Shell-Model Applications in nuclear physics and astrophysics

Gabriel Martínez Pinedo

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- Effective Interaction
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Nucleus as a many body problem

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- Nucleus is made of nucleons (neutrons and protons) that interact by meson exchange. Interaction could be described by a non-relativistic potential.
Nucleus as a many body problem

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We need to solve the Schrödinger equation:

\[
H \Psi = E \Psi
\]

\[
H = \sum_{i=1}^{A} \frac{p_i}{2m} + \sum_{i<j=1}^{A} V_{ij} + \sum_{i<j<k=1}^{A} V_{ijk}
\]
Basic symmetries

- Invariance under translations. Potential depends only on relative coordinates.

\[ J; T^q_k \] = \sim \frac{k}{1} \left( q \right) T^k_{q+1} \]

\[ J^0; T^q_k \] = \sim \frac{q}{q+1} \]
Basic symmetries

- Invariance under translations. Potential depends only on relative coordinates.

- Invariance under rotations (space, spin, isospin).
  - States are eigenstates of total angular momentum ($J = L + S$) and isospin.
  - Tensor operators (multipole operators).

\[
[J_{\pm}, T_{q}^{k}] = \hbar \sqrt{k(k+1) - q(q \pm 1)} T_{q+1}^{k} \\
[J_0, T_{q}^{k}] = \hbar q
\]
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\]
\[
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\]

- Invariance under parity and time reversal
Example of Potential

Potential consistent with two-nucleon scattering data and deuteron structure

Potential AV8':

\[ V(r) = V_c(r) + V_\tau(r)(\tau_1 \cdot \tau_2) + V_\sigma(r)(\sigma_1 \cdot \sigma_2) \]
\[ + V_{\sigma \tau}(r)(\sigma_1 \cdot \sigma_2)(\tau_1 \cdot \tau_2) + V_t(r)S_{12} + V_{t \tau}(r)S_{12}(\tau_1 \cdot \tau_2) \]
\[ + V_b(L \cdot S) + V_{b \tau}(r)(L \cdot S)(\tau_1 \cdot \tau_2) \]
Benchmark test calculations for $^4$He


$E_{\text{exp}} = 28.296$ MeV

\[
\begin{array}{cccc}
\text{Method} & \langle T \rangle & \langle V \rangle & E_b & \sqrt{\langle r^2 \rangle} \\
FY & 102.39(5) & -128.33(10) & -25.94(5) & 1.485(3) \\
CRCGV & 102.30 & -128.20 & -25.90 & 1.482 \\
SVM & 102.35 & -128.27 & -25.92 & 1.486 \\
HH & 102.44 & -128.34 & -25.90(1) & 1.483 \\
GFMC & 102.3(1.0) & -128.25(1.0) & -25.93(2) & 1.490(5) \\
NCSM & 103.35 & -129.45 & -25.80(20) & 1.485 \\
EIHH & 100.8(9) & -126.7(9) & -25.944(10) & 1.486 \\
\end{array}
\]

\[
\begin{array}{cccccc}
\text{Method} & \langle V_c \rangle & \langle V_t \rangle & \langle V_\sigma \rangle & \langle V_{\sigma\tau} \rangle \\
FY & 16.54 & -5.038 & -9.217 & -57.55 \\
CRCGV & 16.54 & -5.035 & -9.215 & -57.51 \\
SVM & 16.54 & -5.036 & -9.213 & -57.51 \\
GFMC & 16.5(5) & -5.03(6) & -9.21(7) & -57.3(5) \\
NCSM & 16.16 & -4.92 & -9.77 & -57.89 \\
\end{array}
\]

\[
\begin{array}{cccccc}
\text{Method} & \langle V_t \rangle & \langle V_{t\tau} \rangle & \langle V_b \rangle & \langle V_{b\tau} \rangle \\
FY & 0.707 & -69.06 & 10.79 & -15.50 \\
CRCGV & 0.708 & -68.99 & 10.60 & -15.30 \\
SVM & 0.707 & -69.03 & 10.78 & -15.49 \\
HH & 0.702 & -69.03 & 10.76 & -15.46 \\
GFMC & 0.71(3) & -68.8(5) & 10.62(15) & -15.40(15) \\
NCSM & 0.68 & -69.13 & 11.23 & -15.80 \\
\end{array}
\]
GFMC calculations for light nuclei

![Diagram showing energy levels for light nuclei with various isotopes and shell-model applications.]
Theoretical models

Limits of nuclear existence

Towards a unified description of the nucleus

Protons

Neutrons

A=10
A=12
A~60

Ab initio few-body calculations

No-Core Shell Model

G-matrix

Density Functional Theory
Selfconsistent Mean Field

r-process

r-process

Figure 2: Top: the nuclear landscape - the territory of RIA physics. The black squares represent the stable nuclei and the nuclei with half-lives comparable to or longer than the age of the Earth (4.5 billion years). These nuclei form the "valley of stability". The yellow region indicates shorter lived nuclei that have been produced and studied in laboratories. By adding either protons or neutrons one moves away from the valley of stability, finally reaching the drip lines where the nuclear binding ends because the forces between neutrons and protons are no longer strong enough to hold these particles together. Many thousands of radioactive nuclei with very small or very large N/Z ratios are yet to be explored. In the (N,Z) landscape, they form the terra incognita indicated in green. The proton drip line is already relatively well delineated experimentally up to Z=83. In contrast, the neutron drip line is considerably further from the valley of stability and harder to approach. Except for the lightest nuclei where it has been reached experimentally, the neutron drip line has to be estimated on the basis of nuclear models - hence it is very uncertain due to the dramatic extrapolations involved. The red vertical and horizontal lines show the magic numbers around the valley of stability. The anticipated paths of astrophysical processes (r-process, purple line; rp-process, turquoise line) are shown. Bottom: various theoretical approaches to the nuclear many-body problem. For the lightest nuclei, ab initio calculations (Green's Function Monte Carlo, no-core shell model) based on the bare nucleon-nucleon interaction, are possible. Medium-mass nuclei can be treated by the large-scale shell model. For heavy nuclei, the density functional theory (based on selfconsistent mean field) is the tool of choice. By investigating the intersections between these theoretical strategies, one aims at nothing less than developing the unified description of the nucleus.
Shell-Model basics

- Shell-Model assumes the existence of shells. Magic numbers are obtained when a shell is completely fill.
- Shells results from the bunching (grouping) of levels coming from a independent particle average potential.
Assume the existence of some single-particle wave functions that are the solution of a Schrödinger equation

\[ h\phi(r) = \{T + U\} \phi_a(r) = \varepsilon_a \phi_a(r) \]

The independent-particle motion hamiltonian is then:

\[ H_0 = \sum_{k=1}^{A} T(k) + U(r_k) \]

Eigenfunctions are the product of single-particle wave functions:

\[ \Phi_{a_1a_2\ldots a_A}(1, 2, \ldots, A) = \prod_{k=1}^{A} \phi_{a_k}(r_k) \]
System identical particles

Wave function should be antisymmetric. For two particles:

\[ \Phi_{ab}(1, 2) = \frac{1}{\sqrt{2}} [\phi_a(1)\phi_b(2) - \phi_a(2)\phi_b(1)] = \frac{1}{\sqrt{2}} \begin{bmatrix} \phi_a(1) & \phi_b(2) \\ \phi_a(2) & \phi_b(1) \end{bmatrix} \]

