

THE K -MATRIX FORMALISM FOR OVERLAPPING RESONANCES

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Abstract: The relation between the potential theory and S -matrix approaches to the overlapping resonance problem is derived in an elementary way by relating both to the K -matrix approach. A generalisation of the formalism to the production of overlapping resonances in final states is given.

We consider a situation in which there are several overlapping resonance states (with the same spin-parity and other relevant quantum numbers) labelled A, B, C, \dots , coupling to various two-particle continuum states labelled i, j, k, \dots . This problem has been extensively studied in the nuclear physics context, using potential theory. In particular, Feshbach¹⁻³) has derived a generalisation (which is of wide applicability) of the one-level Breit-Wigner formula for the case of overlapping resonances. An alternative approach has been to write down a general T -matrix (or S -matrix) as a sum of resonance terms, and then to try and solve the constraints imposed by unitarity on the resonance parameters. Though pursued in the nuclear physics context also⁴)†, this method is of interest in particle physics, where one wishes to avoid potentials, and it has been so studied⁵). The difficulty with it is that the constraints are complicated non-linear conditions which seem hard to solve in general. Nevertheless, the overlapping resonance problem exists in particle physics: for example, the A_2 has been regarded as two overlapping resonances^{5, 6}), the Q -region in $K\pi\pi$ mass may involve two resonances⁷), and there is the famous $K\bar{K}$ system⁸). Now, it is known that the use of the K -matrix (rather than the T -matrix) is a simple way to satisfy the constraints of unitarity⁹), and the existence and properties of the K -matrix can be established on rather general grounds^{9, 10}). Thus one might think that the K -matrix would be a useful approach to the problem in the particle physics context.

McGlinn and Polis¹¹) used the K -matrix discussion of sum rules in the $K\bar{K}$ system. Their suggestion that the Bell-Steinberger sum rule¹²) could not be derived in this way, but only in potential theory, led to a series of papers¹³⁻¹⁵) deriving the sum rule in both the T -matrix and potential theory approaches, and showing the equivalence of these approaches^{16, 17}).

None of these latter papers, however, used the K -matrix, which was the way

† I thank Prof. P. K. Kabir for bringing this reference to my attention.

McGlinn and Polis attacked the problem; one would like to see the connection directly between the K -matrix and potential theory approaches. This connection has actually been given, in the nuclear physics context, by Feshbach³⁾ using his general formalism, which he has also related to the T -matrix^{2,3)}. Our purpose here is to give a simple, purely algebraic, derivation of the equivalence of the T -matrix and potential theory approaches by relating both to the K -matrix, and to put the results in a particle physics context†. Having done this, we go on to outline an extension of the K -matrix formalism to production reactions in which the overlapping resonances appear in final states, along with another particle, or particles.

At the outset, we wish to make clear that we shall be dealing with three different sorts of base states for the resonances. We shall label these three sets by $\{\bar{\alpha}, \bar{\beta}, \dots\}$, $\{\alpha, \beta, \dots\}$, and $\{A, B, \dots\}$. The first is the set of base states used in the potential theory formalism, the second is the set used in the K -matrix formalism, and the third is the set of actual physical states. The precise definitions of these various states will be given in the course of the development.

In potential theory, the T -matrix elements T_{ij} for transition between continuum states i and j via the overlapping resonances forming the intermediate states can be written^{1-3, 8, 13)} (all repeated indices are summed)

$$T_{ij} = f_{i\bar{\alpha}} S'_{\bar{\alpha}\bar{\beta}} f_{\bar{\beta}j}, \quad (1)$$

where the elements of the propagator matrix $S'_{\bar{\alpha}\bar{\beta}}$ are given by

$$S'_{\bar{\alpha}\bar{\beta}} = (M - \frac{1}{2}i\Gamma - m)_{\bar{\alpha}\bar{\beta}}^{-1}, \quad (2)$$

where m is the energy and M is a Hermitian mass matrix and Γ is a Hermitian width matrix

$$(\Gamma)_{\bar{\alpha}\bar{\beta}} = \sum_i 2\pi\rho_i f_{\bar{\alpha}i} f_{i\bar{\beta}}. \quad (3)$$

