Continuous and Discrete (Flavor) Symmetries

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Feruglio et al. (’07), H/Serone (’11)
Blum/H/Lindner (’07), H/Ziegler (’10), Blum/H (’09), H/Meloni (to appear)
Outline

• Observations

• General comments on the flavor symmetry $G_F$ properties, possible symmetries

• Examples
  • $A_4$ and $S_4$: tri-bimaximal mixing
  • Dihedral groups in general, $\theta_C$ and $\mu\tau$ symmetry

• Conclusions
# Observations: Charged Fermion Masses

<table>
<thead>
<tr>
<th>Mass at $M_Z$</th>
<th>in units of $m_t(M_Z)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$u$</td>
<td>$(1.7 \pm 0.4)$ MeV</td>
</tr>
<tr>
<td>$c$</td>
<td>$(0.62 \pm 0.03)$ GeV</td>
</tr>
<tr>
<td>$t$</td>
<td>$(171 \pm 3)$ GeV</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Mass at $M_Z$</th>
<th>in units of $m_b(M_Z)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$d$</td>
<td>$(3.0 \pm 0.6)$ MeV</td>
</tr>
<tr>
<td>$s$</td>
<td>$(54 \pm 8)$ MeV</td>
</tr>
<tr>
<td>$b$</td>
<td>$(2.87 \pm 0.03)$ GeV</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Mass at $M_Z$</th>
<th>in units of $m_{\tau}(M_Z)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$e$</td>
<td>$(0.486570161 \pm 0.000000042)$ MeV</td>
</tr>
<tr>
<td>$\mu$</td>
<td>$(102.7181359 \pm 0.0000092)$ MeV</td>
</tr>
<tr>
<td>$\tau$</td>
<td>$1.74624^{+0.00020}_{-0.00019}$ GeV</td>
</tr>
</tbody>
</table>
Observations: Neutrino Masses

• Mass hierarchy in $\nu$ sector is mild compared to charged fermions

• We know two mass squared differences (at $2\sigma$)
  
  \[
  \delta m^2 \equiv \Delta m^2_{\text{sol}} \equiv m_2^2 - m_1^2 = (7.58^{+0.41}_{-0.42}) \times 10^{-5} \text{ eV}^2
  \]
  
  \[
  \Delta m^2 \equiv m_3^2 - \frac{m_2^2 + m_1^2}{2} = (2.35^{+0.22}_{-0.18}) \times 10^{-3} \text{ eV}^2
  \]

• However $m_1 < m_3$ and $m_3 < m_1$ is possible due to $\Delta m^2 \leq 0$
Observations: Neutrino Masses

• Mass hierarchy in $\nu$ sector is mild compared to charged fermions

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  (Fogli et al. ('11))

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• However $m_1 < m_3$ and $m_3 < m_1$ is possible due to $\Delta m^2 \leq 0$

• Also unknown: the absolute mass scale $m_0$

\[
\sum m_\nu \leq 0.7 \text{ eV} \quad \text{(WMAP+LSS)}
\]

\[
m_\beta < 2.2 \text{ eV} \quad \text{(Mainz)}
\]

\[
|m_{ee}| < (0.2...1) \text{ eV} \quad \text{(Heidelberg-Moscow, IGEX, NEMO)}
\]
Observations: Fermion Masses and Mixings

- The mixing pattern is very peculiar \((\text{Fogli et al. ('11)})\)

\[
\sin^2(\theta_{l12}) = 0.306^{+0.036}_{-0.031}, \quad \sin^2(\theta_{l23}) = 0.42^{+0.18}_{-0.06} \quad \text{and} \quad \sin^2(\theta_{l13}) = 0.021^{+0.015}_{-0.013}
\]

\[
\theta_{l12} = (33.6^{+2.2}_{-2.0})^\circ, \quad \theta_{l23} = (40.4^{+10.4}_{-3.5})^\circ \quad \text{and} \quad \theta_{l13} = (8.3^{+2.6}_{-3.2})^\circ \quad (2 \sigma)
\]

compare to quark sector

\[
\theta_{q12} \approx 13^\circ, \quad \theta_{q23} \approx 2.4^\circ \quad \text{and} \quad \theta_{q13} \approx 0.21^\circ
\]
Observations: Fermion Masses and Mixings

• The mixing pattern is very peculiar \( (\text{Fogli et al. ('11)}) \)

\[
\sin^2(\theta_{12}^l) = 0.306^{+0.036}_{-0.031} \quad \sin^2(\theta_{23}^l) = 0.42^{+0.18}_{-0.06} \quad \text{and} \quad \sin^2(\theta_{13}^l) = 0.021^{+0.015}_{-0.013}
\]

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\]

compare to quark sector

\[
\theta_{12}^q \approx 13^\circ, \quad \theta_{23}^q \approx 2.4^\circ \quad \text{and} \quad \theta_{13}^q \approx 0.21^\circ
\]

(Still) Hint at a Flavor Symmetry !?!
Special Patterns for Lepton Mixings - Before T2K

• \( \mu\tau \) Symmetry

\[
\sin^2(\theta^l_{23}) = \frac{1}{2}, \quad \sin^2(\theta^l_{13}) = 0
\]

\[
|U_{PMNS}| = \begin{pmatrix}
\cos(\theta^l_{12}) & \sin(\theta^l_{12}) & 0 \\
\sin(\theta^l_{12}) & \cos(\theta^l_{12}) & 0 \\
\sqrt{2} & \sqrt{2} & 1 \\
\sqrt{2} & \sqrt{2} & 1 \\
\end{pmatrix}
\]

• Tri-Bimaximal mixing (TB mixing)

\[
\sin^2(\theta^l_{12}) = \frac{1}{3}, \quad \sin^2(\theta^l_{23}) = \frac{1}{2}, \quad \sin^2(\theta^l_{13}) = 0
\]

\[
|U_{PMNS}| = \begin{pmatrix}
\frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} & 0 \\
\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \\
\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \\
\end{pmatrix}
\]
Special Patterns for Lepton Mixings -Before T2K