A-particle Wave function:

\[ \Phi_{a_1a_2...a_A}(1, 2, ..., A) = \sqrt{\frac{1}{A!}} \begin{bmatrix} \phi_{a_1}(r(1)) & \phi_{a_1}(r(2)) & \cdots & \phi_{a_1}(r(A)) \\ \phi_{a_2}(r(1)) & \phi_{a_2}(r(2)) & \cdots & \phi_{a_2}(r(A)) \\ \vdots & \vdots & \ddots & \vdots \\ \phi_{a_A}(r(1)) & \phi_{a_A}(r(2)) & \cdots & \phi_{a_A}(r(A)) \end{bmatrix} \]
Simple single-particle potential

Empirical construction based in an harmonic oscillator potential plus a spin-orbit term to reproduce the magic numbers (M. Goeppert-Mayer and H. Jensen)

\[ U(r) = \frac{1}{2} m \omega^2 r^2 + D l^2 + l \cdot s \]

\[ \varepsilon_{nlj} = \hbar \omega [2(n-1)+l+3/2] + D l (l+1) + C \begin{cases} 
  l + 1 & j = l - 1/2 \\
  -l & j = l + 1/2 
\end{cases} \]

\[ \hbar \omega = \frac{41}{A^{1/3}} \text{ MeV} \]
Solution to the many-nucleon problem

\[ H \Psi(1, 2, \ldots, A) = \left[ \sum_{k=1}^{A} T(k) + \sum_{k<l=1}^{A} W(k, l) \right] \Psi(1, 2, \ldots, A) = E \Psi(1, 2, \ldots, A) \]

Hamiltonian rewrittent:

\[ H = \sum_{k=1}^{A} [T(k) + U(k)] + \left[ \sum_{k<l=1}^{A} W(k, l) - \sum_{k=1}^{A} U(k) \right] = H_0 + V_{\text{res}} \]

Hartree-Fock theory provides method to derive single-particle potential. The criterium is to search for the “best” \( A \)-particle slater determinant such us the value of \( H \) is minimum. Next, one assumes that the resulting residual interaction is small and that:

\[ \Psi(1, 2, \ldots, A) = \Phi_{\alpha_1 \alpha_2 \ldots \alpha_A} (1, 2, \ldots, A) \]
Validity shell structure


\[ \Delta \rho_{Pb} - \rho_{Tl} \]

\[ ^{208}_{82}\text{Pb}_{126} \]

\[ 3p_{1/2} \quad -7368 \]
\[ 3s_{1/2} \quad -8013 \]
\[ 2d_{3/2} \quad -8364 \]
\[ 2d_{5/2} \quad -9696 \]
\[ 1h_{11/2} \quad -9361 \]
\[ 2f_{5/2} \quad -9708 \]
\[ 1f_{13/2} \quad -9001 \]
\[ 1g_{7/2} \quad -11487 \]

Protons \[ ^{132}_{50}\text{Sn}_{82} \] Neutrons
Limits of the IPM

Example $^{16}O$

$$
\text{\ } \Pi \quad \text{\ } \nu
$$

$$
\uparrow \quad \text{\ } \downarrow
$$

$$
\hbar \omega
$$

$$
\uparrow \quad \text{\ } \downarrow
$$

$$
\uparrow \quad \text{\ } \downarrow
$$

$$
\hbar \omega \sim 12 \text{ MeV}
$$

$0^+ ...$

$0^- ... 4^-$

$0^+$
Limits of the IPM

Example $^{16}\text{O}$

$Q_{\beta} = 10419.0$

$Q_{\beta} = 10419.0$

$7.13 \text{ s}$

$16N$

$2^+\; : \; T=1$

$0$

$16O$

$12C$

$28.0\% \quad 9.11$

$0^+\; : \; T=0$

$0$

$16O$

$12C$

$0^+\; : \; T=0$

$0$

$16O$

$12C$

$0^+\; : \; T=0$

$0$

$16O$

$12C$

$0^+\; : \; T=0$

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$0$

$16O$

$12C$

$0^+\; : \; T=0$

$0$
Limits of the IPM

Example $^{16}\text{O}$

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Two-particle wave function

Let’s consider the possible isospin values for two nucleons:

\[ |T = 1, T_z = 1\rangle = |n\rangle|n\rangle, \quad |1, -1\rangle = |p\rangle|p\rangle \]
\[ |1, 0\rangle = \frac{1}{\sqrt{2}} \left[ |n\rangle|p\rangle + |p\rangle|n\rangle \right] \]
\[ |0, 0\rangle = \frac{1}{\sqrt{2}} \left[ |n\rangle|p\rangle - |p\rangle|n\rangle \right] \]

N-particle system: \( T = \text{odd} \) (symmetric), \( T = \text{even} \) (antisymmetric).

We can also couple the angular momentum:

\[ |JM\rangle = \sum_{m_1m_2} \langle j_1m_1j_2m_2|JM\rangle |j_1m_1\rangle |j_2m_2\rangle \]

In \( N = Z \) nuclei protons and neutrons occupy the same orbits. Low lying states fulfill \((J + T = \text{odd number})\).
Possible two-particle states \((j_1 j_2)^J T\):

\[
\begin{align*}
(p_{3/2} p_{3/2})^J T &= 1, 3, 5; T = 0, \\
(p_{3/2} p_{1/2})^J T &= 1, 2; T = 0, \\
(p_{1/2} p_{1/2})^J T &= 1; T = 0,
\end{align*}
\]

\[
\begin{align*}
(p_{3/2} p_{3/2})^J T &= 2, 4, 6; T = 1, \\
(p_{3/2} p_{1/2})^J T &= 1, 2; T = 1, \\
(p_{1/2} p_{1/2})^J T &= 0; T = 1.
\end{align*}
\]
In order to incorporate the correlations, one has to go beyond mean-field.

**Spherical mean-field**

- Breaking symmetries of the system
- Mixing different mean-field configurations

- Hartree-Fock Bogoliubov
- Nilsson
- Deformed Hartree-Fock

- Tamm-Dancoff
- RPA
- Interacting shell-model
Possible solution

- Take the basics formed by the A-particle Slater determinants:

\[ \Phi_a = \Phi_{a_1 a_2 ... a_A} (1, 2, \ldots, A) \]
Possible solution

- Take the basics formed by the A-particle Slater determinants:

  \[ \Phi_a = \Phi_{a_1 a_2 \ldots a_A} (1, 2, \ldots, A) \]

- build the (infinite) matrix:

  \[ \langle \Phi_b | H | \Phi_a \rangle \]
Possible solution

- Take the basics formed by the A-particle Slater determinants:

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- build the (infinite) matrix:

\[ \langle \Phi_b | H | \Phi_a \rangle \]

- diagonalize
Shell-Model approximation

The space of orbits generated by the mean-field potential are grouped in three blocks

- **Inner core**: orbits that are always full.

- **Valence space**: orbits that contain the physical degrees of freedom relevant to a given property. The distribution of the valence particles among these orbitals is governed by the interaction.

