetries that may be present in the Yukawa couplings that we write down might be broken in the scalar potential, but we shall not deal on that issue here.

This paper is organized as follows. In section 2 we provide and derive the full list of symmetries that may occur in the Yukawa couplings of an $SO(10)$ GUT. In section 3 we give the simplest models that realize each of the *discrete non-Abelian* symmetries that we have listed in section 2. We summarize our findings in section 4.

2 Full classification of the symmetries

In this section we shall list all the symmetry groups, both discrete and continuous, which can be used in flavour model building in $SO(10)$ GUTs with an arbitrary number of Higgs multiplets in the irreps **10**, $\overline{\mathbf{126}}$, and **120**. The end result of this section is the following list of groups:

MAIN CONCLUSION OF THE PAPER

$$\begin{aligned}
&\text{discrete Abelian:} && \mathbb{Z}_2, \quad \mathbb{Z}_3, \quad \mathbb{Z}_4, \quad \mathbb{Z}_2 \times \mathbb{Z}_2; && \text{(3a)}\\
&\text{continuous Abelian:} && U(1), \quad U(1) \times \mathbb{Z}_2, \quad U(1) \times U(1); && \text{(3b)}\\
&\text{discrete non-Abelian:} && S_3, \quad D_4, \quad Q_4, \quad A_4, \quad S_4, \quad \Delta(54) \big/ \mathbb{Z}_3^{\text{center}}, \quad \Sigma(36); && \text{(3c)}\\
&\text{continuous non-Abelian:} && O(2), \quad O(2) \times U(1), \quad [U(1) \times U(1)] \rtimes S_3, && \\
& && SU(2), \quad SU(2) \times U(1), \quad SO(3), \quad SU(3). && \text{(3d)}
\end{aligned}$$

In (3c),

$$\mathbb{Z}_3^{\text{center}} = \left\{ \text{diag}\,(1,\ 1,\ 1)\,,\ \text{diag}\,(\omega,\ \omega,\ \omega)\,,\ \text{diag}\left(\omega^2,\ \omega^2,\ \omega^2\right) \right\}, \quad \omega \equiv \exp\,(2i\pi/3) \qquad (4)$$

is the center of $SU(3)$.

Our claim is that trying to enforce any symmetry group which is not in the list (3) unavoidably produces a model whose full symmetry group, including the accidental symmetries, is in the list.

We derive the classification (3) by using the methods developed in [7] and [6], *viz.* we firstly identify all possible Abelian symmetry groups and we then construct the non-Abelian groups as extensions of the Abelian ones.

Readers who are not interested in the detailed derivation of (3) may skip this section.

2.1 Rephasing symmetries

HOW IS THIS RESULT OBTAINED?

We start with symmetries which act on fermion families and on scalars just through rephasings. In analogy with (2), one has

Consider first symmetry involving just phases

$$\Gamma_a \to e^{i\psi_a} S^T \Gamma_a S = \Gamma_a, \quad \text{where } S = \text{diag}\left(e^{i\alpha_1},\ e^{i\alpha_2},\ e^{i\alpha_3}\right). \qquad (5)$$

The phases ψ_a arise from the transformation of the scalar multiplets H_a which couple through the matrices Γ_a; the phases $\alpha_{1,2,3}$ refer to the transformation of the three fermion families.

The transformation (5) does not mix matrices Γ_a corresponding to scalars in different irreps of $SO(10)$, *i.e.* it does not mix the Γ_a linked to H_a in the **10** with those linked to H_a in either the $\overline{\mathbf{126}}$ or the **120**.

We consider the following problem: With n_S symmetric Γ_a and n_A antisymmetric Γ_a, which rephasing symmetry groups can one have?[1] This problem can be systematically solved through the Smith Normal Form (SNF) technique explained in [7, 8]. Adapting it to the problem at hand, we write the equations

One gets a **linear** system of equations for the phases, with **integer coefficients**

$$\sum_l d_{kl}\, \alpha_l \equiv \alpha_i + \alpha_j + \psi_a = 2\pi n_k.$$

One equation for each non-zero Yukawa matrix entry

Equation (6) states that the phase $\alpha_i + \alpha_j + \psi_a$ acquired by a *nonzero* entry $(\Gamma_a)_{ij}$ must be an integer multiple of 2π. To this end, we have introduced indices k that refer to all the nonzero entries of any of the Yukawa-coupling matrices. We have moreover represented all the phases, including the $n_S + n_A$ phases ψ_a, by α_l, where $l = 1, 2, 3, 4, \ldots, (3 + n_S + n_A)$. The integer-valued coefficients d_{kl} may take the values either 0 or 1 or 2.

SNF is the way to go

Equations (6) constitute a system of m linear equations for the $3 + n_S + n_A$ phases α_l, where m is the total number of independent[2] nonzero entries of all the Γ_a. We must now analyze the properties of the matrix $D = \{d_{kl}\}$; namely, we must find its SNF, read out its diagonal values, and from them write the corresponding symmetry group. This procedure is described in more detail in [7, 8].

2.1.1 Single matrix Γ

Let us suppose that there is a single matrix Γ_a. There are then only four phases α_l, with $l = 1, 2, 3, 4$ and $\alpha_4 = \psi_a$; moreover, in every row of D the last entry is always 1. The first three entries of each row of D may be, up to permutations, either (2, 0, 0) or (1, 1, 0). For instance, a row (2, 0, 0, 1) of D corresponds to nonzero Γ_{11}; a row (1, 0, 1, 1) of D corresponds to nonzero Γ_{13} and Γ_{31} (remember that all the matrices Γ_a are either symmetric or antisymmetric).

The number of possible matrices Γ is small, so they can be checked one by one. For example,

$$\text{if } \Gamma \sim \begin{pmatrix} \times & 0 & 0 \\ 0 & \times & 0 \\ 0 & 0 & \times \end{pmatrix}, \quad \text{then } D = \begin{pmatrix} 2 & 0 & 0 & 1 \\ 0 & 2 & 0 & 1 \\ 0 & 0 & 2 & 1 \end{pmatrix}. \tag{7}$$

(In the matrix Γ in (7), and below, a $\times$ represents a nonzero entry.) Through simple manipulations[3] of the matrix D in (7), one arrives at its SNF, which is

$$D_{\text{SNF}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \end{pmatrix}.$$

SPOILER: From here we can see (next page ...) that there is a Z2 x Z2 symmetry

[1] Clearly, the Yukawa interactions $f^T H_a f$ are invariant under the simultaneous global rephasing of all the fermions f by a phase δ and of all the Higgs multiplets H_a by a phase -2δ. We only look for symmetries above and beyond this trivial global rephasing invariance.

[2] Since the Γ_a are either symmetric or antisymmetric, not all their off-diagonal matrix elements are independent.

[3] The allowed manipulations are: effecting permutations of the order of the rows and/or columns of D; flipping the signs of any rows and/or columns of D; and adding any row or column of D to any other row or column.

The crucial point is the following: the manipulations of D which bring it to its SNF leave invariant the set of solutions of the system (6). Namely, that system is transformed into a new one,

$$\sum_l (D_{\mathrm{SNF}})_{kl}\ \tilde{\alpha}_l = 2\pi \tilde{n}_k.$$

Matrix transformations needed to achieve the Smith Normal Form are "nice" ones [unlike Gauss elimination]

Thus, the SNF in (8) yields

$$\tilde{\alpha}_1 = 2\pi\tilde{n}_1, \quad \tilde{\alpha}_2 = \pi\tilde{n}_2, \quad \tilde{\alpha}_3 = \pi\tilde{n}_3, \tag{10}$$

which means that the relevant symmetry group is $\mathbb{Z}_2 \times \mathbb{Z}_2$, because both $\tilde{\alpha}_2$ and $\tilde{\alpha}_3$ are integer multiples of π. The last column of D_{SNF} in (8) is composed of zeros and therefore places no restriction on $\tilde{\alpha}_4$; this free $\tilde{\alpha}_4$ constitutes a $U(1)$ invariance that exists for *any* Yukawa matrices and represents the global rephasing mentioned in footnote 1.

Another example is

$$\Gamma \sim \begin{pmatrix} \times & \times & 0 \\ \times & \times & 0 \\ 0 & 0 & \times \end{pmatrix} \Rightarrow D = \begin{pmatrix} 2 & 0 & 0 & 1 \\ 0 & 2 & 0 & 1 \\ 0 & 0 & 2 & 1 \\ 1 & 1 & 0 & 1 \end{pmatrix} \Rightarrow D_{\mathrm{SNF}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \Rightarrow G = \mathbb{Z}_2. \tag{11}$$

One sees that adding one off-diagonal nonzero entry to the Γ of (7) reduces the symmetry group from $\mathbb{Z}_2 \times \mathbb{Z}_2$ to $\mathbb{Z}_2$.