Further patterns which might be realized

- Golden ratio \( (\phi = \frac{1}{2} (1 + \sqrt{5})) \)

\[
\sin^2(\theta_{12}) = \frac{1}{\sqrt{5}\phi} \approx 0.276, \quad \sin^2(\theta_{23}) = \frac{1}{2}, \quad \sin^2(\theta_{13}) = 0
\]

\[
|U_{PMNS}| = \begin{pmatrix}
\sqrt{\frac{1}{10}(5 + \sqrt{5})} & \sqrt{\frac{2}{5+\sqrt{5}}} & 0 \\
\frac{1}{\sqrt{5+\sqrt{5}}} & \sqrt{\frac{1}{20}(5 + \sqrt{5})} & \frac{1}{\sqrt{2}} \\
\frac{1}{\sqrt{5+\sqrt{5}}} & \sqrt{\frac{1}{20}(5 + \sqrt{5})} & \frac{1}{\sqrt{2}}
\end{pmatrix}
\]

Are these still reasonable Leading Order Patterns ?!
Summary of Observations

• Three families of elementary particles observed
• Strong hierarchy among charged fermions
• Mass hierarchy in the $\nu$ sector is much milder, ordering and $m_0$ unknown
• Only $\theta_C$ is sizable in the quark sector
• Special lepton mixing pattern could be realized? Which one?
• No excessive flavor violation observed, all in accordance with Standard Model

\[\downarrow\]

Necessity of Constraints on Couplings $y_{ij}^u, y_{ij}^l$, etc.
Summary of Observations

- Three families of elementary particles observed
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\[ \downarrow \]

| Necessity of Flavor Symmetry $G_F$ ?! |
Effect of Flavor Symmetry $G_F$

- Yukawa couplings in the SM
  \[ y_{ij}^u Q_i H^c u_j^c \quad \text{or} \quad y_{ij}^d Q_i H d_j^c \]
  with $y_{ij}^{u,d} \in \mathbb{C}$

- Enforce invariance under $G_F$
  $\Leftrightarrow$ Constraints on $y_{ij}^{u,d}$
  $\Leftrightarrow$ Extension of scalar sector needed

\[
H \rightarrow H_k \quad \text{or} \quad H \rightarrow H \phi_k/(M, \Lambda)
\]

multi-Higgs doublets \quad or \quad flavon fields

\[
y_{ij,k}^d Q_i H_k d_j^c \quad \text{or} \quad y_{ij,k}^d Q_i H d_j^c \left( \frac{\phi_k}{(M, \Lambda)} \right)
\]

renormalizable couplings \quad or \quad in general non-renormalizable

but in both cases alignment is needed
Choice of Flavor Symmetry $G_F$

The symmetry $G_F$ could ...

• ... be abelian or non-abelian
Choice of Flavor Symmetry $G_F$

The symmetry $G_F$ could ...

- ... be abelian or non-abelian
- ... be continuous or discrete
Choice of Flavor Symmetry $G_F$

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- ... be spontaneously broken or explicitly
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- ... be broken at low or high energies  
  (low: electroweak scale)  
  (high: seesaw/GUT scale)
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- ... be broken at low or high energies  
  (low: electroweak scale)  
  (high: seesaw/GUT scale)

Its maximal possible size depends on the gauge group.
Possible Symmetries

- **Continuous Groups**

  \[ SU(2), \ U(2), \ SO(3), \ SU(3), \ U(3) \]

- **Discrete Groups**

  - Permutation symmetries, \( S_N \) and \( A_N \) with \( N \in \mathbb{N} \)
  - Dihedral symmetries, \( D_n \) and \( D'_n \) with \( n \in \mathbb{N} \)
  - Further double-valued groups: \( T', \ O', \ I', \ ... \)
  - Subgroups of \( SU(3) \), series of \( \Delta(3n^2) \) and \( \Delta(6n^2) \) groups with \( n \in \mathbb{N} \), as well as finite number of \( \Sigma \) groups
  - Additional groups such as subgroups of the mentioned groups, e.g. \( T_7 \cong Z_7 \rtimes Z_3 \subset \Delta(147) \), subgroups of \( U(3) \), e.g. \( \Sigma(81) \)
Models with Continuous $G_F$

The group $U(2)$ (Barbieri/Hall (’96), Barbieri et al. (’96, ’97, ’99))

- $2 + 1$ structure explains heaviness of the third generation
- Framework of the models: supersymmetric $SU(5) (SO(10))$ GUT
- Symmetry is broken in steps: $U(2) \rightarrow U(1) \rightarrow$ nothing
- $2 + 1$ also appropriate for solving the SUSY flavor problem
- Nine relations among fermion masses and mixings
- However, only $\theta_{23}$ large in lepton sector and $\theta_{12}, \theta_{13}$ small
Models with Continuous $G_F$

The groups $SO(3)$ and $SU(3)$ are promising, ...

(King ('05), King/Ross ('01, '03), de Medeiros Varzielas/Ross ('05))

- ... since they allow to unify all three generations (in $SO(3)$ only left-handed ones)
- ... since the largeness of two mixing angles in the lepton sector indicates that not only two generations are closely related
Models with Continuous $G_F$

General characteristics of $SO(3)$ & $SU(3)$ flavor models

- Hierarchical breaking
  $$SO(3) \to SO(2) \to \text{nothing} \quad \text{and} \quad SU(3) \to SU(2) \to \text{nothing}$$
- SUSY models with SM, PS or (in EDs) $SO(10)$ gauge group
- Masses of SM fermions arise from non-renorm. operators only
- Additionally symmetries ($Z_n$ factors and $U(1)$) to forbid operators
- Different messenger scales allow different expansion parameters in up quark, down quark and charged lepton sector
- Specific vacuum structure necessary to predict TB mixing
- Vacuum alignment achieved by $F$- or $D$-terms
Models with Continuous $G_F$

Lepton mixing in the $SO(3) \& SU(3)$ flavor models

- TB mixing is achieved through CSD \cite{King ('05)}
- Corrections from charged lepton sector are small

Discrete “spin-offs” of the $SO(3) \& SU(3)$ models \cite{de Medeiros Varzielas et al. ('05, '06), King/Malinsky ('06)}

- Use $A_4$, $\Delta(27)$, $\Delta(108)$ as flavor group
- Very similar results compared to the models with continuous symmetries
- In general the vacuum alignment is simpler
- No $D$-terms arise from discrete flavor symmetry
**Possible Discrete Symmetries**

- Permutation symmetries: symmetric groups $S_N$ and alternating groups $A_N$ with $N \in \mathbb{N}$

- Dihedral symmetries: single-valued groups $D_n$ and double-valued groups $D'_n$ with $n \in \mathbb{N}$

- Further double-valued groups: $T'$, $O'$, $I'$, ...