- **External space**: all the remaining orbits that are always empty.
Shell-Model approximation

The exact solution on the infinite Hilbert space spanned by the mean field orbits is approximated in the large scale shell-model calculation by the solution of the Schrödinger equation in the valence space using an effective interaction.

\[ H\Psi = E\Psi \rightarrow H_{\text{eff}}\Psi_{\text{eff}} = E\Psi_{\text{eff}} \]

In general, effective operators have to be introduced to account for the restrictions of the Hilbert space

\[ \langle \Psi | O | \Psi \rangle = \langle \Psi_{\text{eff}} | O_{\text{eff}} | \Psi_{\text{eff}} \rangle \]
A shell model calculation needs the following ingredients:

- A valence space
Shell-Model calculation

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- A valence space
- An effective interaction
A shell model calculation needs the following ingredients:

- A valence space
- An effective interaction
- A code to build and diagonalize the Hamiltonian matrix.
The choice of the valence space:

- In light nuclei the *harmonic oscillator closures determine* the natural valence spaces.

\[
\begin{align*}
^4\text{He} & \rightarrow ^{16}\text{O} & ^{40}\text{Ca} & \rightarrow ^{80}\text{Zr} \\
p \text{ shell} & \rightarrow sd \text{ shell} & pf \text{ shell} & \uparrow \\
\text{Cohen/} & \text{Brown/} & \text{Wildenthal} & \text{Deformed}
\end{align*}
\]
Valence space

- In heavier nuclei:
  \[ jj \] closures due to the spin-orbit term show up

\[ N=28, 50, 82, 126 \]
Valence space

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  - $jj$ closures due to the spin-orbit term show up
    \[ N=28, 50, 82, 126 \]
- the transition $\HO \rightarrow jj$: occurs between $^{40}\text{Ca}$ and $^{100}\text{Sn}$
  - where the protagonism shifts from the $1f_{7/2}$ to the $1g_{9/2}$
Valence space

- In heavier nuclei:
  \( jj \) closures due to the spin-orbit term show up
  \( N=28, 50, 82, 126 \)

- The transition \( HO \rightarrow jj \): occurs between \(^{40}\text{Ca}\) and \(^{100}\text{Sn}\) where the protagonism shifts from the \( 1f_{7/2} \) to the \( 1g_{9/2} \)

- A valence space can be adequate to describe some properties and completely wrong for others

<table>
<thead>
<tr>
<th>(^{48}\text{Cr})</th>
<th>((f_{7/2})^8)</th>
<th>((f_{7/2}p_{3/2})^8)</th>
<th>((fp)^8)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(Q(2^+)(\text{e.fm}^2))</td>
<td>0.0</td>
<td>-23.3</td>
<td>-23.8</td>
</tr>
<tr>
<td>(E(2^+)(\text{MeV}))</td>
<td>0.63</td>
<td>0.44</td>
<td>0.80</td>
</tr>
<tr>
<td>(E(4^+)/E(2^+))</td>
<td>1.94</td>
<td>2.52</td>
<td>2.26</td>
</tr>
<tr>
<td>(\text{BE2}(2^+ \rightarrow 0^+)(e^2\text{.fm}^4))</td>
<td>77</td>
<td>150</td>
<td>216</td>
</tr>
<tr>
<td>(B(GT))</td>
<td>0.90</td>
<td>0.95</td>
<td>3.88</td>
</tr>
</tbody>
</table>
Second quantization

- **Creation and destruction operators:**

\[ a_k^\dagger |0\rangle = |k\rangle, \quad a_k |k\rangle = |0\rangle \]

vacuum \(|0\rangle\), such \(a_k |0\rangle \forall k\).

For fermions antisymmetry given by anti-commutation rules

\[ \{a^\dagger_i, a^\dagger_j\} = \{a_i, a_j\} = 0, \quad \{a^\dagger_i, a_j\} = \delta_{ij} \]

- **Slater determinant:**

\[ \Phi_{\alpha_1 \alpha_2 \ldots \alpha_A} (1, 2, \ldots, A) = a^\dagger_{\alpha_A} \cdots a^\dagger_{\alpha_2} a^\dagger_{\alpha_1} |0\rangle \]
Second quantization

- **One body operators:**

\[
O = \sum_{k=1}^{A} O(k) \rightarrow O = \sum_{\alpha, \beta} \langle \alpha | O | \beta \rangle a_{\alpha}^{\dagger} a_{\beta}
\]

number of particles \( N = \sum_{\alpha} n_{\alpha} = \sum_{\alpha} a_{\alpha}^{\dagger} a_{\alpha} \)

- **Two-body operators:**

\[
O = \sum_{i<j=1}^{A} O(i, j) \rightarrow O = \sum_{\alpha, \beta, \gamma, \delta} \langle \alpha \beta | O | \gamma \delta \rangle a_{\alpha}^{\dagger} a_{\beta}^{\dagger} a_{\delta} a_{\gamma}
\]
The interaction in second quantization

\[ H = \sum_{\alpha} \varepsilon_{\alpha} a_{\alpha}^{\dagger} a_{\alpha} + \frac{1}{4} \sum_{\alpha,\beta,\gamma,\delta} \langle \alpha,\beta \mid V \mid \gamma,\delta \rangle a_{\alpha}^{\dagger} a_{\beta}^{\dagger} a_{\delta} a_{\gamma} = H_0 + V_{\text{res}} \]
The interaction in second quantization

In general it is convenient to work in a couple basis:

Defining $\tilde{a}_{jm} = (-1)^{j+m}a_{j-m}$, and the coupling:

$$\left[a^\dagger_{j_1}a^\dagger_{j_2}\right]^J_J M = \sum_{m_1 m_2} \langle j_1 m_1 j_2 m_2 | J M \rangle a^\dagger_{j_1 m_1} a^\dagger_{j_2 m_2}$$

$$H = \sum_{\alpha} \varepsilon_{\alpha} n_{\alpha} - \frac{1}{4} \sum_{j_1 j_2 j_3 j_4 J T} \langle j_1 j_2 | V | j_3 j_4 \rangle^{J T}$$

$$\sqrt{(2J + 1)(2T + 1)(1 + \delta_{12})(1 + \delta_{23})}$$

$$\left[\left[a^\dagger_{j_1} a^\dagger_{j_2}\right]^J_J T \times \left[\tilde{a}_{j_3} \tilde{a}_{j_4}\right]^J_J T \right]^{00}$$
All the information needed for a shell-model calculation are the independent particle energies ($\varepsilon_\alpha$) and the two-body matrix elements $\langle j_1j_2; JT|V|j_3j_4; JT \rangle$

Example USD interaction ($d_{5/2}, s_{1/2}, d_{3/2}$)

**USD**

```
  3  205  1001  203
-3.94780 -3.16354  1.64658
  0  205  205  205  205  0  5
  0.0000 -1.6321  0.0000 -1.5012  0.0000 -4.2256
-2.8197  0.0000 -1.0020  0.0000 -0.1641  0.0000
  0  205  1001  205  1001  2  3
-1.4474 -3.8598
-0.8183  0.7626
  0  205  203  205  203  1  4
-6.5058 -3.8253 -0.5377 -4.5062
  1.0334 -0.3248  0.5894 -1.4497
  0  1001  1001  1001  1001  0  1
  0.0000 -3.2628
-2.1246  0.0000
  0  1001  203  1001  203  1  2
-4.2930 -1.8194
  0.6066 -0.4064
  0  203  203  203  203  0  3
  0.0000 -1.4151  0.0000 -2.8842
-2.1845  0.0000 -0.0665  0.0000
```

........................
Application

**Spectrum of** $0^+$ **states in** $^{18}O$

*we build a* $3 \times 3$ *matrix:*

$$
\begin{pmatrix}
(d_5^2/2) & (d_3^2/2) & (s_1^1/2) \\
-3.9478 \times 2 & -3.1856 & -0.13247 \\
+(-2.8197) & & \\
-3.1856 & 1.64658 \times 2 & -1.0835 \\
& (+(-2.1845)) & \\
-1.3247 & -1.0835 & -3.1654 \times 2 \\
& & (+(-2.1246))
\end{pmatrix}
$$