The states $\bar{\alpha}, \bar{\beta}, \dots$ are the 'bare' resonance states as they exist before either the coupling V_c to the continuum channels i or direct coupling between the states is turned on. They are in fact, therefore, bound states. ρ_i is the phase-space factor for channel i , and the f are the matrix elements $f_{\bar{\alpha}i} = \langle \bar{\alpha} | V_c | i \rangle$ between the bare states $\bar{\alpha}$ and the con-

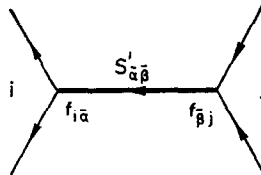


Fig. 1. Transition from state j to state i via the set of overlapping resonance states described by the propagator matrix $S'_{\bar{\alpha}\bar{\beta}}$.

† Rosenfeld⁴⁾ has also given the connection between the K -matrix and S -matrix approaches (though not the potential theory approach), but we feel that our method is rather simpler; in addition, we do not need to assume that the phase-space factors are slowly varying.

tinuum states i . If V_c is Hermitian we have $f_{\bar{\alpha}i} = f_{i\bar{\alpha}}^*$ and if time refersal is valid we can choose the f to be real and such that $f_{\bar{\alpha}i} = f_{i\bar{\alpha}}$. Eq. (1) is easily visualized as fig. 1.

The matrices Γ and M do not, in general, commute, and unless they do they cannot be simultaneously diagonalised by a unitary transformation. But it is always possible to diagonalise one of them, in particular M , by a unitary matrix U :

$$U_{\alpha\bar{\alpha}} M_{\bar{\alpha}\beta} U_{\beta\bar{\beta}}^{-1} = m_{\alpha} \delta_{\alpha\beta}, \quad UU^{\dagger} = I. \tag{4}$$

Thus U effects the transformation from the set of base states $\{\bar{\alpha}, \bar{\beta}, \dots\}$ to a new set $\{\alpha, \beta, \dots\}$. The physical significance of the latter set is that it is the set of states which diagonalise M , the Hermitian part of the "effective Hamiltonian" matrix $M - \frac{1}{2}i\Gamma$; we shall call them the mass-mixed states. Their relevance to the K -matrix will appear below; note that they are not the physical states, which diagonalise $M - \frac{1}{2}i\Gamma$.

If we now define new couplings $f_{i\alpha}, f_{\beta j}$ by

$$f_{\beta j} = U_{\beta\bar{\beta}} f_{\bar{\beta}j}, \quad f_{i\alpha} = f_{i\bar{\alpha}} U_{\bar{\alpha}\alpha}^{-1}, \tag{5}$$

the T -matrix has the form

$$T_{ij} = f_{i\alpha} S'_{\alpha\beta} f_{\beta j}, \tag{6}$$

where

$$\begin{aligned} S'_{\alpha\beta}{}^{-1} &= (UMU^{-1} - \frac{1}{2}i\Gamma U^{-1} - m)_{\alpha\beta} \\ &= m_{\alpha} \delta_{\alpha\beta} - \frac{1}{2}i \sum_i 2\pi\rho_i f_{\alpha i} f_{i\beta} - m\delta_{\alpha\beta} = S_{\alpha\beta}^{-1} - \Sigma_{\alpha\beta}, \quad \text{say,} \end{aligned}$$

with

$$S_{\alpha\beta}^{-1} = (m_{\alpha} - m)\delta_{\alpha\beta}, \quad \Sigma_{\alpha\beta} = \frac{1}{2}i \sum_i 2\pi\rho_i f_{\alpha i} f_{i\beta}.$$