- Subgroups of $SU(3)$: series of $\Delta(3n^2)$ and $\Delta(6n^2)$ groups with $n \in \mathbb{N}$, as well as finite number of $\Sigma$ groups

- Subgroups of $U(3)$ such as $\Sigma(81)$ and subgroups of the listed groups such as $T_7 \cong Z_7 \rtimes Z_3 \subset \Delta(147)$
TB Mixing from Non-Trivial $A_4$ Breaking

(Altarelli/Feruglio ('05), de Medeiros Varzielas et al. ('05), He et al. ('06))

- $G_F = A_4$ spontaneously broken at high energies
- Low energy effective theory: MSSM
- Breaking of $G_F$ is induced by VEVs of flavon fields which are singlets under the SM gauge group
- (MS)SM fermions transform non-trivially under $G_F$
- MSSM Higgs doublets $h_{u,d}$ are singlets under $G_F$
- Elaborate construction of scalar sector needed to ensure vacuum alignment
### Particle Content

#### LEPTONS

<table>
<thead>
<tr>
<th>Field</th>
<th>$L$</th>
<th>$e_L^c$</th>
<th>$\mu_L^c$</th>
<th>$\tau_L^c$</th>
<th>$\phi_T$</th>
<th>$\phi_S$</th>
<th>$\xi, \bar{\xi}$</th>
<th>$\theta$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A_4$</td>
<td>3</td>
<td>1</td>
<td>1''</td>
<td>1'</td>
<td>3</td>
<td>3</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>$Z_3$</td>
<td>$\omega$</td>
<td>$\omega^2$</td>
<td>$\omega^2$</td>
<td>$\omega^2$</td>
<td>1</td>
<td>$\omega$</td>
<td>$\omega$</td>
<td>1</td>
</tr>
<tr>
<td>$U(1)_{FN}$</td>
<td>0</td>
<td>2</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>$-1$</td>
</tr>
</tbody>
</table>

Additionally needed

- $Z_3$ symmetry to separate charged lepton and neutrino sector
- $U(1)_{FN}$ for hierarchy $m_e \ll m_\mu \ll m_\tau$
**Group Theory of $A_4$**

- The group $A_4$ is the symmetry group of a regular tetrahedron, group of even permutations of four objects
- Order of the group is 12
- Irred reps are 1, 1', 1'' and 3
- Kronecker products

\[
\begin{align*}
1 \times \mu &= \mu \quad \forall \quad \mu, \\
1' \times 1' &= 1'', \quad 1'' \times 1'' = 1', \quad 1' \times 1'' = 1 \\
1 \times 3 &= 1' \times 3 = 1'' \times 3 = 3 \\
3 \times 3 &= 1 + 1' + 1'' + 3 + 3
\end{align*}
\]
Group Theory of $A_4$

- The group $A_4$ is the symmetry group of a regular tetrahedron, group of even permutations of four objects
- Order of the group is 12
- Irred reps are $1$, $1'$, $1''$ and $3$
- Generator relations $S^2 = 1$, $T^3 = 1$, $(ST)^3 = 1$

<table>
<thead>
<tr>
<th>rep.</th>
<th>$S$</th>
<th>$T$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>1'</td>
<td>1</td>
<td>$\omega^2$</td>
</tr>
<tr>
<td>1''</td>
<td>1</td>
<td>$\omega$</td>
</tr>
<tr>
<td>3</td>
<td>$\frac{1}{3}$ $\begin{pmatrix} -1 &amp; 2 &amp; 2 \ 2 &amp; -1 &amp; 2 \ 2 &amp; 2 &amp; -1 \end{pmatrix}$</td>
<td>$\begin{pmatrix} 1 &amp; 0 &amp; 0 \ 0 &amp; \omega^2 &amp; 0 \ 0 &amp; 0 &amp; \omega \end{pmatrix}$</td>
</tr>
</tbody>
</table>

$(\omega = e^{\frac{2\pi i}{3}})$

(Altarelli/Feruglio '05), 2nd paper)
Fermion Masses at LO

- Assume vacuum

\[ G_S : \langle \varphi_S \rangle = (v_S, v_S, v_S), \quad \langle \xi \rangle = u, \quad \langle \tilde{\xi} \rangle = 0, \]

\[ G_T : \quad \langle \varphi_T \rangle = (v_T, 0, 0). \]

- Charged lepton sector

\[ M_l = \frac{v_T}{\Lambda} v_d \text{ diag} \left( y_e \frac{\langle \theta \rangle^2}{\Lambda^2}, y_\mu \frac{\langle \theta \rangle}{\Lambda}, y_\tau \right) \]

- Neutrino sector

\[ M_\nu = \frac{v_\nu^2}{\Lambda} \begin{pmatrix}
    a + 2b/3 & -b/3 & -b/3 \\
    -b/3 & 2b/3 & a - b/3 \\
    -b/3 & a - b/3 & 2b/3
\end{pmatrix} \Rightarrow \text{TB mixing!} \]

with \( \frac{v_\nu^2}{\Lambda} \text{ diag}(a + b, a, -a + b) \) and \( a = x_a \frac{u}{\Lambda}, \quad b = x_b \frac{v_S}{\Lambda} \).
**Non-Trivial $A_4$ Breaking**

- Observation
  
  $\langle \varphi_S \rangle = (v_S, v_S, v_S)$, $\langle \xi \rangle = u$ breaks $A_4 \to G_S \cong Z_2$ in $\nu$ sector
  
  $\langle \varphi_T \rangle = (v_T, 0, 0)$ breaks $A_4 \to G_T \cong Z_3$ in $l$ sector

  $A_4$ completely broken in the whole theory
**Non-Trivial $A_4$ Breaking**

- Observation

$$\langle \varphi_S \rangle = (v_S, v_S, v_S), \quad \langle \xi \rangle = u$$

breaks $A_4 \to G_S \cong Z_2$ in $\nu$ sector

$$\langle \varphi_T \rangle = (v_T, 0, 0)$$

breaks $A_4 \to G_T \cong Z_3$ in $l$ sector

$A_4$ completely broken in the whole theory

In detail: $Z_2$ symmetry is generated by the element $S$ of $A_4$

Since for $1$, $1'$ and $1''$ $S = 1$, VEV $\langle \xi \rangle$ preserves $S$.