If there are less than three nonzero entries in Γ, then the system (9) is unable to fix all three $\tilde{\alpha}_{1,2,3}$; this implies a symmetry group which contains $U(1)$ factors. For instance,

$$\Gamma \sim \begin{pmatrix} \times & 0 & 0 \\ 0 & 0 & \times \\ 0 & \times & 0 \end{pmatrix} \Rightarrow D = \begin{pmatrix} 2 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 \end{pmatrix} \Rightarrow D_{\mathrm{SNF}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \Rightarrow G = U(1), \tag{12}$$

since the SNF in (12) yields no constraint on $\tilde{\alpha}_3$.

By examining all the possible matrices Γ in this way, we arrive at the list (13) of possible symmetries.

$$\text{For a symmetric } \Gamma: \quad U(1)\times U(1), \quad U(1)\times\mathbb{Z}_2, \quad U(1), \quad \mathbb{Z}_2\times\mathbb{Z}_2, \quad \mathbb{Z}_2. \tag{13a}$$

$$\text{For an antisymmetric } \Gamma: \quad U(1)\times U(1), \quad U(1). \tag{13b}$$

The difference between symmetric and antisymmetric matrices Γ arises because for antisymmetric Γ there are less possibilities for nonzero matrix elements—only off-diagonal matrix elements may be nonzero. The explicit matrices corresponding to each symmetry group are, up to permutations,

END RESULT OF THIS SIMPLE STAGE (1 Yukawa matrix; just phases)

$$U(1)\times U(1):\ \begin{pmatrix}\times & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix},\quad \begin{pmatrix}0 & \times & 0\\ \times & 0 & 0\\ 0 & 0 & 0\end{pmatrix}; \tag{14a}$$

$$U(1)\times \mathbb{Z}_2:\ \begin{pmatrix}\times & 0 & 0\\ 0 & \times & 0\\ 0 & 0 & 0\end{pmatrix}; \tag{14b}$$

$$U(1):\ \begin{pmatrix}\times & \times & 0\\ \times & 0 & 0\\ 0 & 0 & 0\end{pmatrix}\subset\begin{pmatrix}\times & \times & 0\\ \times & \times & 0\\ 0 & 0 & 0\end{pmatrix},\quad \begin{pmatrix}\times & 0 & 0\\ 0 & 0 & \times\\ 0 & \times & 0\end{pmatrix},\quad \begin{pmatrix}0 & \times & \times\\ \times & 0 & 0\\ \times & 0 & 0\end{pmatrix}; \tag{14c}$$

$$\mathbb{Z}_2\times \mathbb{Z}_2:\ \begin{pmatrix}\times & 0 & 0\\ 0 & \times & 0\\ 0 & 0 & \times\end{pmatrix}; \tag{14d}$$

$$\mathbb{Z}_2:\ \begin{pmatrix}\times & \times & 0\\ \times & 0 & 0\\ 0 & 0 & \times\end{pmatrix}\subset\begin{pmatrix}\times & \times & 0\\ \times & \times & 0\\ 0 & 0 & \times\end{pmatrix}. \tag{14e}$$

Any other matrices Γ—except those which are permutations of one of the matrices in (14)—possess no symmetry at all.

2.1.2 Several matrices Γ_a

If there are several Yukawa-coupling matrices Γ_a, then each row of the matrix D has, up to permutations, one of the following forms

$$(2,\,0,\,0\,|\,0,\,\ldots,\,1,\,\ldots,\,0), \tag{15a}$$
$$(1,\,1,\,0\,|\,0,\,\ldots,\,1,\,\ldots,\,0). \tag{15b}$$

(The form (15a) is available only for symmetric Γ_a.) This allows for more possibilities than the single-Γ_a case. For example, if there are two matrices

$$\Gamma_1 \sim \begin{pmatrix}\times & 0 & 0\\ 0 & 0 & \times\\ 0 & \times & 0\end{pmatrix} \quad\text{and}\quad \Gamma_2 \sim \begin{pmatrix}0 & 0 & 0\\ 0 & \times & 0\\ 0 & 0 & \times\end{pmatrix}, \tag{16}$$

then

$$D = \left(\begin{array}{ccc|cc}2 & 0 & 0 & 1 & 0\\ 0 & 1 & 1 & 1 & 0\\ 0 & 2 & 0 & 0 & 1\\ 0 & 0 & 2 & 0 & 1\end{array}\right) \;\Rightarrow\; D_{\rm SNF} = \begin{pmatrix}1 & 0 & 0 & 0 & 0\\ 0 & 1 & 0 & 0 & 0\\ 0 & 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 4 & 0\end{pmatrix} \;\Rightarrow\; G = \mathbb{Z}_4. \tag{17}$$

The problem that one now faces is how to efficiently check all possible combinations of many matrices Γ_a. In the following we make several observations that simplify, and eventually allow one to solve, that problem.

First observation: If *the same* nonzero entry is present in both Γ_a and $\Gamma_{a'}$, then the two scalar multiplets H_a and $H_{a'}$ must transform in the same way under the symmetry, *viz.* $\psi_a = \psi_{a'}$. But then, the structures Γ_a and $\Gamma_{a'}$ must completely coincide. Thus, any two matrices Γ_a and $\Gamma_{a'}$ either do not have nonzero entries in the same position, or they have nonzero entries at fully identical positions.[4]

For example, the symmetry group of the two matrices

$$\Gamma_1 \sim \begin{pmatrix} 0 & \times & 0 \\ \times & 0 & \times \\ 0 & \times & 0 \end{pmatrix} \quad \text{and} \quad \Gamma_2 \sim \begin{pmatrix} 0 & \times & \times \\ \times & 0 & 0 \\ \times & 0 & 0 \end{pmatrix} \tag{18}$$

is the same as the symmetry group of the merged matrix[5]

$$\Gamma_3 \sim \begin{pmatrix} 0 & \times & \times \\ \times & 0 & \times \\ \times & \times & 0 \end{pmatrix}. \tag{19}$$

Thus, the matrices Γ_a may be grouped in several sets of matrices with identical nonzero matrix elements, and any two sets of matrices do not have any common nonzero matrix element.[6]

Second observation: Matrices Γ_a with a single entry[7] do not modify the symmetry group in any way. This is because, by adjusting the ψ_a which transforms the scalar field H_a, any single-entry Γ_a will be symmetric under any rephasing of the fermion fields that one wishes.

Therefore, the rows of the matrix D corresponding to a single-entry Γ_a may safely be eliminated from D; simultaneously, the column of D corresponding to the phase ψ_a should also be removed. One only needs to check matrices with more than one independent nonzero entry.

Third observation: The matrix D has at most six rows, corresponding to the six possible nonzero entries in the matrices Γ_a. As we have seen in the first observation, the rows may be grouped into a few non-intersecting sets. According to the second observation, none of the sets is allowed to have just one row. There are only five ways of grouping at most six rows in several sets, when none of the sets has just one row:

$$4+2, \quad 3+3, \quad 2+2+2, \quad 3+2, \quad 2+2. \tag{20}$$

[4]This result does not depend on whether the matrices Γ_a and $\Gamma_{a'}$ are symmetric or antisymmetric; if one of them is antisymmetric, then it just does not borrow the nonzero diagonal entries from the symmetric one.

[5]This symmetry group happens to be trivial, as we have seen in the previous subsection. In group-theoretic terms, the two $U(1)$ symmetry groups of Γ_1 and Γ_2 do not have a non-trivial intersection.

[6]Having several Yukawa-coupling matrices with the same texture will in general complicate the analysis of the resulting mass matrices; yet, the *group-theoretic properties*—they are what we care about here—are not sensitive to a proliferation of identical matrices.

[7]Whenever we talk of a single-entry Γ_a, we always have in mind only the *independent nonzero* entries of that Γ_a.