*whose eigenvalues produce the spectrum:*

$O_3^+ \quad 14.1$

$O_2^+ \quad 4.3$

$O_1^+ \quad 0.0$
Large scale Shell-Model basis and codes

$p f$-shell valence space: $1f_{7/2}, 2p_{3/2}, 2p_{1/2}, 1f_{5/2}$

<table>
<thead>
<tr>
<th>nucleus</th>
<th>m-scheme</th>
<th>jj-scheme</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>(ANTOINE)</td>
<td>(NATHAN)</td>
</tr>
<tr>
<td>$^{48}\text{Cr}$</td>
<td>1,963,461</td>
<td>41,355</td>
</tr>
<tr>
<td>$^{54}\text{Fe}$</td>
<td>345,400,174</td>
<td>5,220,621</td>
</tr>
<tr>
<td>$^{56}\text{Fe}$</td>
<td>501,113,392</td>
<td>7,413,488</td>
</tr>
<tr>
<td>$^{56}\text{Ni}$</td>
<td>1,087,455,228</td>
<td>15,443,684</td>
</tr>
</tbody>
</table>

Impossible to store Hamiltonian matrix!
Still possible to compute $H \Psi$.
Diagonalization using an iterative algorithm.
Lanczos algorithm

Construction of a orthonormal basis:
Initial vector $|1\rangle$.

\[
E_{12}|2\rangle = (H - E_{11})|1\rangle
\]
\[
E_{23}|3\rangle = (H - E_{22})|2\rangle - E_{12}|1\rangle
\]
\[
\cdots
\]
\[
E_{NN+1}|N + 1\rangle = (H - E_{NN})|N\rangle - E_{N-1N}|N - 1\rangle
\]

where

\[
E_{NN} = \langle N | H | N \rangle, \quad E_{NN+1} = E_{N+1N}
\]
Lanczos algorithm

\[
\begin{pmatrix}
E_{11} & E_{12} & 0 & 0 & \ldots & 0 \\
E_{12} & E_{22} & E_{23} & 0 & \ldots & 0 \\
0 & E_{23} & E_{33} & E_{34} & \ldots & 0
\end{pmatrix}
\]

- Diagonalize and obtain an approximation to the energy and wave function.
- Do a few more iterations
- Diagonalize again and check that \( \Delta E = E_{N+k} - E_N < \epsilon \)
### Lanczos convergence

**RANDOM STARTING VECTOR**

<p>| | | | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
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<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
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<td>11.335118</td>
<td>29.120687</td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>-21.344259</td>
<td>-7.802025</td>
<td>4.637278</td>
<td>16.927858</td>
</tr>
<tr>
<td>9</td>
<td>-30.092574</td>
<td>-19.653950</td>
<td>-9.343311</td>
<td>0.467972</td>
</tr>
<tr>
<td>12</td>
<td>-32.722076</td>
<td>-24.462806</td>
<td>-17.104890</td>
<td>-9.353111</td>
</tr>
</tbody>
</table>
### Lanczos Convergence

**$^{48}\text{Cr}$**

- $\text{Dim (t=2)} = 6 \times 10^5$
- $\text{Dim (full space)} = 2 \times 10^6$

**STARTING VECTOR: EIGENVECTOR OF A SMALLER SPACE**

<table>
<thead>
<tr>
<th>Iter</th>
<th>DIA</th>
<th>NONDIA</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
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<td>4.642871</td>
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<td>3</td>
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<td>-16.116780</td>
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<tr>
<td>9</td>
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</tr>
<tr>
<td>12</td>
<td>-32.953553</td>
<td>-28.345536</td>
</tr>
<tr>
<td>15</td>
<td>-32.953655</td>
<td>-28.528301</td>
</tr>
</tbody>
</table>
Isospin symmetry

Nuclear spectra are almost unchanged under the exchange of neutrons and protons.

\[
\begin{array}{cccc}
0^+ & 0 & 1 & 0
\end{array}
\]

\[
\begin{array}{cccc}
2^+ & 765 & 783 & 0
\end{array}
\]

\[
\begin{array}{cccc}
4^+ & 1087 & 1098 & 1282
\end{array}
\]

\[
\begin{array}{cccc}
6^+ & 1308 & 1315 & 1581
\end{array}
\]

\[
\begin{array}{cccc}
8^+ & 1627 & 1637 & 1581
\end{array}
\]

\[
\begin{array}{cccc}
10^+ & 1581 & 1595 & 1581
\end{array}
\]

\[
\begin{array}{cccc}
12^+ & 662 & 610 & 0
\end{array}
\]

\[
\begin{array}{cccc}
\text{Fe}^{50}_{26} & 26 & \text{Cr}^{50}_{24} & 24
\end{array}
\]
Isospin representation

Neutron and proton are considered different states of the same particle: the nucleon

\[ |n\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad |p\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \]

Isospin operators:

\[ t = \frac{1}{2} \tau, t_\pm = t_1 \pm t_2 \]

\[ \tau_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \tau_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \tau_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \]

States are eigenstates of total isospin \( T = \sum_{i=1}^{A} t^i \)

\[ T^2 = T(T + 1)|T, T_z\rangle, \quad T_z|T, T_z\rangle \]

\[ T \geq T_z = \frac{|N - Z|}{2} \]
Realistic interactions

<table>
<thead>
<tr>
<th></th>
<th>Exp.</th>
<th>KB</th>
<th>KLS</th>
<th>Bonn A</th>
<th>Bonn B</th>
<th>Bonn C</th>
</tr>
</thead>
<tbody>
<tr>
<td>$^{44}\text{Ca}$</td>
<td>1.16</td>
<td>1.45</td>
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<tr>
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<td>1.10</td>
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</tr>
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</table>

$\langle(f_{7/2}^8|\Psi_{GS}\rangle$

<p>| | | | | | | |</p>
<table>
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<tbody>
<tr>
<td>$^{44}\text{Ca}$</td>
<td>0.468</td>
<td>0.381</td>
<td>0.214</td>
<td>0.345</td>
<td>0.437</td>
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The realistic interactions do not reproduce the shell closure at $N$ or $Z = 28$. Why?
### Realistic interactions

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<tbody>
<tr>
<td>$2^+_1$ excitation energy</td>
<td></td>
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<td></td>
<td></td>
<td></td>
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<td>0.214</td>
<td>0.345</td>
</tr>
</tbody>
</table>

### $^{56}$Ni model space ($f_{7/2} p_{3/2})^{16}$

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<th></th>
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<th></th>
<th></th>
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<tbody>
<tr>
<td>$\langle (f_{7/2})^{16}</td>
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<td>0.04</td>
<td>0.015</td>
<td>0.018</td>
<td>0.011</td>
</tr>
<tr>
<td>$\langle n_{p_{3/2}} \rangle$</td>
<td></td>
<td>4.5</td>
<td>5.2</td>
<td>5.7</td>
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The realistic interactions do not reproduce the shell closure at $N$ or $Z = 28$. Why?
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The realistic interactions do not reproduce the shell closure $N$ or $Z=28$
Realistic interactions

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$\langle (f_{7/2})^8 | \Psi_{GS} \rangle$

<table>
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<tr>
<th>Element</th>
<th>Value</th>
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<th>KLS</th>
<th>Bonn A</th>
<th>Bonn B</th>
<th>Bonn C</th>
</tr>
</thead>
<tbody>
<tr>
<td>$^{56}\text{Ni}$ model space ($f_{7/2} p_{3/2}^{16}$)</td>
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<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$^{56}\text{Ni}$</td>
<td>2.70</td>
<td>0.39</td>
<td>0.31</td>
<td>0.43</td>
<td>0.42</td>
<td>0.42</td>
</tr>
</tbody>
</table>

$\langle (f_{7/2})^{16} | \Psi_{GS} \rangle$

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<td>0.015</td>
<td>0.018</td>
<td>0.011</td>
<td>0.019</td>
<td></td>
</tr>
</tbody>
</table>

$\langle n_{p3/2} \rangle$

<table>
<thead>
<tr>
<th>Element</th>
<th>Value</th>
<th>KB</th>
<th>KLS</th>
<th>Bonn A</th>
<th>Bonn B</th>
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</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>4.5</td>
<td>5.2</td>
<td>5.7</td>
<td>5.2</td>
<td>5.0</td>
<td></td>
</tr>
</tbody>
</table>

The realistic interactions do not reproduce the shell closure $N$ or $Z=28$

Why?
The structure of the Hamiltonian

From the work of M. Dufour and A. Zuker (PRC 54 1996 1641)

Separation theorem:
Any effective interaction can be split in two parts:

\[ H = H_m(\text{monopole}) + H_M(\text{multipole}) \]

\( H_m \) contains all the terms that are affected by a spherical Hartree-Fock variation, hence responsible of the global saturation properties and of the evolution of the spherical single particle field.