For $\langle \varphi_S \rangle$ the vector has to fulfill

$$\langle \varphi_S \rangle \propto v_{+1} \quad \text{with} \quad S v_{+1} = +1 v_{+1}$$

$$\begin{pmatrix}
-1 & 2 & 2 \\
2 & -1 & 2 \\
2 & 2 & -1
\end{pmatrix}
\begin{pmatrix}
1 \\
1 \\
1
\end{pmatrix} = 
\begin{pmatrix}
1 \\
1 \\
1
\end{pmatrix}$$

i.e. \( \frac{1}{3} \)
Non-Trivial $A_4$ Breaking

• Observation

$\langle \varphi_S \rangle = (v_S, v_S, v_S)$, $\langle \xi \rangle = u$ breaks $A_4 \rightarrow G_S \cong Z_2$ in $\nu$ sector

$\langle \varphi_T \rangle = (v_T, 0, 0)$ breaks $A_4 \rightarrow G_T \cong Z_3$ in $l$ sector

$A_4$ completely broken in the whole theory

In detail: $Z_3$ symmetry is generated by the element $T$ of $A_4$

For $\langle \varphi_T \rangle$ the vector has to fulfill

$\langle \varphi_T \rangle \propto v_+ \quad \text{with} \quad Tv_+ = +1 v_+$

i.e. \[
\begin{pmatrix}
1 & 0 & 0 \\
0 & \omega^2 & 0 \\
0 & 0 & \omega
\end{pmatrix}
\begin{pmatrix}
1 \\
0 \\
0
\end{pmatrix} = 
\begin{pmatrix}
1 \\
0 \\
0
\end{pmatrix}
\]
Non-Trivial $A_4$ Breaking

- Observation

\[
\langle \varphi_S \rangle = (v_S, v_S, v_S), \quad \langle \xi \rangle = u \quad \text{breaks} \quad A_4 \rightarrow G_S \cong Z_2 \quad \text{in } \nu \text{ sector}
\]

\[
\langle \varphi_T \rangle = (v_T, 0, 0) \quad \text{breaks} \quad A_4 \rightarrow G_T \cong Z_3 \quad \text{in } l \text{ sector}
\]

$A_4$ completely broken in the whole theory

<table>
<thead>
<tr>
<th>$A_4$</th>
<th>$D_2$</th>
<th>$Z_3$</th>
<th>$Z_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>1'</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>1''</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>3</td>
<td>1</td>
<td>1 +...</td>
<td>1 +...</td>
</tr>
</tbody>
</table>

with 1 being a total singlet of the subgroup
Non-Trivial $A_4$ Breaking

- Observation

\[ \langle \varphi_S \rangle = (v_S, v_S, v_S), \quad \langle \xi \rangle = u \]
breaks \quad \begin{align*}
A_4 & \rightarrow G_S \cong Z_2 \\
& \text{in } \nu \text{ sector}
\end{align*}

\[ \langle \varphi_T \rangle = (v_T, 0, 0) \]
breaks \quad \begin{align*}
A_4 & \rightarrow G_T \cong Z_3 \\
& \text{in } l \text{ sector}
\end{align*}

$A_4$ completely broken in the whole theory

Mismatch of different subgroups generates non-trivial (large) mixing and even more predicts exact values of mixing angles (TB mixing) (independent of choice of parameters, apart from order of eigenvalues)

\[ \Rightarrow \]

Interpretation of Mixing

Note:
In the $T'$ extension of this model small quark mixings are explained through the breaking to the same subgroup in up and down quark sector.

*(Feruglio et al. ('07))
Is $S_4$ better for TB Mixing? 

(Lam ('06,'07,'08))

- TB mixing only results from $A_4$, if two flavons $\xi', \xi''$ in 1' and 1'' are not present in the neutrino sector
- However: $\langle \xi' \rangle \neq 0, \langle \xi'' \rangle \neq 0$ leave $Z_2 \subset A_4$ invariant
- Matrix $M_\nu$

$$M_\nu = \frac{v^2_u}{\Lambda} \begin{pmatrix}
    a + \frac{2b}{3} & -\frac{b}{3} & -\frac{b}{3} \\
    -\frac{b}{3} & \frac{2b}{3} & a - \frac{b}{3} \\
    -\frac{b}{3} & a - \frac{b}{3} & \frac{2b}{3}
\end{pmatrix}$$

is not only invariant under $S$

$$S = \frac{1}{3} \begin{pmatrix}
    -1 & 2 & 2 \\
    2 & -1 & 2 \\
    2 & 2 & -1
\end{pmatrix}, \text{ i.e. } S^T M_\nu S = M_\nu$$
**Is $S_4$ better for TB Mixing?**

- TB mixing only results from $A_4$, if two flavons $\xi', \xi''$ in $1'$ and $1''$ are not present in the neutrino sector.
- However: $\langle \xi' \rangle \neq 0, \langle \xi'' \rangle \neq 0$ leave $Z_2 \subset A_4$ invariant.
- Matrix $M_\nu$

$$
M_\nu = \frac{v^2_u}{\Lambda} \begin{pmatrix}
 a + 2b/3 & -b/3 & -b/3 \\
 -b/3 & 2b/3 & a - b/3 \\
 -b/3 & a - b/3 & 2b/3
\end{pmatrix}
$$

- but also under $P_{23}$

$$
P_{23} = \begin{pmatrix}
 1 & 0 & 0 \\
 0 & 0 & 1 \\
 0 & 1 & 0
\end{pmatrix}, \quad \text{i.e.} \quad P_{23}^T M_\nu P_{23} = M_\nu
$$
Is $S_4$ better for TB Mixing?

- TB mixing only results from $A_4$, if two flavons $\xi', \xi''$ in $1'$ and $1''$ are not present in the neutrino sector.
- However: $\langle \xi' \rangle \neq 0, \langle \xi'' \rangle \neq 0$ leave $Z_2 \subset A_4$ invariant.
- Matrix $M_\nu$ is invariant under $S$ and $P_{23}$.

\[
\downarrow \quad M_\nu \text{ is invariant under } Z_2 \times Z_2
\]

But only $S$ is an element of $A_4$ and not $P_{23}$.

\[
\downarrow \quad \text{The actual symmetry must be larger: } S_4
\]

- $S_4 \rightarrow Z_2 \times Z_2$ in $\nu$ sector
- $S_4 \rightarrow Z_3$ in $l$ sector
- $S_4 \text{ completely broken in the whole theory}$
**Probably the best is** $S_4 \times Z_3$  

**(H/Serone ('11))**

**Problem:** Whether the charged lepton masses can still be *naturally* generated by, e.g., a $U(1)_{FN}$, is not obvious.