In matrix form, therefore, ¹⁸⁾†

$$S' = S + S'\Sigma S,$$

which we can iterate as

$$S' = S + S\Sigma S + S\Sigma S\Sigma S + \dots,$$

and so we obtain

$$\begin{aligned} T_{ij} &= f_{i\alpha} S'_{\alpha\beta} f_{\beta j} = f_{i\alpha} S_{\alpha\beta} f_{\beta j} + f_{i\alpha} S_{\alpha\gamma} \Sigma_{\gamma\delta} S_{\delta\beta} f_{\beta j} + \dots \\ &= f_{i\alpha} S_{\alpha\beta} f_{\beta j} + f_{i\alpha} S_{\alpha\gamma} (i\pi \sum_k \rho_k f_{\gamma k} f_{k\delta}) S_{\delta\beta} f_{\beta j} + \dots = K_{ij} + i\pi \sum_k K_{ik} \rho_k K_{kj} + \dots, \end{aligned}$$

which is the iterated form of

$$T = K(1 - i\pi\rho K)^{-1}. \tag{7}$$

Eq. (7) is the standard expression for the T -matrix in terms of the K -matrix ⁹⁾, where^{††}

$$K_{ij} = f_{i\alpha} S_{\alpha\beta} f_{\beta j} = \sum_{\alpha} f_{i\alpha} \frac{1}{m_{\alpha} - m} f_{\alpha j}. \tag{8}$$

† I thank Professor A. N. Kamal for showing me this reference.

†† Corresponding to the results on pp. 416-417 of ref. ³⁾.

As before, with Hermitian potentials $K = K^\dagger$, and with time-reversal invariance K is real and symmetric. A common approximation in resonance theory is to take the $f_{i\alpha}$ as constant.

The physical meaning of (8) and of the states $\{\alpha, \beta, \dots\}$, is simple. If we put into the K -matrix a sum of poles at the masses of the resonance states after they have been mass-mixed (i.e. the masses of the states which diagonalise M), with coupling constants from the continuum channels to these mass-mixed states, then we get a T -matrix which is exactly equivalent to the potential theory one. This therefore sums up the inter-relationships between the three formalisms.

How many real parameters are needed to describe the resonances? Let there be N channels and R resonances. With Hermitian potentials and time-reversal invariance, M is a real symmetric matrix with $\frac{1}{2}R(R+1)$ parameters; Γ contains NR real parameters $f_{\alpha i}$, making $\frac{1}{2}R(R+1) + NR$ parameters in all. On the other hand, the K -matrix given by (8) contains Rm_α and $NRf_{\alpha i}$, or $R + NR$ parameters in all. This is the necessary number of parameters as found by Rosenfeld⁴⁾ and the reason (1) contains more is simply that the states $\bar{\alpha}$ are unphysical anyway, a unitary transformation of them being always allowed. The parametrisation of (1) is of course unitary, but it contains redundant parameters; (8) represents the minimal parametrisation.

As Stodolsky¹⁵⁾ has pointed out, the form of Σ can be traced directly to unitarity. Defining the S -matrix by

$$S_{ij} = \delta_{i\bar{k}} + 2\pi i \sqrt{\rho_i} T_{ij} \sqrt{\rho_j}, \quad (9)$$

the unitarity relation $SS^\dagger = I$ is

$$T - T^\dagger = 2\pi i T \rho T^\dagger, \quad (10)$$

which, if $T_{ij} = f_{i\alpha} S'_{\alpha\beta} f_{\beta j}$, can be reduced to

$$S'_{\alpha\beta}{}^{\dagger-1} - S'_{\alpha\beta}{}^{-1} = 2\pi i \sum_i f_{\alpha i} \rho_i f_{i\beta}. \quad (11)$$

Thus the antihermitian part of S' , namely Σ , is given by $\Sigma_{\alpha\beta} = i\pi \sum_i f_{\alpha i} \rho_i f_{i\beta}$ as above. Furthermore, writing (10) in the form $\text{Im } T^{-1} = -\pi\rho$ (assuming a symmetric T -matrix) we see that a unitary T -matrix can always be written in the form (7) provided only that K is real. Thus unitarity is the essential ingredient throughout.

