Symmetries in *N*~*Z* nuclei

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Symmetry in quantum mechanics

Symmetries of the nuclear shell model

Interacting boson model

Applications to *N*~*Z* nuclei

Symmetry in quantum mechanics

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Symmetries, groups and algebras

Degeneracy and state labelling

Dynamical symmetry breaking

Selection rules

Example: isospin SU(2) symmetry

Symmetry

- Greek origin: «with proportion, with order».
- *Oxford Dictionary of Current English:*
  - Symmetry: «…right correspondence of parts; quality of harmony or balance (in size, design etc.) between parts».
- Symmetry in physics via group theory (mathematical theory of symmetry).
- Two main protagonists of group theory:
  - Evariste Galois (1831): **Galois** theory.
  - Sophus Lie (1873): **Lie** algebra.

Definition of symmetry

- Assume a hamiltonian *H* which commutes with operators g_i that form a Lie algebra *G*:

$$\forall g_i \in G : \quad \left[H, g_i\right] = 0$$

- $\Rightarrow$ *H* has *symmetry G* or is *invariant under G*.
- Three mathematical concepts:
  – Group.
  – Lie group: infinite, continuous and connected.
  – Lie algebra: from a subset of group elements close to the identity element.

Definition of a group

- A set $\{g\}\equiv G$ and a *multiplication* $\circ$ satisfying
  - Closure:

    $$\forall g, g' \in G: \quad g \circ g' \in G$$

  - Associativity:

    $$\forall g, g', g'' \in G: \quad g \circ (g' \circ g'') = (g \circ g') \circ g''$$

  - Existence of an identity element e:

    $$\forall g \in G: \quad e \circ g = g \circ e = g$$

  - Existence of unique inverse for every element g:

    $$\forall g \in G: \quad g^{-1} \circ g = g \circ g^{-1} = e$$

- Examples: permutations S_n, rotations D_n,...

Lie groups

- A Lie group contains an *infinite* number of elements characterized by a set of *continuous* variables.
- Additional conditions:
  - Connection to the identity element.
  - Analytic multiplication function.
- Example: rotations in 2 dimensions, SO(2).

$$g(\theta) = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

Lie algebras

- Idea: obtain properties of the *infinite* number of elements g of a Lie *group* in terms of those of a *finite* number of elements g_i (called generators) of a Lie *algebra*.
- All properties of a Lie algebra follow from the commutation relations between its generators:

$$\left[g_i, g_j\right] \equiv g_i \circ g_j - g_j \circ g_i = \sum_k c_{ij}^k g_k$$

- Generators satisfy the **Jacobi** identity:

$$\left[g_i,\left[g_j, g_k\right]\right]+\left[g_j,\left[g_k, g_i\right]\right]+\left[g_k,\left[g_i, g_j\right]\right]=0$$

Rotations in 2 dimensions, SO(2)

- Matrix representation of finite elements:

$$g(\varphi) = \begin{pmatrix} \cos\varphi & \sin\varphi \\ -\sin\varphi & \cos\varphi \end{pmatrix}$$

- Infinitesimal element and generator:

$$\lim_{\varphi \to 0} g(\varphi) \approx \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \varphi \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \equiv e + \varphi g_1$$

- Exponentiation leads back to finite elements:

$$g(\varphi) = \lim_{n \to \infty} \left(e + \frac{\varphi}{n} g_1 \right)^n = \exp(e + \varphi g_1) = \exp \begin{pmatrix} 1 & \varphi \\ -\varphi & 1 \end{pmatrix} = g(\varphi)$$

Rotations in 3 dimensions, SO(3)

- Matrix representation of finite elements:

$$g(\theta_1,\theta_2,\theta_3) = \begin{pmatrix} \cos\theta_3 & \sin\theta_3 & 0 \\ -\sin\theta_3 & \cos\theta_3 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos\theta_2 & 0 & \sin\theta_2 \\ 0 & 1 & 0 \\ -\sin\theta_2 & 0 & \cos\theta_2 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta_1 & -\sin\theta_1 \\ 0 & \sin\theta_1 & \cos\theta_1 \end{pmatrix}$$

- Generators:

$$g_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \quad g_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \quad g_3 = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

- The *structure constants* are given by the 3-space **Levi-Civita** tensor ε_{ijk}:

$$[g_i, g_j] = \sum_{k=1}^{3} \varepsilon_{ijk} g_k \quad (\varepsilon_{123} = \varepsilon_{231} = \varepsilon_{312} = -\varepsilon_{132} = -\varepsilon_{321} = -\varepsilon_{213} = 1)$$