**Solution:**

$S_4 \times Z_3$

The additional $Z_3$ factor solves the problem of the charged lepton mass hierarchy, because now the $S_4$ breaking is

$$S_4 \times Z_3 \to Z_2 \times Z_2 \times Z_3 \quad \text{in } \nu \text{ sector}$$

$$S_4 \times Z_3 \to Z_3^{(D)} \quad \text{in } l \text{ sector}$$

$S_4 \times Z_3$ completely broken in the whole theory

**Details:**

<table>
<thead>
<tr>
<th>Field</th>
<th>$L$</th>
<th>$e_R$</th>
<th>$\mu_R$</th>
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<th>$\nu_R$</th>
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<td>1</td>
<td>$\omega^2$</td>
<td>$\omega$</td>
<td>1</td>
</tr>
</tbody>
</table>
\( S_4 \times Z_3 \rightarrow Z_3^{(D)} \)

- Take the generator \( T \) of \( S_4 \) for \( 3 \)

\[
T = \begin{pmatrix}
1 & 0 & 0 \\
0 & \omega^2 & 0 \\
0 & 0 & \omega
\end{pmatrix}
\]

and break it together with \( Z_3 \) to their diagonal subgroup \( Z_3^{(D)} \)

- Leptons transform under the remnant \( Z_3^{(D)} \) as

\[
L_1 \sim 1 \quad L_2 \sim \omega^2 \quad L_3 \sim \omega \\
e_R \sim 1 \quad \mu_R \sim \omega^2 \quad \tau_R \sim \omega
\]

- As consequence, the charged lepton mass matrix is diagonal.
Should we throw this away in light of T2K?

- In all models TB mixing holds only at LO and gets corrected.
- For protecting $\theta_{12}$ corrections should be $\delta \lesssim 0.05$.
- In most models (but not all!) then also $\sin \theta_{13} \sim \delta \approx 0.05$.
- However, usually

\[
\sin \theta_{13} \approx c \times \delta
\]

with $c$ a complex number. $c$ largish still saves most models.

Early exception: Lin ('09)

- If you do not like this, maybe you like models with bimaximal mixing at LO, because then $\delta \sim 0.2$ also for $\theta_{12}$, but tricks needed for keeping $\delta_{\theta_{23}}$ smaller. (Altarelli/Feruglio/Merlo ('09), Meloni ('11))

- T2K signal has to be confirmed by others/more statistics!
Dihedral Symmetries: Idea of the Prediction

It has been shown in a study on the general properties of dihedral groups used as flavor symmetries that in case of a non-trivial breaking one element of the mixing matrix is given only in terms of group theoretical quantities. \textit{(Blum et al. (’07))}

The element of the mixing matrix $|V_{\alpha\beta}|$ is

$$|V_{\alpha\beta}| = \left| \cos \left( \pi \left( \frac{k_1 - k_2}{n} \right) j \right) \right|$$

for

- $n$ being the index of the dihedral group $D_n$
- $j$ being the index of the representation $2j$ under which two of the three generations of left-handed fermion fields transform
- $k_1$ and $k_2$ being two integers which specify the different subgroups $Z_2 = \langle B A^{k_{1,2}} \rangle$ preserved in the two different sectors
Dihedral Symmetries: Idea of the Prediction

Step 1:
Take a dihedral group $D_n$ generated by $A$ and $B$ and break it to $Z_2 = \langle BA^{k_1} \rangle$ in fermion sector 1 and to $Z_2 = \langle BA^{k_2} \rangle$ in fermion sector 2

Step 2:
Assign left-handed fields $L \sim 1_s + 2_j$ and right-handed ones are either in singlet representations $L^c \sim 1_i^p$ or also $L^c \sim 1_l + 2_m$

Step 3:
Consider a generic model in which you have two sets of scalars $\Phi_1$ and $\Phi_2$ (for fermion sector 1 and sector 2) in all representations $\mu$ of $D_n$ and allow non-zero VEVs only for those which leave invariant $Z_2 = \langle BA^{k_1} \rangle$ and $Z_2 = \langle BA^{k_2} \rangle$, respectively, i.e.

sector 1 : $D_n \langle \Phi_1 \rangle \rightarrow Z_2 = \langle BA^{k_1} \rangle$
sector 2 : $D_n \langle \Phi_2 \rangle \rightarrow Z_2 = \langle BA^{k_2} \rangle$
Dihedral Symmetries: Idea of the Prediction

Step 4:
Mass matrices $M_i$ have the form ($i = 1, 2$)

$$M_i = \begin{pmatrix} 0 & A_i & B_i \\ C_i & D_i & E_i \\ -C_i e^{-i\varphi_{ij}} & D_i e^{-i\varphi_{ij}} & E_i e^{-i\varphi_{ij}} \end{pmatrix}$$

with $\varphi_i = \frac{2\pi k_i}{n}$
Step 4:
Mass matrices $\mathcal{M}_i$ have the form ($i = 1, 2$)

$$\mathcal{M}_i = \begin{pmatrix}
A_i & C_i & C_i e^{-i\varphi_i m} \\
B_i & D_i & E_i \\
B_i e^{-i\varphi_i j} & E_i e^{-i\varphi_i (j-m)} & D_i e^{-i\varphi_i (j+m)}
\end{pmatrix}$$

with $\varphi_i = \frac{2\pi k_i}{n}$
Dihedral Symmetries: Idea of the Prediction

Step 4:
Mass matrices $M_i$ have the form ($i = 1, 2$)

$$M_i = \begin{pmatrix}
A_i & C_i & C_i e^{-i\varphi_i m} \\
B_i & D_i & E_i \\
B_i e^{-i\varphi_{ij}} & E_i e^{-i\varphi_i (j-m)} & D_i e^{-i\varphi_i (j+m)}
\end{pmatrix}$$

with $\varphi_i = \frac{2\pi k_i}{n}$

Step 5:
Such matrices have $M_i M_i^\dagger$ of the form

$$M_i M_i^\dagger = \begin{pmatrix}
a_i & b_i e^{i\zeta_i} & b_i e^{i(\zeta_i+\varphi_{ij})} \\
b_i e^{-i\zeta_i} & c_i & d_i e^{i\varphi_{ij}} \\
b_i e^{-i(\zeta_i+\varphi_{ij})} & d_i e^{-i\varphi_{ij}} & c_i
\end{pmatrix}$$
Dihedral Symmetries: Idea of the Prediction