Fourth observation: The 2-, 3-, and 4-entry matrices with non-trivial symmetry groups have already been given in (14b)–(14e):[8]

$$\text{4-entry}: \quad \begin{pmatrix} \times & \times & 0 \\ \times & \times & 0 \\ 0 & 0 & \times \end{pmatrix}, \begin{pmatrix} \times & 0 & \times \\ 0 & \times & 0 \\ \times & 0 & \times \end{pmatrix}, \begin{pmatrix} \times & 0 & 0 \\ 0 & \times & \times \\ 0 & \times & \times \end{pmatrix} \quad (\text{symmetry } \mathbb{Z}_2); \tag{21a}$$

$$\text{3-entry}: \quad \begin{pmatrix} \times & \times & 0 \\ \times & \times & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} \times & 0 & \times \\ 0 & 0 & 0 \\ \times & 0 & \times \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & \times & \times \\ 0 & \times & \times \end{pmatrix} \quad (\text{symmetry } U(1)); \tag{21b}$$

$$\begin{pmatrix} \times & 0 & 0 \\ 0 & \times & 0 \\ 0 & 0 & \times \end{pmatrix} \quad (\text{symmetry } \mathbb{Z}_2 \times \mathbb{Z}_2); \tag{21c}$$

$$\text{2-entry}: \quad \begin{pmatrix} \times & 0 & 0 \\ 0 & 0 & \times \\ 0 & \times & 0 \end{pmatrix}, \begin{pmatrix} 0 & \times & 0 \\ \times & 0 & 0 \\ 0 & 0 & \times \end{pmatrix}, \begin{pmatrix} 0 & 0 & \times \\ 0 & \times & 0 \\ \times & 0 & 0 \end{pmatrix} \quad (\text{symmetry } U(1)); \tag{21d}$$

$$\begin{pmatrix} \times & 0 & 0 \\ 0 & \times & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & \times & 0 \\ 0 & 0 & \times \end{pmatrix}, \begin{pmatrix} \times & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \times \end{pmatrix} \quad (\text{symmetry } U(1) \times \mathbb{Z}_2); \tag{21e}$$

$$\begin{pmatrix} 0 & \times & \times \\ \times & 0 & 0 \\ \times & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & \times \\ 0 & 0 & \times \\ \times & \times & 0 \end{pmatrix}, \begin{pmatrix} 0 & \times & 0 \\ \times & 0 & \times \\ 0 & \times & 0 \end{pmatrix} \quad (\text{symmetry } U(1)). \tag{21f}$$

The symmetry group of a collection of such matrices is the intersection of the symmetry groups of the individual matrices. Therefore, in order to find symmetry groups beyond the ones already listed in (14b)–(14e), we only need to intersect the $U(1)$ groups coming from matrices of the types (21b), (21d), (21e), and (21f).

We are left with very few possible combinations of non-intersecting 3- or 2-entry matrices in combinations of the types $3+2$, $2+2$, and $2+2+2$. We must check those combinations one by one.

The only (but for permutations of the rows and columns) combination of the type $3+2$ is

$$\Gamma_1 \sim \begin{pmatrix} \times & \times & 0 \\ \times & \times & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \Gamma_2 \sim \begin{pmatrix} 0 & 0 & \times \\ 0 & 0 & \times \\ \times & \times & 0 \end{pmatrix}. \tag{22}$$

One easily sees that the corresponding symmetry group is $U(1)$.

There are only two possible combinations of the type $2+2+2$:

$$\Gamma_1 \sim \begin{pmatrix} \times & 0 & 0 \\ 0 & \times & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \Gamma_2 \sim \begin{pmatrix} 0 & \times & 0 \\ \times & 0 & 0 \\ 0 & 0 & \times \end{pmatrix}, \quad \Gamma_3 \sim \begin{pmatrix} 0 & 0 & \times \\ 0 & 0 & \times \\ \times & \times & 0 \end{pmatrix}; \tag{23a}$$

$$\Gamma_1 \sim \begin{pmatrix} \times & 0 & 0 \\ 0 & 0 & \times \\ 0 & \times & 0 \end{pmatrix}, \quad \Gamma_2 \sim \begin{pmatrix} 0 & 0 & \times \\ 0 & \times & 0 \\ \times & 0 & 0 \end{pmatrix}, \quad \Gamma_3 \sim \begin{pmatrix} 0 & \times & 0 \\ \times & 0 & 0 \\ 0 & 0 & \times \end{pmatrix}. \tag{23b}$$

[8]Remember that any matrix Γ which is not in (14) does not possess any symmetry.

The corresponding symmetry groups are $\mathbb{Z}_2$ and $\mathbb{Z}_3$, respectively.

There are several possible combinations of the type 2+2. Some of them are subsets of (23b) and have the same $\mathbb{Z}_3$ symmetry group. The combination (16) has symmetry group $\mathbb{Z}_4$, as we have already seen. All the other $2+2$ combinations give symmetry groups which are either $U(1)$ or $\mathbb{Z}_2$.

We have thus arrived at the conclusion that Yukawa-coupling matrices in $SO(10)$ models can only have the Abelian symmetries in (14), plus $\mathbb{Z}_3$ and $\mathbb{Z}_4$. No other group can be the *full* rephasing-symmetry group of any collection of either symmetric or antisymmetric Yukawa matrices.

2.2 $\mathbb{Z}_3 \times \mathbb{Z}_3$ (complications: SU(3) vs PSU(3))

In the previous section we have assumed that any Abelian symmetry acts through rephasing in both the fermion and scalar sectors. We now relax this assumption and consider symmetry transformations which act through rephasing in the fermion sector and through an *arbitrary unitary matrix* in the scalar sector. Any single transformation can be brought to this form through an appropriate basis change of the fermion generations. We want to discover whether symmetry groups of this type exist.

In this case, the transformation (5) must be generalized to

$$\Gamma_a \to S^T \Gamma_a S = \sum_b v_{ab} \Gamma_b, \tag{24}$$

with coefficients v_{ab} forming a unitary transformation matrix $V = \{v_{ab}\}$ in the space of the scalars H_a. Since V is unitary, it has eigenvectors. By performing a basis change in the space of the H_a we can arrive at matrices $\bar{\Gamma}_a$ which are the eigenvectors of V; the eigenvalues have modulus 1 because V is unitary, *i.e.* they are phases $e^{i\psi_a}$. In this way we reduce (24) to (5).

Thus, although the condition (24) appears to offer more freedom, that freedom in reality corresponds only to a meaningless change of basis in scalar space. The available rephasing-symmetry groups are exactly the same as before, *viz.* $\mathbb{Z}_3$, $\mathbb{Z}_4$, and the groups in (14).

Still, we must remember that an overall rephasing of the scalar fields just corresponds to the trivial $U(1)$ transformation mentioned in footnote 1. Therefore, we must consider the action of symmetry transformations up to such a rephasing. This means looking not just for Abelian symmetry groups belonging to either $U(3)$ or $SU(3)$, but also for Abelian groups belonging to $PSU(3) \simeq U(3) \,/\, U(1)^{\text{center}} \simeq SU(3) \,/\, \mathbb{Z}_3^{\text{center}}$, where

$$U(1)^{\text{center}} = \left\{ \operatorname{diag}\left(e^{i\theta},\ e^{i\theta},\ e^{i\theta}\right) \right\} \tag{25}$$

is the center of $U(3)$ and $\mathbb{Z}_3^{\text{center}}$ is the center of $SU(3)$. It turns out that this allows for *only one*[9] further Abelian group: the group $\mathbb{Z}_3 \times \mathbb{Z}_3 = \Delta(27) \,/\, \mathbb{Z}_3^{\text{center}}$.

[9] The proof of this fact can be found in [7].

The non-Abelian group $\Delta(27)$ is the subgroup of $SU(3)$ generated by the matrices

$$A_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{pmatrix}, \tag{26}$$

$$D = \begin{pmatrix} 0 & e^{i\psi_1} & 0 \\ 0 & 0 & e^{i\psi_2} \\ e^{-i(\psi_1+\psi_2)} & 0 & 0 \end{pmatrix}, \tag{27}$$

where the phases ψ_1 and ψ_2 are arbitrary. This group contains the center of $SU(3)$, *viz.* (4). It is easy to check that the factor group of $\Delta(27)$ by its center $\mathbb{Z}_3^{\mathrm{center}}$ is the Abelian group $\mathbb{Z}_3 \times \mathbb{Z}_3$. This Abelian group cannot be represented just through a rephasing; its faithful irreducible representation is not one-dimensional but rather three-dimensional.

Let us look for Yukawa-coupling matrices transforming as a representation of $\mathbb{Z}_3 \times \mathbb{Z}_3$. We firstly identify matrices $\Gamma_{1,2,3}$ such that

$$A_3 \Gamma_1 A_3 = \Gamma_1, \quad A_3 \Gamma_2 A_3 = \omega^2 \Gamma_2, \quad A_3 \Gamma_3 A_3 = \omega \Gamma_3. \tag{28}$$

We find

$$\Gamma_1 = \begin{pmatrix} f_1 & 0 & 0 \\ 0 & 0 & g_1 \\ 0 & g_1 & 0 \end{pmatrix}, \quad \Gamma_2 = \begin{pmatrix} 0 & 0 & g_2 \\ 0 & f_2 & 0 \\ g_2 & 0 & 0 \end{pmatrix}, \quad \Gamma_3 = \begin{pmatrix} 0 & g_3 & 0 \\ g_3 & 0 & 0 \\ 0 & 0 & f_3 \end{pmatrix}. \tag{29}$$

We then enforce D-invariance of $\{\Gamma_1,\ \Gamma_2,\ \Gamma_3\}$. One easily finds that

$$D^T \Gamma_1 D = \Gamma_2, \quad D^T \Gamma_2 D = \Gamma_3, \quad D^T \Gamma_3 D = \Gamma_1, \tag{30}$$

provided

$$f_2 = f_1\, e^{2i\psi_1}, \quad g_2 = g_1\, e^{-i\psi_1}, \quad f_3 = f_2\, e^{2i\psi_2}, \quad g_3 = g_2\, e^{-i\psi_2}. \tag{31}$$

Thus, the matrices (29) are $\mathbb{Z}_3 \times \mathbb{Z}_3$-invariant provided phases ψ_1 and ψ_2 exist such that (31) apply.