Important property:

\[ \langle CS \pm 1 | H | CS \pm 1 \rangle = \langle CS \pm 1 | H_m | CS \pm 1 \rangle \]
The structure of the Hamiltonian

For all the realistic $G$-matrices,

- $H_m$ is not accurate enough.
- $H_M$ is almost the same.

The monopole part has to be empirically corrected to reproduce the structure of the “simple” nuclei $|CS \pm 1\rangle$
Multipole Hamiltonian

Can be characterized as the sum of simple terms:

- $L = 0$ isovector and isoscalar pairing
- Elliot’s quadrupole-quadrupole force.
- $(\sigma_1 \cdot \sigma_2)(\tau_1 \cdot \tau_2)$
- Octupole and hexadecapole terms $r^\lambda Y_\lambda \cdot r^\lambda Y_\lambda$

All the interactions contains similar terms:

<table>
<thead>
<tr>
<th>Interaction</th>
<th>$JT = 01$</th>
<th>$JT = 10$</th>
<th>$\lambda \tau = 20$</th>
<th>$\lambda \tau = 40$</th>
<th>$\lambda \tau = 11$</th>
</tr>
</thead>
<tbody>
<tr>
<td>KB3</td>
<td>-4.75</td>
<td>-4.46</td>
<td>-2.79</td>
<td>-1.39</td>
<td>+2.46</td>
</tr>
<tr>
<td>FPD6</td>
<td>-5.06</td>
<td>-5.08</td>
<td>-3.11</td>
<td>-1.67</td>
<td>+3.17</td>
</tr>
<tr>
<td>GOGNY</td>
<td>-4.07</td>
<td>-5.74</td>
<td>-3.23</td>
<td>-1.77</td>
<td>+2.46</td>
</tr>
</tbody>
</table>
The Monopole Hamiltonian

$H_m$ contains terms that depend in $n$ and $T$. Assume that we have a constant potential and single particle energy:

$$E = n\varepsilon + \frac{n(n-1)}{2}V$$
The Monopole Hamiltonian

If we consider also isospin:

\[ E = n\varepsilon + \frac{n(n - 1)}{2} V + bT(T + 1) \]
In general we have different orbits each with different average interactions:

\[
E = \sum_i n_i \varepsilon_i + \sum_i \frac{n_i(n_i - 1)}{2} V_{ii} + \sum_{i \neq j} n_i n_j V_{ij}
\]
The Monopole Hamiltonian

If we consider also the fact that we have isospin:

\[ H_m = \sum_i \varepsilon_i n_i + \sum_{ij} \left[ \frac{1}{1 + \delta_{ij}} a_{ij} n_i (n_j - \delta_{ij}) + \frac{1}{1 + \delta_{ij}} b_{ij} \left( T_i \cdot T_j - \frac{3n_i}{4} \delta_{ij} \right) \right] \]

Where \( a \) and \( b \) are defined from the centroids (average interaction):

\[ V_{ij}^T = \frac{\sum_J (2J + 1) W_{ijij}^{JT}}{\sum_J (2J + 1)} \]

\[ a_{ij} = \frac{3V_{ij}^1 + V_{ij}^0}{4}, \quad b_{ij} = V_{ij}^1 - V_{ij}^0 \]
Influence of monopole hamiltonian

The evolution of effective spherical single particle energies with the number of particles in the valence space can be extracted from $H_m$. In the case of identical particles the expression is:

$$\varepsilon_j(n) = \varepsilon_j(n = 1) + \sum_i V_{ij} n_i$$

The monopole hamiltonian $H_m$ also governs the relative position of the various T-values in the same nucleus, via the terms:

$$b_{ij} T_i \cdot T_j$$

Even small defects in the centroids can produce large changes in the relative position of the different configurations due to the appearance of quadratic terms involving the number of particles in the different orbits.
Effect of monopole corrections

\[ \frac{3}{2} - \frac{1}{2} - \frac{7}{2} - \frac{5}{2} - \frac{5}{2} - \frac{3}{2} - \frac{7}{2} - \frac{9}{2} \]

\[ \frac{3}{2} - \frac{1}{2} - \frac{7}{2} - \frac{5}{2} - \frac{5}{2} - \frac{3}{2} - \frac{7}{2} - \frac{9}{2} \]

\[ \frac{3}{2} - \frac{1}{2} - \frac{9}{2} + \frac{5}{2} \]

\[ \frac{1}{2}, \frac{3}{2} \]

\[ 7/2^+ \]

\[ 3/2^- \]

\[ 5/2^- \]

\[ 7/2^- \]

\[ 9/2^- \]

\[ 5/2^- \]

\[ 7/2^- \]

\[ 9/2^+ \]

\[ 3/2^- \]

\[ 5/2^- \]

\[ 7/2^- \]

\[ 9/2^- \]

\[ 5/2^- \]

\[ 7/2^- \]

\[ 9/2^- \]
Figure 3. Comparison of bare single-particle energies ($\varepsilon$) and effective single-particle energies ($\varepsilon + H_m$) of $^{56}\text{Ni}$. Far right column shows experimental energy levels of $^{57}\text{Ni}$ and the shell-gap evaluated by using the binding energies: $\Delta = 2BE(^{56}\text{Ni}) - BE(^{57}\text{Ni}) - BE(^{55}\text{Ni})$. Data are taken from ref. [16]
Limits of IPM

Example $^{16}$O

$\hbar \omega \sim 12 \text{ MeV}$
Limits of IPM

Example $^{16}\text{O}$

$Q_{\beta-} = 10419.0$

$\approx ^{16}\text{N}$

$6.5 \times 10^{-7}\% \quad 8.9$

$0.0012\% \quad 6.2$

$1.06\% \quad 4.3$

$4.8\% \quad 5.1$

$66.2\% \quad 4.5$

$0.012\% \quad 10.0$

$0+: T=0 \quad 0$

$2+: T=1 \quad 0$

$2+: T=0 \quad 0$

$1+: T=0 \quad 0$

$1+: T=0 \quad 0$

$0+: T=0 \quad 0$

$0+: T=0 \quad 0$

$12\text{C}_6$

$28.0\% \quad 9.11$

$7.13 \text{s}$

$\alpha$

$9844.5 \quad 0.62 \text{keV}$

$9585 \quad 420 \text{keV}$

$8871.9 \quad 125 \text{fs}$

$7116.85 \quad 8.3 \text{fs}$

$6917.1 \quad 4.70 \text{fs}$

$6129.89 \quad 18.4 \text{ps}$

$6049.4 \quad 67 \text{ps}$

$\text{stable}$
Limits of IPM

Example $^{16}\text{O}$
Monopole explanation

\[ E_{0\text{ph}} = 12\varepsilon_p + \frac{12 \cdot 11}{2} V_{pp} \]
\[ E_{1\text{ph}} = 11\varepsilon_p + \varepsilon_r + 55V_{pp} + 11V_{pr} \]
\[ E_{2\text{ph}} = 10\varepsilon_p + 2\varepsilon_r + 45V_{pp} + 20V_{pr} + V_{rr} \]
\[ E_{4\text{ph}} = 8\varepsilon_p + 4\varepsilon_r + 28V_{pp} + 32V_{pr} + 6V_{rr} \]
Monopole explanation

\[ \Delta_{1\text{ph}} = \varepsilon_r - \varepsilon_p + 11(V_{pr} - V_{pp}) \]
\[ \Delta_{2\text{ph}} = 2(\varepsilon_r - \varepsilon_p) + V_{pp} + 20(V_{pr} - V_{pp}) + (V_{rr} - V_{pp}) \]
\[ \Delta_{4\text{ph}} = 4(\varepsilon_r - \varepsilon_p) + 32(V_{pr} - V_{pp}) + 6(V_{rr} - V_{pp}) \]
Given $H_m$, the effective single particle energies [from Otsuka et al, Phys. Rev. Lett. 87, 82502 (2001)],
Evolution SPE far off stability