Step 6:
Mixing matrices $U_i$ are

$$U_i = \begin{pmatrix}
0 & \cos \theta_i e^{i\xi_i} & \sin \theta_i e^{i\xi_i} \\
-\frac{1}{\sqrt{2}} e^{i\varphi_{ij}} & -\frac{\sin \theta_i}{\sqrt{2}} & \frac{\cos \theta_i}{\sqrt{2}} \\
\frac{1}{\sqrt{2}} & -\frac{\sin \theta_i}{\sqrt{2}} e^{-i\varphi_{ij}} & \frac{\cos \theta_i}{\sqrt{2}} e^{-i\varphi_{ij}}
\end{pmatrix} \quad \text{with} \quad \varphi_i = \frac{2\pi k_i}{n}$$

Step 7:
Physical mixings arise from $V = U_1^t U_2^*$. You can check that one element is of the form

$$|V_{\alpha\beta}| = \frac{1}{2} \left| 1 + e^{i(\varphi_1 - \varphi_2)j} \right| = \left| \cos \left( (\varphi_1 - \varphi_2) \frac{j}{2} \right) \right| = \left| \cos \left( \frac{\pi (k_1 - k_2) j}{n} \right) \right|$$

Rest of elements depends also on $\theta_{1,2}$ and $\xi_{1,2}$. 
**$D_4$ Model for $\mu \tau$ Symmetry**

It has been shown in a study on the general properties of dihedral groups used as flavor symmetries that in case of a non-trivial breaking one element of the mixing matrix is given only in terms of group theoretical quantities.

In this model this is true for the PMNS matrix element $|U_{\mu 3}|$ which is

$$|U_{\mu 3}| = \left| \cos \left( \frac{\pi ((0, 2) - (1, 3))}{4} \right) \right| = \left| \cos \left( \frac{\pi}{4} \right) \right| = 1/\sqrt{2}$$

$n = 4$, since $G_F$ contains $D_4$

$j = 1$, since $L_D \sim 2$

$k_1 = 0, 2$: $Z_2 = \langle B \rangle$ or $Z_2 = \langle B A^2 \rangle$ in neutrino sector

$k_2 = 1, 3$: $Z_2 = \langle B A \rangle$ or $Z_2 = \langle B A^3 \rangle$ in charged lepton sector
\textbf{$D_{14}$ Model: Idea of the Prediction (Blum/H ('09))}

It has been shown in a study on the general properties of dihedral groups used as flavor symmetries that in case of a non-trivial breaking one element of the mixing matrix is given only in terms of group theoretical quantities.

In this model this is true for the CKM matrix element $|V_{ud}|$ which is

$$|V_{ud}| = \left| \cos \left( \pi \left( \frac{0 - 1}{14} \right) \right) \right| = \left| \cos \left( \frac{\pi}{14} \right) \right| \approx 0.97493$$

for

$n = 14$, since $G_F$ contains $D_{14}$

$j = 1$, since $Q_D \sim 2_1$

$k_1 = 0$ and $k_2 = 1$ or $k_2 = 13$ so that $Z_2 = \langle B \rangle$ is conserved in the up and $Z_2 = \langle B A \rangle$ or $Z_2 = \langle B A^{13} \rangle$ in the down quark sector
Group Theory of $D_{14}$

- $D_{14}$ belongs to the dihedral groups and is the symmetry group of a regular planar 14-gon
- Its order is 28, i.e. it has 28 distinct elements
- It has 4 + 6 real irreducible reps., $1_j$, $i = 1, \ldots, 4$ and $2_j$, $j = 1, \ldots, 6$
- Generator relations of $D_{14}$

\[
A^{14} = 1, \quad B^2 = 1, \quad ABA = B
\]

- Generators

\begin{align*}
1_1 & : \quad A = 1, \quad B = 1 \\
1_2 & : \quad A = 1, \quad B = -1 \\
1_3 & : \quad A = -1, \quad B = 1 \\
1_4 & : \quad A = -1, \quad B = -1
\end{align*}
**Group Theory of $D_{14}$**

- $D_{14}$ belongs to the dihedral groups and is the symmetry group of a regular planar 14-gon
- Its order is 28, i.e. it has 28 distinct elements
- It has $4 + 6$ real irreducible representations, $1_i$, $i = 1, \ldots, 4$ and $2_j$, $j = 1, \ldots, 6$
- Generator relations of $D_{14}$
  \[
  A^{14} = 1 \ , \ B^2 = 1 \ , \ A \ B \ A = B
  \]
- Generators
  \[
  A = \begin{pmatrix}
  e^{\left(\frac{\pi i}{7}\right)j} & 0 \\
  0 & e^{-\left(\frac{\pi i}{7}\right)j}
  \end{pmatrix} \quad \text{and} \quad
  B = \begin{pmatrix}
  0 & 1 \\
  1 & 0
  \end{pmatrix} \quad , \quad j = 1, \ldots, 6
  \]
Visualization of $D_{14}$
Visualization of $D_{14}$
Visualization of $D_{14}$
**$Z_2$ Subgroups of $D_{14}$**

One type of $Z_2$ subgroup is generated by the element $BA^k$ with $k = 0, \ldots, 13$

\[
\begin{align*}
1_1 & : \quad BA^k = 1 \quad \checkmark \\
1_2 & : \quad BA^k = -1 \quad \_ \\
1_3 & : \quad BA^k = (-1)^k \quad \checkmark \text{ for } k \text{ even} \\
1_4 & : \quad BA^k = (-1)^{k+1} \quad \checkmark \text{ for } k \text{ odd}
\end{align*}
\]

Scalars $\varphi_{1,2}$ forming a $2_j$ under $D_{14}$ leave a $Z_2$ invariant, if

\[
\begin{pmatrix}
\langle \varphi_1 \rangle \\
\langle \varphi_2 \rangle
\end{pmatrix} \propto 
\begin{pmatrix}
e^{-\frac{\pi i j}{7}} \\
1
\end{pmatrix}
\Rightarrow \begin{cases}
Z_2 \text{ group in up quark sector with } k_1 = 0 \\
\text{and in down quark sector with } k_2 \neq 0 \text{ odd}
\end{cases}
\]
**$D_{14}$ Quark Model**

(Blum/H (’09))