We have thus found that $\mathbb{Z}_3 \times \mathbb{Z}_3$ is another possible Abelian symmetry of $SO(10)$ Yukawa-coupling matrices. The full list of possible Abelian symmetries is thus

$$\mathbb{Z}_2, \quad \mathbb{Z}_3, \quad \mathbb{Z}_4, \quad \mathbb{Z}_2 \times \mathbb{Z}_2, \quad \mathbb{Z}_3 \times \mathbb{Z}_3, \tag{32a}$$

$$U(1), \quad U(1) \times \mathbb{Z}_2, \quad U(1) \times U(1). \tag{32b}$$

It so happens that the set of matrices $\Gamma_{1,2,3}$ in (29) with the proviso (31) is accidentally invariant under a larger symmetry group. Let us define

$$B = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & -e^{i\psi_2} \\ 0 & -e^{-i\psi_2} & 0 \end{pmatrix}. \tag{33}$$

Then, the matrices (29) which satisfy (31) also satisfy

$$B^T \Gamma_1 B = \Gamma_1, \quad B^T \Gamma_2 B = \Gamma_3, \quad B^T \Gamma_3 B = \Gamma_2. \tag{34}$$

Thus, $\{\Gamma_1,\ \Gamma_2,\ \Gamma_3\}$ is automatically B-invariant and its symmetry group is not just $\mathbb{Z}_3 \times \mathbb{Z}_3$: it is actually $\Delta(54)\ /\ \mathbb{Z}_3^{\mathrm{center}}$, where $\Delta(54)$ is the subgroup of $SU(3)$ generated by A_3, D, and B. For this reason, the group $\mathbb{Z}_3 \times \mathbb{Z}_3$ does not appear in (3a); rather, $\Delta(54)\ /\ \mathbb{Z}_3^{\mathrm{center}}$ is in (3c).

AT THIS POINT ALL ABELIAN SYMMETRIES HAVE BEEN FOUND. WHAT IS LEFT IS TO USE THIS INFORMATION TO FIND THE NON-ABELIAN ONES AS WELL

2.3 Discrete non-Abelian symmetries

We next want to find the discrete *non-Abelian* symmetry groups that may be used in the Yukawa sector of $SO(10)$ models. Our analysis follows closely [6], where the analogous problem was solved for the scalar sector of the three-Higgs-doublet model. Indeed, our results are exactly the same as in [6]; we therefore repeat only briefly the argument in that paper.

Any non-Abelian *discrete* group G contains (usually many) Abelian subgroups A. We must firstly have the *full* list of all possible discrete Abelian groups A. We already know that list to be (32a). Thus, we want to know which non-Abelian discrete groups exist which only have Abelian subgroups in (32a).

We note that all the groups in (32a) have group orders with prime factors 2 and 3 only. Therefore, by Cauchy's lemma, the order of any non-Abelian group which only has Abelian subgroups in (32a) must be of the form $2^a 3^b$. Now, according to Burnside's theorem, any group with order $2^a 3^b$ contains a normal Abelian subgroup. The fact that we are looking for subgroups of $PSU(3)$ allows one to derive a stronger conclusion [6]: G contains a normal *maximal* Abelian subgroup. Let now A denote that subgroup. Then, the group G has structure

$$G = A \,.\, K, \quad \text{where} \quad K \subseteq \mathrm{Aut}(A), \tag{35}$$

i.e. the group G is constructed as an extension of A through a subgroup of the automorphism group of A. Since we already have the full list (32a) of possible A, we have to

Finite group theories are used here to systematize the job ahead
See Refs. 6,7 for details

1. find their automorphism groups $\mathrm{Aut}(A)$,
2. find all the subgroups K of the automorphism groups,
3. for each pair A and K, construct all the extensions of A through K.

At the end we will still need to check whether the resulting models have not acquired any accidental symmetries, especially continuous ones. We leave that task to section 3.

We now follow the steps above for each of the groups in (32a):

1. $\mathrm{Aut}\,(\mathbb{Z}_2) = \{e\}$,[10] hence no non-Abelian extension of $\mathbb{Z}_2$ is possible.
2. $\mathrm{Aut}\,(\mathbb{Z}_3) = \mathbb{Z}_2$, therefore the only possible non-Abelian extension of $\mathbb{Z}_3$ is $\mathbb{Z}_3 \rtimes \mathbb{Z}_2 = S_3$.
3. $\mathrm{Aut}\,(\mathbb{Z}_4) = \mathbb{Z}_2$, therefore there are two possible non-Abelian extensions of $\mathbb{Z}_4$: $\mathbb{Z}_4 \rtimes \mathbb{Z}_2 = D_4$ and $\mathbb{Z}_4 \,.\, \mathbb{Z}_2 = Q_4$.
4. $\mathrm{Aut}\,(\mathbb{Z}_2 \times \mathbb{Z}_2) = S_3$. The group S_3 has subgroups $\mathbb{Z}_2$, $\mathbb{Z}_3$, and S_3. Therefore, the possible non-Abelian extensions of $\mathbb{Z}_2 \times \mathbb{Z}_2$ are $(\mathbb{Z}_2 \times \mathbb{Z}_2) \rtimes \mathbb{Z}_2 = D_4$, $(\mathbb{Z}_2 \times \mathbb{Z}_2) \rtimes \mathbb{Z}_3 = A_4$, and $(\mathbb{Z}_2 \times \mathbb{Z}_2) \rtimes S_3 = S_4$.
5. $\mathrm{Aut}\,(\mathbb{Z}_3 \times \mathbb{Z}_3) = GL(2,3)$. This is the group of general linear transformations of a two-dimensional space over the finite field $\mathbb{F}_3$, *i.e.* the group of invertible 2×2 matrices with matrix elements which are integers modulo 3. The group $GL(2,3)$ has order 48 and has group elements of order 2, 3, 4, and 6 [9]. It turns out, however, that combining an

[10] The symbol e denotes the identity transformation.

element of order 3 with $\mathbb{Z}_3 \times \mathbb{Z}_3$ always leads to a continuous symmetry. Therefore, only two choices for discrete extensions of $\mathbb{Z}_3 \times \mathbb{Z}_3$ remain:

$$(\mathbb{Z}_3 \times \mathbb{Z}_3) \rtimes \mathbb{Z}_2 = \Delta(54) \big/ \mathbb{Z}_3^{\text{center}} , \tag{36a}$$
$$(\mathbb{Z}_3 \times \mathbb{Z}_3) \rtimes \mathbb{Z}_4 = \Sigma\,(36)\,. \tag{36b}$$

The two groups (36) are subgroups of $PSU(3)$ of order 18 and 36, respectively. Their preimages in $SU(3)$ are $\Delta(54)$ and $\Sigma(36\phi)$, respectively, which have order 54 and 108, respectively.

We have thus finished the derivation of (3c).

2.4 Continuous non-Abelian groups

When studying rephasing symmetries, we have identified three continuous Abelian groups: $U(1)$, $U(1) \times Z_2$, and $U(1) \times U(1)$. We now want to see how they can be extended to non-Abelian groups.[11]

We start by a more accurate description of $U(1)$ subgroups when we pass from $SU(3)$ to $PSU(3)$. There are two types of $U(1)$'s in $SU(3)$. The first one is parameterized as

$$U(1)_1 : \quad \text{diag}\left(1,\ e^{i\alpha},\ e^{-i\alpha}\right), \quad \alpha \in [0, 2\pi)\,, \tag{37}$$

and, provided $\alpha \neq 0$ and $\alpha \neq \pi$, it has three distinct eigenvalues. The second type, $U(1)_2$, is parameterized diag $(e^{2i\alpha},\ e^{-i\alpha},\ e^{-i\alpha})$ and, provided $\alpha \neq 0$ and $\alpha \neq \pm 2\pi/3$, it has a twice-degenerate eigenvalue. Because of this difference in the dimensionality of their subspaces, the two groups $U(1)_1$ and $U(1)_2$ cannot be mapped onto each other by any basis transformation of $SU(3)$. The center of $SU(3)$ is in $U(1)_2$ but not in $U(1)_1$. We factor it out by defining

$$U(1)_2 : \quad \text{diag}\left(e^{2i\alpha/3},\ e^{-i\alpha/3},\ e^{-i\alpha/3}\right), \quad \alpha \in [0, 2\pi)\,. \tag{38}$$

With this definition, $U(1)_1$ and $U(1)_2$ both belong to $PSU(3)$, they only intersect at the unit matrix, and they serve as basis vectors on the torus of rephasing transformations. The matrices (14c) are invariant under the two different $U(1)$s as

$$U(1)_1 : \quad \begin{pmatrix} \times & 0 & 0 \\ 0 & 0 & \times \\ 0 & \times & 0 \end{pmatrix}; \quad U(1)_2 : \quad \begin{pmatrix} 0 & 0 & 0 \\ 0 & \times & \times \\ 0 & \times & \times \end{pmatrix}, \quad \begin{pmatrix} 0 & \times & \times \\ \times & 0 & 0 \\ \times & 0 & 0 \end{pmatrix}. \tag{39}$$

We must extend the two $U(1)$'s to non-Abelian groups separately.