N=20

![Graph showing the evolution of SPE far off stability](Image)

- Proton number axis
- Effective SPE (MeV) axis
- Lines representing different shell states:
  - 1p3/2
  - 0f7/2
  - 1s1/2
  - 0d3/2
  - 0d5/2

Shell-Model Applications in nuclear physics and astrophysics – p. 47
Vanishing of shell closure at $N=20$
Computation of transition operators

Given a one-body transition operator $O$, how do we compute

$$\langle \Psi_f | O | \Psi_i \rangle$$

here $\Psi_i$ and $\Psi_f$ are many-body wave functions obtained from shell-model diagonalization

- **One body operators:**
  \[
  O = \sum_{i=1}^{A} o(r(i)) \quad \rightarrow \quad O = \sum_{i,j} \langle i | O | j \rangle a_i^\dagger a_j
  \]

- **We need to know**
  - the value of our one body operator between single particle wave functions $\langle i | O | j \rangle$
  - the one body density matrix elements $\langle \Psi_f | a_i^\dagger a_j | \Psi_i \rangle$
Computation of transition operators

- For $\langle i | O | j \rangle$, one needs (eventually) to know the radial part of the wave function: usually harmonic oscillator, sometimes wood-saxon.

$$\langle i | O | j \rangle = \int d^3r \ \phi_i^*(r) O(r) \phi_j(r)$$

- For the one body density matrix elements (same procedure as for the Hamiltonian):

$$a_5^\dagger a_2 |001011\rangle = |011001\rangle$$

Now we know the procedure to compute:

- **EL transitions**: $r^L Y_{L0}$

- **$\beta$ decay**:
  - Fermi decay: $\tau_\pm$
  - Gamow-Teller decay: $\sigma \tau_\pm$
Example calculation

\( \beta \) decay half-life calculation

- Determine initial state \( |\Psi_i\rangle \).
- Determine all possible final states \( |\Psi_f\rangle \).
- Compute matrix elements \( \langle \Psi_f | O | \Psi_i \rangle \)

\[
\lambda_f = \ln 2 \frac{K}{f(Z, W_f^0)} [B_f(F) + B_f(GT)]
\]

- Determine total decay rate:

\[
\lambda = \frac{\ln 2}{T_{1/2}} = \sum_f \lambda_f
\]
Lanczos Strength Functions

- $|\Psi\rangle$ ground state given nucleus.
- Sum rule state (or doorway): $|\Omega\rangle = \Omega|\Psi\rangle$.
- Total sum rule (sum over all final states of the matrix element squared) is the norm of state $|\Omega\rangle$:

$$
\langle \Omega | \Omega \rangle = \langle \Psi | \Omega^{\dagger} \Omega | \Psi \rangle = \sum_{i} |\langle i | \Omega | \Psi \rangle|^2
$$

- We can think of the state $|\Omega\rangle$ as a (probability) distribution over the eigenvalues $|i\rangle$ of $H$ with values

$$
|\langle i | \Omega \rangle|^2 = |\langle i | \Omega | \Psi \rangle|^2
$$
Any distribution can be characterized by the moments of the distribution.

\[
\tilde{E} = \langle \Omega | H | \Omega \rangle = \sum_i E_i \langle i | \Omega | \Psi \rangle^2
\]

\[
m_n = \langle \Omega | (H - \tilde{E})^n | \Omega \rangle = \sum_i (E_i - \tilde{E})^n \langle i | \Omega | \Psi \rangle^2
\]

Gaussian distribution characterized by two moments \((\tilde{E}, \sigma^2 = m_2)\)

\[
g(E) = \frac{1}{\sigma \sqrt{2\pi}} \exp\left(-\frac{(E-\tilde{E})^2}{2\sigma^2}\right)
\]
Lanczos Strength Functions

In general we only need a finite number of momenta. We can define a basis of $|\alpha\rangle$ states.

$$m_n = \langle \Omega | (H - \bar{E})^n | \Omega \rangle = \sum_{\alpha}^{N} (E_{\alpha} - \bar{E})^n |\alpha \rangle |\Omega \rangle |\Psi \rangle |^2 \quad (\forall n \leq M)$$

$$E_{\alpha} \approx \langle \alpha | H | \alpha \rangle$$

With $N$ states we can reproduce $2N$ moments of the distribution.
Lanczos Strength Functions

Example: Beta half-life

\[
\lambda \sim \sum_i f(Z, W_0^i) |\langle i | \Omega | \Psi \rangle|^2
\]

\[
f(Z, W_0^i) \approx \int_1^{W_0^i} W^2(W_0^i - 1)^2dW \approx \frac{(W_0^i)^5}{30}
\]
Lanczos Strength Functions

Lanczos method provides a natural way of determining the basis \( |\alpha\rangle \).

Initial vector \( |1\rangle = \frac{|\Omega\rangle}{\sqrt{\langle \Omega | \Omega \rangle}} \).

Each Lanczos iteration gives information about two new moments of the distribution.

\[
E_{12} |2\rangle = (H - E_{11})|1\rangle \\
E_{23} |3\rangle = (H - E_{22})|2\rangle - E_{12}|1\rangle \\
\ldots \\
E_{NN+1}|N+1\rangle = (H - E_{NN})|N\rangle \\
- E_{N-1N}|N-1\rangle
\]

where
\[
E_{NN} = \langle N|H|N\rangle, \quad E_{NN+1} = E_{N+1N}
\]

Diagonalizing Lanczos matrix after \( N \) iterations gives an approximation to the distribution with the same lowest \( 2N \) moments.
Evolution Strength Distribution

M1 Strength on $^{56}$Fe

1 Iteration

Graph showing the B(M1) distribution with energy (E) in MeV on the x-axis and B(M1) in units of $\mu_N^2$ on the y-axis.
Evolution Strength Distribution

M1 Strength on $^{56}$Fe

2 Iterations

E (MeV)

$B(M1) (\mu_N^2)$

0 5 10 15 20
Evolution Strength Distribution

M1 Strength on $^{56}\text{Fe}$

5 Iterations

$B(M1) (\mu_N^2)$ vs. $E$ (MeV)

10th Euro Summer School on Exotic Beams
Evolution Strength Distribution

M1 Strength on $^{56}$Fe

![Graph showing M1 strength distribution for $^{56}$Fe with 10 iterations.](image-url)
M1 Strength on $^{56}\text{Fe}$

20 Iterations
Evolution Strength Distribution

M1 Strength on $^{56}\text{Fe}$

50 Iterations

$B(\text{M1}) (\mu^2)$ vs. $E \text{ (MeV)}$
Evolution Strength Distribution

M1 Strength on $^{56}$Fe

100 Iterations
Evolution Strength Distribution

M1 Strength on $^{56}$Fe

250 Iterations
Evolution Strength Distribution

M1 Strength on $^{56}$Fe

400 Iterations
Evolution of Strength Distribution

GT Strength on $^{48}$Sc

![Graph showing GT strength distribution for $^{48}$Sc.](image)

- Left graph: 50 iterations
- Right graph: 9470 iterations

Energy (MeV) vs. $B(GT)$
Evolution of Strength Distribution

GT Strength on $^{48}\text{Sc}$

![Graph 1](50 it.)