The states of the system which have definite mass and lifetime are neither $\{\bar{\alpha}, \bar{\beta}, \dots\}$ nor $\{\alpha, \beta, \dots\}$, but rather the eigenstates $\{A, B, \dots\}$ of the full operator $M - \frac{1}{2}i\Gamma$:

$$(M - \frac{1}{2}i\Gamma)|A\rangle = m_A|A\rangle, \quad (12)$$

where m_A is complex. These are the states which may be identified with the physical particles, having a definite mass $\text{Re}(m_A)$ and lifetime $\tau^{-1} = -\frac{1}{2}\text{Im}(m_A)$. These masses m_A also occur, of course, as the poles of the T -matrix in the form (1) or (6), since $\det S'_{\alpha\beta} = \det S'_{\bar{\alpha}\bar{\beta}} = 0$ at the eigenvalues $m = m_A$. It would seem that McGlenn and Polis¹¹⁾ failed to distinguish clearly between the K -matrix states $|\alpha\rangle$ and the physical states $|A\rangle$. Their sum rule, though correct, relates to the unphysical states $|\alpha\rangle$ rather

than the physical ones $|A\rangle$. The useful sum rule of Bell and Steinberger ¹²⁾ is obtained very simply from (2):

$$(m_B^* - m_A)\langle B|A\rangle = i\langle B|\Gamma|A\rangle = 2\pi i \sum_i f_{Bi} \rho_i f_{iA}, \tag{13}$$

where

$$f_{Bi} = \sum_\alpha \langle B|\alpha\rangle f_{\alpha i}, \quad f_{iA} = \sum_\beta f_{i\beta} \langle \beta|A\rangle. \tag{14}$$

Eq. (13) holds independently of any symmetry restrictions on the f .

One should note that because $M - \frac{1}{2}i\Gamma$ is not hermitian, $\langle A| \neq (|A\rangle)^\dagger$. The transformation matrix $\langle B|\alpha\rangle$, which relates the K -matrix and T -matrix states, is therefore not unitary, though it is (complex) orthogonal. Rosenfeld ⁴⁾ has introduced and used this matrix. We note that in Rosenfeld's case the phase-space factors ρ_i are taken as constant and absorbed into the f ; with this difference, eq. (14) corresponds to his eq. (10). Also, in terms of the physical states, T may be written as

$$T_{ij} = \sum_A f_{iA} \frac{1}{m_A - m} f_{Aj} \equiv f_{iA} S'_{AB} f_{Bj}, \tag{15}$$

where $S'^{-1}_{AB} = (m_A - m)\delta_{AB}$. Comparing (6), and following, with (15) we find

$$m_A \delta_{AB} + \frac{1}{2}i \sum_i 2\pi \rho_i f_{Ai} f_{iB} = \langle A|\alpha\rangle m_\alpha \delta_{\alpha\beta} \langle \beta|B\rangle, \tag{16}$$

which corresponds to Rosenfeld's (9).

We have discussed so far the case of transitions between the same states i, j, \dots as occur in the definition of $\Gamma_{\bar{\alpha}\bar{\beta}}$ [eq. (3)]. It is simple to consider the slightly more general case in which, in eq. (1), $f_{\bar{\beta}j}$ is replaced by $f_{\bar{\beta}p}$ where p (the production channel) does not occur in the sum in $\Gamma_{\bar{\alpha}\bar{\beta}}$. For example, we might have the production process illustrated in fig. 2, where the production end of the propagator $S'_{\bar{\alpha}\bar{\beta}}$ is hooked on to a