Both $U(1)_1$ and $U(1)_2$ are generated by rephasing transformations S_α, *viz.* (37) and (38). Let us suppose that there is another symmetry R of the model. We want to know what options there are for the enlarged group $G = \langle R,\ S_\alpha \rangle$. We consider the transformation $S_\alpha^R = R^{-1} S_\alpha R$. This is also a symmetry of the model. There are two options: (A) either S_α^R is in the original $U(1)$, and then it is equal to some S_β, or (B) it is not in the original $U(1)$. In case A, the invertible transformation R induces a group automorphism; the group of automorphisms of

[11] No non-Abelian symmetry group of the form $U(1) \times G$, where G is discrete non-Abelian, may exist, because G would necessarily have only $\mathbb{Z}_2$ Abelian subgroups and no such non-Abelian G exists.

$U(1)$ is just $\mathbb{Z}_2$. Thus, case A subdivides into two: either $\beta = \alpha$ (case A1) or $\beta = -\alpha$ (case A2). Let us now examine in turn each of the three cases A1, A2, and B.

In case A1, R commutes with S_α. Therefore, $G = \langle R,\ S_\alpha \rangle$ is Abelian; if it is larger than the original $U(1)$, then we already know that it can only be either $U(1) \times \mathbb{Z}_2$ or $U(1) \times U(1)$.

In case A2 we obtain the non-Abelian group $U(1) \rtimes \mathbb{Z}_2 \simeq O(2)$. This is possible only when $U(1)$ is $U(1)_1$, *cf.* (37), and the $\mathbb{Z}_2$ transformation is the permutation of second and third generations. It is not possible to build a group $U(1)_2 \rtimes \mathbb{Z}_2$, because there is no unitary transformation capable of mapping

$$\mathrm{diag}\left(e^{2i\alpha/3},\ e^{-i\alpha/3},\ e^{-i\alpha/3}\right) \quad \text{into} \quad \mathrm{diag}\left(e^{-2i\alpha/3},\ e^{i\alpha/3},\ e^{i\alpha/3}\right); \tag{40}$$

such a transformation would have to be antiunitary. Therefore, $O(2) \simeq U(1)_1 \rtimes \mathbb{Z}_2$ can be further enlarged to $O(2) \times U(1)_2$. In fact, *any* single-entry matrix is invariant under that group.

In case B, S_α^R does not belong to the initial $U(1)$. Therefore, it defines a different $U(1)$ subgroup of the full symmetry group. Let us now look at the group algebra rather than at the group itself. The generators of S_α and of S_α^R, which we denote t and t', respectively, define a two-dimensional subspace in the entire (8+1)-dimensional space spanned by the generators of $u(3)$. If t and t' commute, then we once again have a $U(1) \times U(1)$ symmetry group; this is possible only when R acts by permutation. In this way we can obtain $[U(1)_1 \times U(2)_2] \rtimes S_3$, which is the symmetry group of three single-entry Yukawa-coupling matrices with equal entries.

If t and t' do not commute, then we must close their subalgebra by including other generators. There exist very few subalgebras of $su(3)$, and they lead to the following non-Abelian groups: $SU(2)$ and $SO(3)$ (which have the same algebra) and $SU(2) \times U(1)$. In this way we finish the derivation of (3d).

3 Minimal models with discrete non-Abelian symmetry

In the previous section we have already written down the Yukawa-coupling matrices for models with Abelian symmetries. Those matrices may in general, as we have pointed out, be accompanied by an arbitrarily large number of single-entry matrices, which do not alter the symmetry group in any way because their intrinsic Abelian symmetry group always is the most general possible, *viz.* $U(1)_1 \times U(1)_2$.

Unfortunately, models with just an Abelian symmetry typically have a rather large number of free parameters. In this section we want to reduce this large freedom by looking for minimal models with *non-Abelian* symmetries. Specifically, we shall look for models with *discrete* symmetries, since the ones with continuous symmetries are in general much too restricted.

3.1 Models based on S_3

The group S_3 is generated by two transformations t_3 and t_2 such that $(t_3)^3 = (t_2)^2 = e$ (the identity transformation) and $t_2 t_3 t_2 = (t_3)^{-1}$. In a triplet representation of the generators, we may make a basis transformation in family space such that $t_3 \to A_3$, where A_3 is the matrix (26). Then, $t_2 \to B$, where B is the matrix (33), which contains an arbitrary phase ψ_2.[12]

[12] In this paper we always adopt symmetry generators with determinant +1, *viz.* belonging to $SU(3)$.

We have already seen that a $\mathbb{Z}_3$-invariant model contains the three Yukawa-coupling matrices (29). In order to extend $\mathbb{Z}_3$ to S_3 one must enforce invariance under B of the set of those three matrices. The matrix Γ_1 is B-invariant by itself alone, while $B^T \Gamma_2 B = e^{i\delta} \Gamma_3$, $B^T \Gamma_3 B = e^{-i\delta} \Gamma_2$ provided

$$f_2 \, e^{i(2\psi_2 - \delta)} = f_3, \quad g_2 \, e^{-i(\psi_2 + \delta)} = g_3. \tag{41}$$

Thus, the set $\{\Gamma_2, \ \Gamma_3\}$ is S_3-invariant if the conditions (41) are satisfied. Since the phases ψ_2 and δ are arbitrary, those conditions simply translate into

$$|f_2| = |f_3| \, , \quad |g_2| = |g_3| \, . \tag{42}$$

One may change the relative phase between the second and third fermion families in order to change ψ_2. One may also rephase the Higgs multiplets and thereby change δ. In this way one may, for instance, achieve $f_2 = f_3$ and $g_2 = g_3$. We still have some rephasing freedom to set, for instance, both f_1 and $f_2 = f_3$ real while g_1 and $g_2 = g_3$ remain complex; or, alternatively, to set f_1 and g_1 real while $f_2 = f_3$ and $g_2 = g_3$ remain complex. Anyway, there are six degrees of freedom in $\Gamma_{1,2,3}$. (When the Higgs fields acquire vevs, additional degrees of freedom appear.)

One may ask whether some of the matrix elements in $\Gamma_{1,2,3}$ may be zero. It turns out that, if either f_2 or g_2 is zero, then the symmetry group is promoted to $O(2)$. On the other hand, either f_1 or g_1 may be zero without leading to an enhanced symmetry. The case $f_1 = g_1 = 0$, *i.e.* $\Gamma_1 = 0$, is possible from the group-theoretical point of view, but it will make the charged-lepton and down-type-quark mass matrices proportional to each other, which is phenomenologically ruled out.

Another possibility is to assume that Γ_1 is antisymmetric. Then $f_1 = 0$ and the off-diagonal matrix elements are g_1 and $-g_1$. This is possible if the scalar multiplet H_1 transforms as a $\mathbf{1}'$ of S_3, since then $B^T \Gamma_1 B = -\Gamma_1$.

The full list of models with S_3 symmetry and having no more than three Higgs multiplets is given in Table 1. For model 1 and model 2 the Yukawa-coupling matrices are

$$\Gamma_1 = \begin{pmatrix} f_1 & 0 & 0 \\ 0 & 0 & g_1 \\ 0 & g_1 & 0 \end{pmatrix}, \quad \Gamma_2 = \begin{pmatrix} 0 & 0 & g_2 \\ 0 & f_2 & 0 \\ g_2 & 0 & 0 \end{pmatrix}, \quad \Gamma_3 = \begin{pmatrix} 0 & g_2 & 0 \\ g_2 & 0 & 0 \\ 0 & 0 & f_2 \end{pmatrix}, \tag{43}$$

with (for instance) real f_1 and f_2 and complex g_1 and g_2. For model 3 and model 4,

$$\Gamma_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & g_1 \\ 0 & -g_1 & 0 \end{pmatrix}, \quad \Gamma_2 = \begin{pmatrix} 0 & 0 & g_2 \\ 0 & f_2 & 0 \\ g_2 & 0 & 0 \end{pmatrix}, \quad \Gamma_3 = \begin{pmatrix} 0 & g_2 & 0 \\ g_2 & 0 & 0 \\ 0 & 0 & f_2 \end{pmatrix}, \tag{44}$$

with real g_1 and g_2 and complex f_2.

3.2 Models based on D_4

There are two ways to construct D_4 as an extension of an Abelian group: $D_4 = \mathbb{Z}_4 \rtimes \mathbb{Z}_2$ and $D_4 = (\mathbb{Z}_2 \times \mathbb{Z}_2) \rtimes \mathbb{Z}_2$.