- Flavor group $G_F = D_{14} \times Z_3 \times U(1)_{FN}$
- Framework: MSSM
- Quarks transform non-trivially under $G_F$
- MSSM Higgs doublets $h_{u,d}$ are singlets under $G_F$
- Necessity of gauge singlets (flavons) transforming under $G_F$

$$\left\{\psi_{1,2}^u, \chi_{1,2}^u, \xi_{1,2}^u, \eta^u\right\} \text{ and } \left\{\psi_{1,2}^d, \chi_{1,2}^d, \xi_{1,2}^d, \eta^d, \sigma\right\}$$

- FN field $\theta$ is only charged under $U(1)_{FN}$
- Structure of Yukawa couplings

$$\frac{\theta^2}{\Lambda^4}(Q_D u^c \chi^u \xi^u)h_u \text{ and } \frac{1}{\Lambda}Q_3(b^c \eta^d)h_d$$
### Setup of Quark Model

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<tr>
<th>Field</th>
<th>$Q_D$</th>
<th>$Q_3$</th>
<th>$u^c$</th>
<th>$c^c$</th>
<th>$t^c$</th>
<th>$d^c$</th>
<th>$s^c$</th>
<th>$b^c$</th>
<th>$h_{u,d}$</th>
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Leading Order in Up Quark Sector

\[ Q_3 t^c h_u \]

\[ \frac{1}{\Lambda} (Q_D \psi^u) t^c h_u \]

\[ \frac{1}{\Lambda} Q_3 (c^c \eta^u) h_u \]

\[ \frac{\theta^2}{\Lambda^4} (Q_D u^c \chi^u \xi^u) h_u + \frac{\theta^2}{\Lambda^4} (Q_D u^c (\xi^u)^2) h_u + \frac{\theta^2}{\Lambda^4} (Q_D \psi^u \eta^u u^c) h_u \]

\[ \frac{1}{\Lambda^2} (Q_D c^c \chi^u \xi^u) h_u + \frac{1}{\Lambda^2} (Q_D c^c (\xi^u)^2) h_u + \frac{1}{\Lambda^2} (Q_D \psi^u) (\eta^u c^c) h_u \]
Leading Order in Down Quark Sector

\[
\begin{aligned}
1 \phi \\
1 \Lambda Q_3 (b^c \eta^d) h_d \\
\frac{\theta}{\Lambda^2} Q_3 s^c \sigma h_d \\
\frac{\theta}{\Lambda^2} (Q_D \psi^d) s^c h_d
\end{aligned}
\]
Vacuum Structure

Up quark sector \( \langle \eta^u \rangle \neq 0 \)

\[
\begin{pmatrix}
\langle \psi^u_1 \rangle \\
\langle \psi^u_2 \rangle
\end{pmatrix} = v^u \begin{pmatrix} 1 \\ 1 \end{pmatrix},
\begin{pmatrix}
\langle \chi^u_1 \rangle \\
\langle \chi^u_2 \rangle
\end{pmatrix} = w^u \begin{pmatrix} 1 \\ 1 \end{pmatrix},
\begin{pmatrix}
\langle \xi^u_1 \rangle \\
\langle \xi^u_2 \rangle
\end{pmatrix} = z^u \begin{pmatrix} 1 \\ 1 \end{pmatrix}
\]

Down quark sector \( \langle \eta^d \rangle \neq 0 \) and \( \langle \sigma \rangle \neq 0 \)

\[
\begin{pmatrix}
\langle \psi^d_1 \rangle \\
\langle \psi^d_2 \rangle
\end{pmatrix} = v^d \begin{pmatrix} e^{-\frac{\pi i k}{7}} \\ 1 \end{pmatrix},
\begin{pmatrix}
\langle \chi^d_1 \rangle \\
\langle \chi^d_2 \rangle
\end{pmatrix} = w^d e^{\frac{\pi i k}{7}} \begin{pmatrix} e^{-\frac{2\pi i k}{7}} \\ 1 \end{pmatrix},
\begin{pmatrix}
\langle \xi^d_1 \rangle \\
\langle \xi^d_2 \rangle
\end{pmatrix} = z^d e^{\frac{2\pi i k}{7}} \begin{pmatrix} e^{-\frac{4\pi i k}{7}} \\ 1 \end{pmatrix}
\]
Results for Quarks at Leading Order

Assume $\frac{\langle \Phi^u \rangle}{\Lambda} \approx \epsilon$, $\frac{\langle \Phi^d \rangle}{\Lambda} \approx \epsilon$, $t = \frac{\langle \theta \rangle}{\Lambda} \approx \epsilon \approx \lambda^2 \approx 0.04$

then $M_u$ and $M_d$ read

$$M_u = \begin{pmatrix}
-\alpha_1^u t^2 \epsilon^2 & \alpha_2^u \epsilon^2 & \alpha_3^u \epsilon \\
\alpha_1^u t^2 \epsilon^2 & \alpha_2^u \epsilon^2 & \alpha_3^u \epsilon \\
0 & \alpha_4^u \epsilon & y_t
\end{pmatrix}
\langle h_u \rangle$$

$$M_d = \begin{pmatrix}
0 & \alpha_1^d t \epsilon & 0 \\
0 & \alpha_1^d e^{-\pi i k/7} t \epsilon & 0 \\
0 & \alpha_2^d t \epsilon & y_b \epsilon
\end{pmatrix}
\langle h_d \rangle$$
Results for Quarks at Leading Order

Quark Masses

\[ m_u^2 : m_c^2 : m_t^2 \sim \epsilon^8 : \epsilon^4 : 1, \quad m_u^2 : m_s^2 : m_b^2 \sim 0 : \epsilon^2 : 1, \]
\[ m_b^2 : m_t^2 \sim \epsilon^2 : 1 \text{ for small } \tan \beta \]

CKM matrix

\[ |V_{CKM}| = \begin{pmatrix}
| \cos(\frac{k \pi}{14})| & | \sin(\frac{k \pi}{14})| & 0 \\
| \sin(\frac{k \pi}{14})| & | \cos(\frac{k \pi}{14})| & 0 \\
0 & 0 & 1
\end{pmatrix} + \begin{pmatrix}
0 & O(\epsilon^4) & O(\epsilon^2) \\
O(\epsilon^2) & O(\epsilon^2) & O(\epsilon) \\
O(\epsilon) & O(\epsilon) & O(\epsilon^2)
\end{pmatrix} \]

\[ k = 1 \text{ or } k = 13 \text{ leads to } |V_{ud}| \approx 0.97493 \]