![Graph 2](9470 it.)
Fermi matrix elements

\[ B(F) = \frac{1}{2J_i + 1} \sum_{M_i,M_f} |\langle J_f M_f; T_f T_{z_f} | \sum_{k=1}^{A} t^{k}_{\pm} | J_i M_i; T_i T_{z_i} \rangle|^2 \]

\[ B(F) = [T_i(T_i + 1) - T_{z_i}(T_{z_i} \pm 1)] \delta_{J_i,J_f} \delta_{T_i,T_f} \delta_{T_{z_f},T_{z_i} \pm 1} \]
Fermi matrix elements

\[
B(F) = \frac{1}{2J_i + 1} \sum_{M_i, M_f} |\langle J_f M_f; T_f T_{z_f} | \sum_{k=1}^{A} t^k_{\pm} | J_i M_i; T_i T_{z_i} \rangle|^2
\]

\[
B(F) = [T_i (T_i + 1) - T_{z_i} (T_{z_i} \pm 1)] \delta_{J_i, J_f} \delta_{T_i, T_f} \delta_{T_{z_f}, T_{z_i} \pm 1}
\]

Energetics:

\[
E_{\text{IAS}} = Q_{\beta} + \text{sign}(T_{z_i}) [E_C(Z + 1) - E_C(Z) - (m_n - m_H)]
\]
Fermi matrix elements

\[ B(F) = \frac{1}{2J_i + 1} \sum_{M_i,M_f} |\langle J_f M_f; T_f T_{z_f} | \sum_{k=1}^{A} t_{+}^{k} | J_i M_i; T_i T_{z_i} \rangle|^2 \]

\[ B(F) = [T_i (T_i + 1) - T_{z_i} (T_{z_i} \pm 1)] \delta_{J_i,J_f} \delta_{T_i,T_f} \delta_{T_{z_f},T_{z_i} \pm 1} \]

Energetics:

\[ E_{\text{IAS}} = Q_{\beta} + \text{sign}(T_{z_i}) [E_C(Z + 1) - E_C(Z) - (m_n - m_H)] \]

Selection rule:

\[ \Delta J = 0 \quad \Delta T = 0 \quad \pi_i = \pi_f \]
Fermi matrix elements

\[ B(F) = \frac{1}{2J_i + 1} \sum_{M_i, M_f} |\langle J_f M_f; T_f T_{z_f} | \sum_{k=1}^{A} t_k^\pm | J_i M_i; T_i T_{z_i} \rangle|^2 \]

\[ B(F) = [T_i (T_i + 1) - T_{z_i} (T_{z_i} \pm 1)] \delta_{J_i, J_f} \delta_{T_i, T_f} \delta_{T_{z_i}, T_{z_i} \pm 1} \]

Energetics:

\[ E_{IAS} = Q_\beta + \text{sign}(T_{z_i})[E_C(Z + 1) - E_C(Z) - (m_n - m_H)] \]

Selection rule:

\[ \Delta J = 0 \quad \Delta T = 0 \quad \pi_i = \pi_f \]

Sum rule (sum over all the final states):

\[ S(F) = S_-(F) - S_+(F) = 2T_{z_i} = (N - Z) \]
Gamow-Teller matrix elements

\[ B(GT) = \frac{g_A^2}{2J_i + 1} |\langle J_f; T_f T_{zf} || \sum_{k=1}^{A} \sigma^k t^k_\pm || J_i; T_i T_{zi} \rangle|^2 \]

\[ g_A = -1.2720 \pm 0.0018 \]

Selection rule:

\[ \Delta J = 0, 1 \ (\text{no} J_i = 0 \rightarrow J_f = 0) \quad \Delta T = 0, 1 \quad \pi_i = \pi_f \]

Ikeda sum rule:

\[ S(GT) = S_-(GT) - S_+(GT) = 3(N - Z) \]
SN1987A

Type II supernova in LMC (~ 55 kpc)

- $E_{\text{grav}} \approx 10^{53}$ erg
- $E_{\text{rad}} \approx 8 \times 10^{49}$ erg
- $E_{\text{kin}} \approx 10^{51}$ erg = 1 foe

neutrinos from SN1987A

\[ E_{\nu} \approx 2.7 \times 10^{53} \text{ erg} \]
Evolution Massive Stars

Evolution 20 solar mass star
Late stages core evolution

PRE-SUPERNova

COLLAPSE

NEUTRON TRAPPING

CORE BOUNCE

NEUTRINO BREAKOUT

EXPLOSION

COOLING
Semileptonic Weak Processes in Stars

orbital $e^-$ capture

$\beta^-$ decay

$\beta^+$ decay

bound-state $\beta^-$ decay

continuum charged (anti)lepton capture

(anti)neutrino capture

(anti)neutrino scattering

$$O_F \sim e^{i q \tau} \quad \quad O_{GT} \sim e^{i q \sigma \tau}$$
Presupernova evolution

- $T = 0.1–0.8$ MeV, $\rho = 10^7–10^{10}$ g cm$^{-3}$. Composition of iron group nuclei ($A = 45–65$)

- Important processes:
  - Electron capture:
    $$ e^- + (N, Z) \rightarrow (N + 1, Z - 1) + \nu_e $$
  - $\beta^-$ decay:
    $$ (N, Z) \rightarrow (N - 1, Z + 1) + e^- + \bar{\nu}_e $$

- Dominated by allowed transitions (Fermi and Gamow-Teller)

- Evolution decreases number of electrons ($Y_e$) and Chandrasekhar mass ($M_{\text{Ch}} \approx 1.4(2Y_e)^2 \ M_\odot$)

  - Fenomenological model (Fuller, Fowler, Newman, 1985)

  - Charge exchange reactions ($n, p$), ($d, ^{2}\text{He}$)

  - Microscopic model (Shell-Model) (Langanke & Martínez-Pinedo, 2001)
GT in charge exchange reactions

GT strength could be measured in CE reactions:

- $\text{GT}_- \text{ proved in } (p, n), (^3\text{He}, t)$.
- $\text{GT}_+ \text{ proved in } (n, p), (t, ^3\text{He}), (d, ^2\text{He})$.

Mathematical relationship ($E_p \geq 100$ MeV/nucleon):

$$\frac{d\sigma}{d\Omega dE}(0^\circ) \approx S(E_x)B(\text{GT})$$

$$B(\text{GT}) = \left(\frac{g_A}{g_V}\right)^2 \frac{\langle f \| \sum_k \sigma^k t^k \| i \rangle^2}{2J_i + 1}$$

Ikeda sum rule:

$$S_- - S_+ = 3(N - Z)$$
Independent Particle Model

GT+ strength in $^{58}\text{Ni}$ measured in $(n,p)$.

Independent particle model (FFN).

$^{58}\text{Ni}$ (El-Kateb et al, 1994)

$^{58}\text{Ni}$ (El-Kateb et al, 1994)
Gamow-Teller strength

$GT_{+}$ strength measured in charge-exchange $(n, p)$ experiments (TRIUMF).