	H_1	(H_2, H_3)
model 1	$(\overline{\mathbf{126}}, \mathbf{1})$	$(\mathbf{10}, \mathbf{2})$
model 2	$(\mathbf{10}, \mathbf{1})$	$(\overline{\mathbf{126}}, \mathbf{2})$
model 3	$(\mathbf{120}, \mathbf{1}')$	$(\mathbf{10}, \mathbf{2})$
model 4	$(\mathbf{120}, \mathbf{1}')$	$(\overline{\mathbf{126}}, \mathbf{2})$

Table 1: Minimal $SO(10)$ models with symmetry S_3. In each parenthesis, the first number denotes the $SO(10)$ irrep and the second number denotes the S_3 irrep.

3.2.1 $D_4 = \mathbb{Z}_4 \rtimes \mathbb{Z}_2$

This group is generated by two transformations t_4 and t_2 such that $(t_4)^4 = (t_2)^2 = e$ and $t_2 t_4 t_2 = (t_4)^{-1}$. As before, in a triplet representation one can make $t_4 \to A_4$ diagonal through an appropriate basis change:

$$A_4 = \mathrm{diag}\,(1,\ i,\ -i)\,. \tag{45}$$

Then, $t_2 \to B$ with the matrix B in (33).

We next write down the Yukawa matrices (16), which define the group $\mathbb{Z}_4$:

$$\Gamma_1 = \begin{pmatrix} f_1 & 0 & 0 \\ 0 & 0 & g_1 \\ 0 & g_1 & 0 \end{pmatrix}, \quad \Gamma_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & f_2 & 0 \\ 0 & 0 & f_3 \end{pmatrix}. \tag{46}$$

Clearly, $A_4 \Gamma_1 A_4 = \Gamma_1$ and $A_4 \Gamma_2 A_4 = -\Gamma_2$. We require that $\{\Gamma_1,\ \Gamma_2\}$ be invariant under B. We find that $B^T \Gamma_1 B = \Gamma_1$ automatically, but imposing $B^T \Gamma_2 B = \sigma \Gamma_2$ is only possible when $\sigma = \pm 1$ and

$$f_2\, e^{2i\psi_2} = \sigma f_3. \tag{47}$$

This implies $|f_2| = |f_3|$. The relative phase between f_2 and f_3 may be offset by ψ_2, making $f_2 = f_3$. Finally, one may use the fermion and Higgs rephasing freedom to make f_1 and $f_2 = f_3$ real, while g_1 remains complex.

The above minimal D_4 model is built with two non-equivalent D_4 singlets, *viz.* both Γ_1 and Γ_2 transform into themselves under either A_4 or B. There are four different singlets of D_4, denoted $\mathbf{1}_{pq}$, where the subscripts $p = \pm 1$ and $q = \pm 1$ reflect the actions of A_4 and B, respectively. They correspond to the following matrices:

$$\Gamma^{(D_4)}_{++} = \begin{pmatrix} f & 0 & 0 \\ 0 & 0 & g \\ 0 & g & 0 \end{pmatrix}, \quad \Gamma^{(D_4)}_{+-} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & h \\ 0 & -h & 0 \end{pmatrix}, \quad \Gamma^{(D_4)}_{-\pm} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & l & 0 \\ 0 & 0 & \pm l\, e^{2i\psi_2} \end{pmatrix}. \tag{48}$$

Combining these, one can construct several models with two distinct singlets. Most of those models have an accidental continuous symmetry; the only choice for which the full symmetry group remains D_4 and is not accidentally augmented to a continuous symmetry is precisely $(\mathbf{1}_{++},\ \mathbf{1}_{-\pm})$, *i.e.* the matrices (46) with $|f_2| = |f_3|$.

We may add more D_4 singlets, either in the same or in different one-dimensional irreps of D_4, and this leads to several more non-equivalent models. Unfortunately, all of them share the problem that their mass matrices have a block-diagonal form which leads to the decoupling of the first generation from the other two, producing CKM and PMNS matrices in disagreement with the phenomenology.

These problems can be avoided in a minimal way by using one $\mathbf{1}_{++}$ and one doublet of D_4; the corresponding models are shown in Table 2. The corresponding Yukawa-coupling matrices are

$$\Gamma_1 = \begin{pmatrix} f_1 & 0 & 0 \\ 0 & 0 & g_1 \\ 0 & g_1 & 0 \end{pmatrix}, \quad \Gamma_2 = \begin{pmatrix} 0 & g_2 & 0 \\ \pm g_2 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \Gamma_3 = \begin{pmatrix} 0 & 0 & g_2 \\ 0 & 0 & 0 \\ \pm g_2 & 0 & 0 \end{pmatrix}, \tag{49}$$

where the plus sign holds for models 5 and 6 and the minus sign holds for models 7 and 8. In both cases one may choose, for instance, real f_1 and g_2 while g_1 remains complex.

	H_1	(H_2, H_3)
model 5	$(\overline{\mathbf{126}}, \mathbf{1})$	$(\mathbf{10}, \mathbf{2})$
model 6	$(\mathbf{10}, \mathbf{1})$	$(\overline{\mathbf{126}}, \mathbf{2})$
model 7	$(\overline{\mathbf{126}}, \mathbf{1})$	$(\mathbf{120}, \mathbf{2})$
model 8	$(\mathbf{10}, \mathbf{1})$	$(\mathbf{120}, \mathbf{2})$

Table 2: Minimal $SO(10)$ models with symmetry $O(2)$. In each parenthesis, the first number denotes the $SO(10)$ irrep and the second number denotes the $O(2)$ irrep. For symmetry D_4, which is a subgroup of $O(2)$, one should write $\mathbf{1}_{++}$ instead of $\mathbf{1}$.

However, the matrices (49) have an accidental $U(1)_1$ continuous symmetry: they are actually $O(2)$-symmetric. The continuous group $O(2) = U(1)_1 \rtimes \mathbb{Z}_2$, where $\mathbb{Z}_2$ is generated by B, contains the discrete groups D_n, $\forall n \in \mathbb{N}$, and therefore the matrices (49) are invariant under any D_n, in particular D_4 and $D_3 = S_3$. In order to obtain a model that is invariant just under D_4 one may rely on the scalar potential; some terms may be present in it that break $O(2)$ down to its subgroup D_4.[13]

An alternative possibility to obtain a model that is just D_4-invariant, and not also $O(2)$-invariant, requires the use of four scalar multiplets, by combining two distinct D_4 singlets, for example by adding to (49) one further matrix

$$\Gamma_4 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & f_2 & 0 \\ 0 & 0 & f_2 \end{pmatrix}. \tag{50}$$

The symmetry group of this set of four Yukawa-coupling matrices will be D_4 and the mass matrices will not be block-diagonal.

3.2.2 $D_4 = (\mathbb{Z}_2 \times \mathbb{Z}_2) \rtimes \mathbb{Z}_2$

The minimal structure with symmetry $\mathbb{Z}_2 \times \mathbb{Z}_2$ is the single matrix (14d):

$$\Gamma_0 = \begin{pmatrix} f_1 & 0 & 0 \\ 0 & f_2 & 0 \\ 0 & 0 & f_3 \end{pmatrix}, \tag{51}$$

[13] If we had used for H_1 any other singlet of D_4 instead of $\mathbf{1}_{++}$, then the full symmetry group of the ensuing matrices would have been $U(1)_2 \times (U(1)_1 \rtimes \mathbb{Z}_2) = U(1)_2 \times O(2)$. Again, this larger symmetry might be broken down to D_4 through the scalar potential.

which is invariant under the $\mathbb{Z}_2 \times \mathbb{Z}_2$ group $\{1,\ P_1,\ P_2,\ P_3\}$, where

$$P_1 = \mathrm{diag}\,(+1,\ -1,\ -1)\,, \quad P_2 = \mathrm{diag}\,(-1,\ +1,\ -1)\,, \quad P_3 = \mathrm{diag}\,(-1,\ -1,\ +1) = P_1 P_2. \tag{52}$$

The automorphisms of $\mathbb{Z}_2 \times \mathbb{Z}_2$ form the S_3 group of the permutations of P_1, P_2, and P_3. One can pick the $\mathbb{Z}_2$ subgroup of this S_3 which effects $P_2 \leftrightarrow P_3$. This $\mathbb{Z}_2$ subgroup is generated by the matrix B of (33). Under that matrix, Γ_0 transforms as

$$B^T \Gamma_0 B = \mathrm{diag}\left(f_1,\ f_3\, e^{-2i\psi_2},\ f_2\, e^{2i\psi_2}\right). \tag{53}$$

Therefore, invariance of Γ_0 means $f_3 = \pm f_2 \exp\left(2i\psi_2\right)$ and implies $|f_2| = |f_3|$. As before, one may use the rephasing freedom to set $\psi_2 = 0$ and $f_2 = f_3$.

However, $\Gamma_0 = \mathrm{diag}\left(f_1,\ f_2,\ f_2\right)$ has a $U(1)$ symmetry, given by the arbitrary rotation between the second and third generations. Besides, Γ_0 by itself alone leads to diagonal fermion mass matrices, hence no mixing. Therefore we must accompany Γ_0 with a doublet of D_4. We thereby reproduce the cases considered in the previous subsection, albeit in a different weak basis.