Experimental value: \[ |V_{ud}|_{exp} = 0.97419^{+0.00022}_{-0.00022} \]
Completion: Add Leptons

**Goals:**

- Add leptons in minimal way
- Predict $\mu T$ symmetry in the lepton sector
- Do not disturb quark sector
Goals:

- Add leptons in minimal way
- Predict $\mu^\tau$ symmetry in the lepton sector
- Do not disturb quark sector

Solution

<table>
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<tr>
<th>Field</th>
<th>$L_1$</th>
<th>$L_D$</th>
<th>$e^c$</th>
<th>$\mu^c$</th>
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<td>$\omega^4_7$</td>
<td>$\omega^4_7$</td>
<td>$\omega^2_7$</td>
</tr>
</tbody>
</table>
Neutrino Sector

Majorana mass matrix for right-handed neutrinos

\[
\begin{bmatrix}
1 \\ \phi
\end{bmatrix}
\]

\[
\begin{align*}
(11) & \quad \nu_1^c \nu_1^c \sigma \\
(23) & \quad (\nu_D^c \nu_D^c) \sigma
\end{align*}
\]

leads to

\[
\mathcal{M}_R = \begin{pmatrix}
\alpha_1^M & 0 & 0 \\
0 & 0 & \alpha_2^M \\
0 & \alpha_2^M & 0
\end{pmatrix} \epsilon \Lambda
\]
Neutrino Sector

Dirac neutrino mass matrix

\[
\begin{bmatrix}
1 & \phi
\end{bmatrix}
\]

(12), (13) \( \frac{1}{\Lambda} (L_1 \nu_D^c \xi^u) h_u \)

(21), (31) \( \frac{1}{\Lambda} (L_D \nu_1^c \chi^u) h_u \)

(23), (32) \( \frac{1}{\Lambda} (L_D \nu_D^c \psi^u) h_u \)

leads to

\[
\mathcal{M}_\nu^D = \begin{pmatrix}
0 & \alpha_1^D & \alpha_1^D \\
\alpha_2^D & 0 & \alpha_3^D \\
-\alpha_2^D & \alpha_3^D & 0
\end{pmatrix}
\]

\( \epsilon \langle h_u \rangle \)
Neutrino Sector

Light neutrino mass matrix

\[ \mathcal{M}_\nu = \begin{pmatrix} 2x^2/v & x & x \\ x & z & v - z \\ x & v - z & z \end{pmatrix} \left( \epsilon \langle h_u \rangle^2 / \Lambda \right) \]

- $\mu \tau$ symmetric neutrino mixing
- $\theta_{12}^\nu$ is given by \[ \tan(\theta_{12}^\nu) = \sqrt{2} \left| \frac{x}{v} \right| \]
- Normal ordering with $m_1 = 0$ is predicted and
\[
m_2^2 = \frac{(|v|^2 + 2|x|^2)^2}{|v|^2} \left( \frac{\epsilon \langle h_u \rangle^2}{\Lambda} \right)^2, \quad m_3^2 = |v - 2z|^2 \left( \frac{\epsilon \langle h_u \rangle^2}{\Lambda} \right)^2
\]
- Additional relation \[ |m_{ee}| = m_2 \sin^2(\theta_{12}^\nu) = \sqrt{\delta m^2} \sin^2(\theta_{12}^\nu) \]
**Charged Lepton Sector**

Alignment of new flavon $\chi_{1,2}^e$

\[
\begin{pmatrix}
\langle \chi_1^e \rangle \\
\langle \chi_2^e \rangle
\end{pmatrix}
= v^e
\begin{pmatrix}
1 \\
0
\end{pmatrix}
\]

for which just one additional driving field $\sigma^{0e}$ is needed

\[
w_{f,l} = a_l \sigma^{0e} \chi_1^e \chi_2^e
\]

Nota bene: Also this alignment preserves a $Z_2$ subgroup of $D_{14}$, because $2_2$ is unfaithful.
Yukawa Operators for Charged Leptons

\[
\begin{align*}
(3\alpha) & \quad \frac{1}{\Lambda} (L_D \chi^e) \alpha^c h_d \\
(2\alpha) & \quad \frac{1}{\Lambda^2} (L_D \chi^e \xi^u) \alpha^c h_d \\
(1\alpha) & \quad \frac{1}{\Lambda^3} (L_1 \chi^e \psi^u \xi^u) \alpha^c h_d \\
(1\alpha) & \quad \frac{1}{\Lambda^3} (L_1 \eta^u) (\chi^e \chi^u) \alpha^c h_d
\end{align*}
\]
**Charged Lepton Mass Matrix**

For $\frac{v^e}{\Lambda} \approx \epsilon \approx \lambda^2$ we get

\[
M_e = \begin{pmatrix}
\alpha_1^e \epsilon^3 & \alpha_2^e \epsilon^3 & \alpha_3^e \epsilon^3 \\
\alpha_4^e \epsilon^2 & \alpha_5^e \epsilon^2 & \alpha_6^e \epsilon^2 \\
\alpha_7^e \epsilon & \alpha_8^e \epsilon & \alpha_9^e \epsilon
\end{pmatrix} \langle h_d \rangle
\]

- Charged lepton masses $m_e : m_\mu : m_\tau \sim \epsilon^2 : \epsilon : 1$
- Charged lepton mixing angles $\theta_{12}^e \sim \epsilon$, $\theta_{13}^e \sim \epsilon^2$, $\theta_{23}^e \sim \epsilon$

Lepton mixings are nearly $\mu\tau$ symmetric

\[
\sin^2(\theta_{23}^l) = \frac{1}{2} + O(\epsilon), \quad \sin(\theta_{13}^l) = O(\epsilon), \quad \sin^2(\theta_{12}^l) = O(1)
\]
Conclusions

• There are many hints at a flavor symmetry: existence of three generations, fermion mass hierarchies, mixing patterns, no excessive flavor violation observed

• Discrete symmetries are very useful to understand fermion mixing, especially if non-trivially broken in different sectors of the theory

• Impact of T2K?! ... one might argue that many mixing patterns/models are now disfavored ... but: in all models corrections exist, T2K results need to be confirmed by others/more statistics

• Still we should search for new symmetries giving easily rise to $\theta_{13} \neq 0$ at LO, not giving up the non-trivial breaking pattern of the flavor group!
Thank you.