![Diagram showing Gamow-Teller strength for various isotopes](image-url)
GT$^+_\text{+}$ strength measured in $(d,^{2}\text{He})$

High resolution Gamow-Teller distributions on $^{51}\text{V}, ^{58}\text{Ni} (^{64}\text{Ni},\ldots)$ measured at KVI (Groningen) by EuroSupernova Collaboration.

Shell-Model Calculation
G. Martinez-Pinedo

$^{51}\text{V}(d,^{2}\text{He})^{51}\text{Ti}$

$E_{\text{lab}}=171$ MeV

$\Theta_{\text{cm}}<1^\circ$

$2.14$ MeV

$3.62$ MeV

$4.88$ MeV

$^{51}\text{V}(n,p)$ Alford et al. (1993)
Shell-model (LMP) vs FFN rates
Most important nuclei to determine the electron capture rate

- $^{55}\text{Fe}$
- $^{57}\text{Co}$
- $^{57}\text{Fe}$
- $^{61}\text{Ni}$
- $^{57}\text{Fe}$
- $^{53}\text{Cr}$
- $^{57}\text{Fe}$
- $^{53}\text{Cr}$
- $^{55}\text{Co}$
- $^{57}\text{Fe}$
- $^{56}\text{Fe}$
- $^{53}\text{Cr}$
- $^{1}\text{H}$

**Graphs:**

- **15 M$_{\odot}$**
- **25 M$_{\odot}$**
Collapse phase

Important processes:

- Neutrino transport (Boltzmann equation):
  \[ \nu + A \rightleftharpoons \nu + A \] (trapping)
  \[ \nu + e^- \rightleftharpoons \nu + e^- \] (thermalization)
  
  cross sections \( \sim E_\nu^2 \)

- electron capture on protons:
  \[ e^- + p \rightleftharpoons n + \nu_e \]

What is the role of electron capture on nuclei?

\[ e^- + (N, Z) \rightleftharpoons (N + 1, Z - 1) + \nu_e \]

What is the role of inelastic neutrino-nucleus scattering?

\[ \nu + A \rightleftharpoons \nu + A^* \]
Collapse abundances

\[ T = 17.84 \text{ GK}, \rho = 3.39 \times 10^{11} \text{ g/cm}^3, Y_e = 0.379 \]
(Un)blocking electron capture at $N=40$

Independent particle treatment (Bruenn)

- $N=40$
- Blocked

- $f_{5/2}$
- $p_{1/2}$
- $f_{7/2}$
- $p_{3/2}$

Neutrons | Protons

- $g_{9/2}$

Unblocked

Correlations

Finite $T$

GT

Core

Weak Rates for nuclei with $A=65–112$ computed using the Shell Model Monte Carlo plus RPA approach
Electron capture: nuclei vs protons

Electron capture rates

Energetics

Abundances

\[ R_h = \sum_i Y_i \lambda_i = Y_h \langle \lambda_h \rangle \]

\[ R_p = Y_p \lambda_p \]
Electron capture on nuclei dominates over capture on protons
Consequences

With Rampp & Janka (General Relativic model)
15 M⊙ presupernova model from A. Heger & S. Woosley
Consequences

With Hix, Liebendörfer, Mezzacappa, Messer
(Newtonian Gravity)

Enclosed Mass ($M_\odot$)

Bruenn
LMS

$Y_e$

Enclosed Mass ($M_\odot$)

velocity ($10^4$ km s$^{-1}$)

Enclosed Mass ($M_\odot$)

$\rho$ (g cm$^{-3}$)

$10^10$

$10^12$

$10^14$

$10^8$

$10^6$

$10^4$

$10^2$

$10^0$

$0.0$ $0.2$ $0.4$ $0.6$ $0.8$ $1.0$ $1.2$

$(g \text{ cm}^{-3})$

$T$ (MeV)
Shock evolution

With Rampp & Janka (General Relativic model)

![Graph showing shock evolution with time (t) and shock radius (R_{shock}) with Bruenn and LMS models.](Image)
Neutrino interactions in the collapse

Bruenn and Haxton (1991)
Based on results for $^{56}$Fe

- Elastic scattering:
  \[ \nu + A \rightleftharpoons \nu + A \] (trapping)

- Absorption:
  \[ \nu_e + (N, Z) \rightleftharpoons e^- + (N - 1, Z + 1) \]

- $\nu$-$e$ scattering:
  \[ \nu + e^- \rightleftharpoons \nu + e^- \]

- Inelastic $\nu$-nuclei scattering:
  \[ \nu + A \rightleftharpoons \nu + A^* \]
\( \nu_e \) absorption cross section on \(^{56}\text{Fe}\)

\(^{56}\text{Fe}(\nu_e, e^-)^{56}\text{Co} \text{ measured by KARMEN collaboration:} \)

\[
\sigma_{\text{exp}} = 2.56 \pm 1.08(\text{stat}) \pm 0.43(\text{syst}) \times 10^{-40} \text{ cm}^2
\]

\[
\sigma_{\text{th}} = 2.38 \times 10^{-40} \text{ cm}^2
\]
Neutrino nucleosynthesis

Neutrinos interact with abundant nuclear species

- Neutral current ($\nu, \nu'$): Nucleus excited to particle unbound states that decay by particle emission.
- Charged current ($\nu_e, e^-$) and ($\bar{\nu}_e, e^+$).

<table>
<thead>
<tr>
<th>Product</th>
<th>Parent</th>
<th>Reaction</th>
</tr>
</thead>
<tbody>
<tr>
<td>$^{11}$B</td>
<td>$^{12}$C</td>
<td>($\nu, \nu' n), (\nu, \nu' p)$</td>
</tr>
<tr>
<td>$^{15}$N</td>
<td>$^{16}$O</td>
<td>($\nu, \nu' n), (\nu, \nu' p)$</td>
</tr>
<tr>
<td>$^{19}$F</td>
<td>$^{20}$Ne</td>
<td>($\nu, \nu' n), (\nu, \nu' p)$</td>
</tr>
<tr>
<td>$^{138}$La</td>
<td>$^{138}$Ba</td>
<td>($\nu, e^-$)</td>
</tr>
<tr>
<td>$^{138}$La</td>
<td>$^{139}$La</td>
<td>($\nu, \nu' n$)</td>
</tr>
<tr>
<td>$^{180}$Ta</td>
<td>$^{180}$Hf</td>
<td>($\nu, e^-$)</td>
</tr>
<tr>
<td>$^{181}$Ta</td>
<td>$^{181}$Hf</td>
<td>($\nu, \nu' n$)</td>
</tr>
</tbody>
</table>

$^{11}$B/$^{10}$B traces galactic evolution:

- Big ratio (100) predicted by $\nu$-process.
- Cosmic rays spallation reactions on C give a ratio of 2.
- Solar system ratio is 4.

$^{138}$La, $^{180}$Ta production is sensitive to neutrino oscillations.
Nucleosynthesis with and without $\nu$

With Heger (Los Alamos)
The production of $^{138}\text{La}$

Produced by $^{139}\text{La}(\gamma, n)$, $^{138}\text{Ba}(\nu_e, e^-)$
The production of $^{180}\text{Ta}$

Produced by $^{181}\text{Ta}(\gamma, n)$, $^{180}\text{Hf}(\nu_e, e^-)$
Presupernova abundances

$T = 9.01 \text{ GK}, \rho = 6.80e+09 \text{ g/cm}^3, Y_e = 0.433$

$A = 65 \text{ (LMP)}$

Log (Mass Fraction)

-5 -4 -3 -2

Z (Proton Number)

N (Neutron Number)