3.3 Models based on $\mathbb{Z}_4 \,.\, \mathbb{Z}_2 = Q_4$

The quaternion group Q_4, which is of order eight, is generated by t_4, which satisfies $\left(t_4\right)^4 = e$, and a second generator t satisfying $t^{-1} t_4 t = \left(t_4\right)^{-1}$ but $t^2 \neq e$. Still, t must be such that t and t_4 do generate a finite group; this is achieved if $t^2 = \left(t_4\right)^2$, since $\left(t_4\right)^2$ generates the center $\mathbb{Z}_2$ of Q_4. An explicit triplet representation of Q_4 through $SU(3)$ matrices is

$$t_4 \to A_4 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & i & 0 \\ 0 & 0 & -i \end{pmatrix}, \quad t \to C = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & e^{i\psi} \\ 0 & -e^{-i\psi} & 0 \end{pmatrix}. \tag{54}$$

The phase ψ in C is arbitrary.

The group Q_4 has four inequivalent singlet representations $\mathbf{1}_{pq}$, where $p = \pm 1$ and $q = \pm 1$ just as in D_4, with $t_4 \to p$ and $t \to q$. The crucial difference between D_4 and Q_4 is that Q_4 is a subgroup of $SU(2)$ but D_4 is not. As a consequence, the invariant $\mathbf{1}_{++}$ of D_4 lies in the symmetric part of the product of two doublets, while the invariant $\mathbf{1}_{++}$ of Q_4 is in the *anti*symmetric part of the product. This is seen in the matrices

$$\Gamma_{++}^{(Q_4)} = \begin{pmatrix} f & 0 & 0 \\ 0 & 0 & g \\ 0 & -g & 0 \end{pmatrix}, \quad \Gamma_{+-}^{(Q_4)} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & h \\ 0 & h & 0 \end{pmatrix}, \quad \Gamma_{-\pm}^{(Q_4)} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & l & 0 \\ 0 & 0 & \pm l\, e^{2i\psi} \end{pmatrix}, \tag{55}$$

which satisfy $A_4\, \Gamma_{pq}^{(Q_4)} A_4 = p\, \Gamma_{pq}^{(Q_4)}$ and $C^T\, \Gamma_{pq}^{(Q_4)}\, C = q\, \Gamma_{pq}^{(Q_4)}$. One observes that $\Gamma_{++}^{(Q_4)}$, which is Q_4-invariant, is antisymmetric in the product of doublets but symmetric in the product of singlets. Now, since the Yukawa-coupling matrices in an $SO(10)$ GUT are always either symmetric or antisymmetric, a scalar multiplet in the $\mathbf{1}_{++}$ of Q_4 will always couple through a Yukawa-coupling matrix $\Gamma_{++}^{(Q_4)}$ which features either $f = 0$ or $g = 0$.

This fact has drastic consequences, namely, all the $SO(10)$ Yukawa-coupling matrices with Q_4 symmetry transform in a well-defined way under $U(1)_2$. Therefore, a set of $SO(10)$ Yukawa-coupling matrices with Q_4 symmetry alone, unaccompanied by any $U(1)$ symmetries, notably a symmetry of type $U(1)_2$, is not possible. Any $SO(10)$ model featuring Q_4 symmetry must rely on the scalar potential to break its accidental $U(1)_2$—and possibly also $U(1)_1$—symmetry. This feature contrasts Q_4 with D_4 models.

3.4 Models based on A_4 or S_4

The $\mathbb{Z}_2 \times \mathbb{Z}_2$ group $\{1_{3\times 3},\ P_1,\ P_2,\ P_3\}$ has a group of automorphisms S_3, formed by the permutations of P_1, P_2, and P_3. The $\mathbb{Z}_3$ subgroup of this S_3 is generated by a 3×3 matrix D such that $D^3 = 1_{3\times 3}$ and $D^{-1} P_{1,2,3}\, D = P_{2,3,1}$. One easily finds that that matrix D is the one in (27). Requiring $D^T \Gamma_0 D = e^{i\delta} \Gamma_0$ gives

$$f_1 = e^{i(\delta - 2\psi_1)} f_2 = e^{-i(\delta + 2\psi_1 + 2\psi_2)} f_3 \quad \text{and} \quad e^{3i\delta} = 1. \tag{56}$$

One may thus define three matrices,

$$\Gamma_0^{(A_4)} = f_0 \, \mathrm{diag}\left(1,\ e^{2i\psi_1},\ e^{2i(\psi_1+\psi_2)}\right), \tag{57a}$$

$$\Gamma_1^{(A_4)} = f_1 \, \mathrm{diag}\left(1,\ \omega^2 e^{2i\psi_1},\ \omega\, e^{2i(\psi_1+\psi_2)}\right), \tag{57b}$$

$$\Gamma_2^{(A_4)} = f_2 \, \mathrm{diag}\left(1,\ \omega\, e^{2i\psi_1},\ \omega^2 e^{2i(\psi_1+\psi_2)}\right), \tag{57c}$$

corresponding to $e^{i\delta} = 1$, $e^{i\delta} = \omega$, and $e^{i\delta} = \omega^2$, respectively.

The group S_3 is generated by D in (27) together with B in (33). Since $B^T \Gamma_0^{(A_4)} B = \Gamma_0^{(A_4)}$, the matrix $\Gamma_0^{(A_4)}$ is S_4-invariant. The matrices $\Gamma_1^{(A_4)}$ and $\Gamma_2^{(A_4)}$ transform into each other under the action of B, provided $f_1 = f_2$. Therefore, the following is a doublet of S_4:

$$\left\{ f_1 \, \mathrm{diag}\left(1,\ \omega^2 e^{2i\psi_1},\ \omega\, e^{2i(\psi_1+\psi_2)}\right),\ f_1 \, \mathrm{diag}\left(1,\ \omega\, e^{2i\psi_1},\ \omega^2 e^{2i(\psi_1+\psi_2)}\right) \right\}. \tag{58}$$

It is clear that by using only Yukawa-coupling matrices $\Gamma_{0,1,2}^{(A_4)}$ one can only obtain diagonal fermion mass matrices, impeding fermion mixing and making the model incompatible with phenomenology. In order to allow for mixing one needs to include A_4/S_4 triplets. There are two triplets of single-entry matrices, a symmetric one and an antisymmetric one:

$$\Gamma_3 = \begin{pmatrix} 0 & g & 0 \\ \pm g & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \Gamma_4 = g \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & e^{i(\psi_1+\psi_2)} \\ 0 & \pm e^{i(\psi_1+\psi_2)} & 0 \end{pmatrix}, \quad \Gamma_5 = g \begin{pmatrix} 0 & 0 & \pm e^{i\psi_2} \\ 0 & 0 & 0 \\ e^{i\psi_2} & 0 & 0 \end{pmatrix}. \tag{59}$$

One of these triplets may accompany the matrix (57a). The resulting models are given in table 3. The corresponding Yukawa-coupling matrices are $\Gamma_0^{(A_4)}$ and $\Gamma_{3,4,5}$; in the latter, the plus sign holds for models 9 and 10 and the minus signs is for models 11 and 12.

The phases ψ_1 and ψ_2 may be rephased away while f_0 is made real; the parameter g in (59) remains, in general, complex.

A model with the smaller symmetry A_4 requires an extra scalar multiplet, coupling with either of the matrices (57b) or (57c). Thus, while a model with symmetry S_4 requires just four Yukawa-coupling matrices, a model with the smaller symmetry A_4 needs at least five Yukawa-coupling matrices.[14,15]

[14]This is reminiscent of the situation with $O(2)$ and D_4, studied in subsection 3.2.1. A model with $O(2)$ symmetry needs only three Yukawa-coupling matrices, a model with the smaller symmetry $D_4 \subset O(2)$ needs at least four Yukawa-coupling matrices.

[15]Alternatively, a model may have symmetry S_4 in its four Yukawa-coupling matrices but that symmetry may be broken down to A_4 in the scalar potential.

	H_1	(H_2, H_3, H_4)
model 9	$(\overline{\mathbf{126}}, \mathbf{1})$	$(\mathbf{10}, \mathbf{3})$
model 10	$(\mathbf{10}, \mathbf{1})$	$(\overline{\mathbf{126}}, \mathbf{3})$
model 11	$(\overline{\mathbf{126}}, \mathbf{1})$	$(\mathbf{120}, \mathbf{3}')$
model 12	$(\mathbf{10}, \mathbf{1})$	$(\mathbf{120}, \mathbf{3}')$

Table 3: $SO(10)$ models with symmetry S_4. In each parenthesis, the first number denotes the $SO(10)$ irrep and the second number denotes the S_4 irrep.