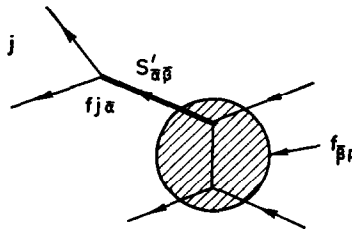


Fig. 2. Production process leading to state j , which is coupled via the overlapping resonances to the production mechanism.

state which is not in $\{i\}$; in particular, $f_{\bar{\beta}p}$ might be dependent on momentum transfer. This would then describe the problem of final-state interactions involving overlapping resonances ¹⁹⁾. We would then have, for the transition from production state p to final state j ,

$$F_{jp} = f_{j\bar{\alpha}} S'_{\bar{\alpha}\bar{\beta}} f_{\bar{\beta}p}. \tag{17}$$

For the case of two resonances, this gives the results of Lassilla and Ruuskanen ⁶), if width mixing effects (i.e. those due to off-diagonal terms in Γ) are neglected. Considering again the iterated form of (17), in the basis α, β, \dots , we find

$$F_{jp} = f_{j\alpha} S_{\alpha\beta} f_{\beta j} + i\pi \sum_k f_{j\alpha} S_{\alpha\gamma} f_{\gamma k} \rho_k f_{k\delta} S_{\delta\beta} f_{\beta p} + \dots,$$

which is the iterated form of

$$F_{jp} = P_{jp} + i\pi \sum_k K_{jk} \rho_k F_{kp}, \quad (18)$$

where K is still the elastic K -matrix, and P_{jp} is an analogous production K -vector with elements

$$P_{jp} = \sum_{\alpha} f_{j\alpha} \frac{1}{m_{\alpha} - m} f_{\alpha p}, \quad (19)$$

for fixed p^{\dagger} . As in standard K -matrix theory, it is simple to verify that (18) guarantees the correct coupled unitarity relations among the two body channels, i, j, \dots , if P as well as K have no normal threshold branch points. We can also write (18) as

$$F = (1 - i\pi K\rho)^{-1} P, \quad (20)$$

demonstrating that the same physical poles will of course occur in this problem as in the previous one. If the production coupling $f_{\alpha p}$ is actually proportional to the coupling to a particular state of $\{i\}$ then P is proportional to the appropriate K -matrix element, and F to the corresponding T -matrix element. This would be the case, for instance, if in fig. 2 the incident and exchanged particle formed a state k included in $\{i\}$, and if all energy and momentum-transfer dependences were ignored.

In practice, of course, the resonances are always accompanied by background. This is easy to incorporate into the K -matrix framework, since constant terms can always be added to the poles in the K -matrix (or P -vector) elements, without destroying unitarity of the T -matrix (or F -vector). One would then have a simple, general, unitary parametrization.

A final point concerns the application of symmetries, such as $SU(3)$ in the strong interaction case. Normally, in coupled-channel problems, it is reckoned reasonable to put in $SU(3)$ symmetric coupling constants and masses into the K -matrix, while the phase-space factors ρ_i are evaluated using the physical (split) masses ²²). However, in the present case involving mass-mixing effects, distortions from symmetry can be very large if the unmixed states are initially nearly degenerate. It is much more reasonable to assume that the unmixed states $\bar{\alpha}, \bar{\beta}, \dots$ are the ones which obey symmetry relations. In fact, to assume that it is the K -matrix couplings $f_{\alpha i}$ and masses m_{α}

[†] Note that P has poles — in fact, the same ones as K . If P in (20) is taken to be simply a constant, unwanted zeros will appear in the numerator of F in (20) at the masses m_{α} . This difficulty was discovered by Graves-Morris ²⁰) using the analytic S -matrix approach to the f.s.i. problem, given by Landshoff and Olive ²¹). It would seem that the potential theory connection established here shows that (19) is the right way to deal with the difficulty.

which obey symmetry relations is equivalent to assuming that the unmixed couplings $f_{\bar{a}i}$ and masses $m_{\bar{a}}$ are symmetric, and that there is no direct coupling between the states $\bar{\alpha}$.

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