3.5 Models based on $\Delta(54)$

We have seen in subsection 2.2 that one may use the group $\Delta(27)$ for the Yukawa-coupling matrices in an $SO(10)$ GUT. That subgroup of $SU(3)$ is generated by the matrices A_3 in (26) and D in (27). The group $\Delta(27)$ has nine triplet irreps $\mathbf{1}_{pq}$, with $p, q \in \{0,\ 1,\ 2\}$, under which $A_3 \to \omega^p$ and $D \to \omega^q$. Unfortunately, though, these singlet irreps cannot be realized in matrix form, *i.e.* there is no matrix X such that $A_3 X A_3 = \omega^p X$ and $D^T X D = \omega^q X$ simultaneously. So, one must realize $\Delta(27)$ solely through triplets

$$\Gamma_1 = \begin{pmatrix} f & 0 & 0 \\ 0 & 0 & g \\ 0 & g & 0 \end{pmatrix}, \quad \Gamma_2 = \begin{pmatrix} 0 & 0 & g\, e^{-i\psi_1} \\ 0 & f\, e^{2i\psi_1} & 0 \\ g\, e^{-i\psi_1} & 0 & 0 \end{pmatrix},$$
$$\Gamma_3 = \begin{pmatrix} 0 & g\, e^{-i(\psi_1+\psi_2)} & 0 \\ g\, e^{-i(\psi_1+\psi_2)} & 0 & 0 \\ 0 & 0 & f\, e^{2i(\psi_1+\psi_2)} \end{pmatrix}. \tag{60}$$

These triplets actually are symmetric under a larger group than $\Delta(27)$, namely $\Delta(54)$.

In order to construct a viable $\Delta(54)$-symmetric model one must pick two triplets. Minimal examples are given in table 4. The corresponding Yukawa-coupling matrices are

	(H_1, H_2, H_3)	(H_4, H_5, H_6)
model 13	$(\overline{\mathbf{126}}, \mathbf{3})$	$(\mathbf{10}, \mathbf{3})$
model 14	$(\mathbf{10}, \mathbf{3})$	$(\mathbf{120}, \mathbf{3}')$
model 15	$(\overline{\mathbf{126}}, \mathbf{3})$	$(\mathbf{120}, \mathbf{3}')$

Table 4: Minimal $SO(10)$ models with symmetry $\Delta(54)$. In each parenthesis, the first number denotes the $SO(10)$ irrep and the second number denotes the $\Delta(54)$ irrep.

$$\Gamma_1 = \begin{pmatrix} f & 0 & 0 \\ 0 & 0 & g \\ 0 & g & 0 \end{pmatrix}, \quad \Gamma_2 = \begin{pmatrix} 0 & 0 & gx^* \\ 0 & fx^2 & 0 \\ gx^* & 0 & 0 \end{pmatrix}, \quad \Gamma_3 = \begin{pmatrix} 0 & gx^*y^* & 0 \\ gx^*y^* & 0 & 0 \\ 0 & 0 & fx^2y^2 \end{pmatrix},$$
$$\Gamma_4 = \begin{pmatrix} f' & 0 & 0 \\ 0 & 0 & g' \\ 0 & g' & 0 \end{pmatrix}, \quad \Gamma_5 = \begin{pmatrix} 0 & 0 & g'x^* \\ 0 & f'x^2 & 0 \\ g'x^* & 0 & 0 \end{pmatrix}, \quad \Gamma_6 = \begin{pmatrix} 0 & g'x^*y^* & 0 \\ g'x^*y^* & 0 & 0 \\ 0 & 0 & f'x^2y^2 \end{pmatrix}, \tag{61}$$

for model 13, and

$$\Gamma_1 = \begin{pmatrix} f & 0 & 0 \\ 0 & 0 & g \\ 0 & g & 0 \end{pmatrix}, \quad \Gamma_2 = \begin{pmatrix} 0 & 0 & gx^* \\ 0 & fx^2 & 0 \\ gx^* & 0 & 0 \end{pmatrix}, \quad \Gamma_3 = \begin{pmatrix} 0 & gx^*y^* & 0 \\ gx^*y^* & 0 & 0 \\ 0 & 0 & fx^2y^2 \end{pmatrix},$$
$$\Gamma_4 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & g' \\ 0 & -g' & 0 \end{pmatrix}, \quad \Gamma_5 = \begin{pmatrix} 0 & 0 & -g'x^* \\ 0 & 0 & 0 \\ g'x^* & 0 & 0 \end{pmatrix}, \quad \Gamma_6 = \begin{pmatrix} 0 & g'x^*y^* & 0 \\ -g'x^*y^* & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{62}$$

for models 14 and 15, where $x = \exp(i\psi_1)$ and $y = \exp(i\psi_2)$.

It is easy to check that, through rephasings of the generations, the phases ψ_1 and ψ_2 may be eliminated while f is rendered real in both (61) and (62); but, g, g', and—in (61)—f' will in general remain complex.

3.6 Models based on $\Sigma(36)$

The symmetry group $\Delta(54) \,/\, \mathbb{Z}_3^{\text{center}}$ may be further enlarged, to $\Sigma(36)$, if we require the sets $\{\Gamma_1,\ \Gamma_2,\ \Gamma_3\}$ and $\{\Gamma_4,\ \Gamma_5,\ \Gamma_6\}$ of either (61) or (62) to be invariant under the action of

$$\frac{i}{\sqrt{3}} \begin{pmatrix} 1 & e^{i\psi_1} & e^{i(\psi_1+\psi_2)} \\ e^{-i\psi_1} & \omega^2 & \omega e^{i\psi_2} \\ e^{-i(\psi_1+\psi_2)} & \omega e^{-i\psi_2} & \omega^2 \end{pmatrix}. \tag{63}$$

It is easy to check that this only happens when

$$\text{either} \quad \frac{g}{f} = e^{i(2\psi_1+\psi_2)}\, \frac{-1+\sqrt{3}}{2} \quad \text{or} \quad \frac{g}{f} = e^{i(2\psi_1+\psi_2)}\, \frac{-1-\sqrt{3}}{2} \tag{64}$$

for (62); for (61), the condition (64) must apply and moreover

$$\text{either} \quad \frac{g'}{f'} = e^{i(2\psi_1+\psi_2)}\, \frac{-1+\sqrt{3}}{2} \quad \text{or} \quad \frac{g'}{f'} = e^{i(2\psi_1+\psi_2)}\, \frac{-1-\sqrt{3}}{2} \tag{65}$$

must also hold.

Just as in models based on $\Delta(54) \,/\, \mathbb{Z}_3^{\text{center}}$, the phases ψ_1 and ψ_2 may be rephased away together with the phase of f. Then, g and f will be both real, while g' and f' will be complex but have the same phase (possibly apart from π).

There are only three discrete subgroups of $PSU(3)$ which contain $\Sigma(36)$ as a subgroup: $\Sigma(72)$, $\Sigma(216)$, and $\Sigma(360)$.[16] By looking at their generators [10], one easily sees that the $\Sigma(36)$-invariant models delineated above cannot be made invariant under any larger discrete subgroup of $PSU(3)$.

4 Discussion and conclusions

Through the long history of model building within $SO(10)$ GUTs equipped with flavour symmetry groups, virtually all the studies have focused on specific (discrete) groups and have

[16] We thank Patrick Otto Ludl for informing us about this fact.

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studied their phenomenological consequences. One might expect that, the more scalars in different irreps of $SO(10)$ one would introduce, the more complicated flavour symmetries one would be able to achieve and the more elaborate Yukawa sectors one would construct, with apparently limitless complexity. In this paper we have shown that things are much more certain: only a limited number of flavour symmetries may be achieved, no matter how large the scalar sector of the $SO(10)$ GUT is. We have given the full classification of all possible flavour symmetry groups that may be imposed on the Yukawa matrices, for an arbitrary number of scalars in the **10**, $\overline{\mathbf{126}}$, or **120** of $SO(10)$. We have used methods from finite group theory to identify all the possible non-Abelian discrete groups whose Yukawa sector does not possess an accidental continuous symmetry.

We have also given examples of minimal models based on each discrete group. Which of those examples might constitute truly viable phenomenological models, *viz.* able to fit the data, remains yet to be studied. In any case, we now know that there exist no other essentially new possibilities beyond those that we have found.

One point that we have delegated to future studies is the structure of the scalar sector for each model. The scalar potential must be compatible with the required symmetry and its minimization must lead to a vacuum that breaks the symmetry just enough to produce a realistic fermion mixing, but not so much that all predictive power gets lost. This might not be easy to achieve: scalar potentials with large symmetry groups tend to have minima which break the symmetry only partially, and the residual symmetries might render the fermion mass matrices much too restrictive [11].

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MY COMMENTS:
1- This analysis uses very little SO(10) specific data [should then apply to other cases];
2- But on the other hand, the scalar sector was not looked at (i.e., other symmetries may indeed be meaningful once the full Lagrangian is considered)
3- I am not sure, but even the addition of vector fermions (eg, 16+16b) could invalided the analysis here

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