Rotations in 3 dimensions, SO(3)

- Exponentiation leads back to finite elements:

$$g(\alpha_1,\alpha_2,\alpha_3) = \exp\left(e + \sum_{i=1}^{3} \alpha_i g_i\right)$$

- Quantum mechanical angular momentum operators are related to SO(3) generators:

$$J_k = -ig_k, \quad \left[J_j, J_k\right] = i\sum_{l=1}^{3} \epsilon_{jkl} J_l$$

Consequences of symmetry

- *H* has symmetry *G* implies
  - *H* has degeneracy structure: if $|\psi\rangle$ is an eigenstate of *H* with energy *E*, so is $g_i|\psi\rangle$:

$$H|\psi\rangle = E|\psi\rangle \Rightarrow Hg_i|\psi\rangle = g_i H|\psi\rangle = Eg_i|\psi\rangle$$

  - Degeneracy structure and labels of eigenstates of *H* are determined by the symmetry *G*:

$$H|\Gamma\gamma\rangle = E(\Gamma)|\Gamma\gamma\rangle$$

$$g_i|\Gamma\gamma\rangle = \sum_{\gamma'} a^{\Gamma}_{\gamma\gamma'}(g_i)|\Gamma\gamma'\rangle$$

  - Degenerate eigenstates correspond to *irreducible* representations of *G* (**Wigner**'s principle).

The hydrogen atom

- The hamiltonian of the hydrogen atom is

$$H = -\frac{\hbar^2}{2M}\nabla^2 - \frac{\kappa}{r}, \quad \kappa = Ze^2$$

- Eigenstates $\psi_{nlm}(r,\theta,\phi)$ [for $n=1,2,\ldots$, $l=0,1,\ldots n-1$, $m=-l,-l+1,\ldots+l$] with energy $-M\kappa^2/2h^2n^2$.
- *H* has rotational symmetry SO(3):

$$\left[H,\vec{L}\right] \equiv \left[H,\vec{r}\times\vec{p}\right] = 0 \Rightarrow m\text{-degeneracy}$$

- What is the origin of additional *l*-degeneracy? According to Wigner's principle it *must* be related to a symmetry.

Symmetry of the hydrogen atom

- The **Runge-Lenz** vector ***A***:

$$\vec{A} = \frac{1}{2M}\left(\vec{p} \times \vec{L} - \vec{L} \times \vec{p}\right) - \kappa \frac{\vec{r}}{r} \Rightarrow \left[H, \vec{A}\right] = 0$$

- ***L*** and ***A*** close under commutation (almost):

$$\left[L_j, L_k\right] = i\hbar \sum_{l=1}^{3} \varepsilon_{jkl} L_l, \left[L_j, A_k\right] = i\hbar \sum_{l=1}^{3} \varepsilon_{jkl} A_l, \left[A_j, A_k\right] = i\hbar \sum_{l=1}^{3} \varepsilon_{jkl} \frac{-2H}{M} L_l$$

- ***L*** and ***B***≡*(-2H/M)*$^{-1/2}$***A*** form an SO(4) algebra:

$$\left[L_j, L_k\right] = i\hbar \sum_{l=1}^{3} \varepsilon_{jkl} L_l, \left[L_j, B_k\right] = i\hbar \sum_{l=1}^{3} \varepsilon_{jkl} B_l, \left[B_j, B_k\right] = i\hbar \sum_{l=1}^{3} \varepsilon_{jkl} L_l$$

- ⇒ The hamiltonian of the hydrogen atom has SO(4) symmetry.

W. Pauli, Z. Phys. **36** (1926) 336

Spectrum of the hydrogen atom

- Isomorphism of SO(4) and SO(3)⊕SO(3):

$$\vec{F}^{\pm} = \tfrac{1}{2}\left(\vec{L} \pm \vec{B}\right) \Rightarrow \left[F_j^{\pm}, F_k^{\pm}\right] = i\hbar \sum_{l=1}^{3} \varepsilon_{jkl} F_l^{\pm}, \quad \left[F_j^{+}, F_k^{-}\right] = 0$$

- Since ***L*** and ***B*** are orthogonal:

$$\left\langle \vec{F}^{\pm} \cdot \vec{F}^{\pm} \right\rangle = j_{\pm}\left(j_{\pm} + 1\right) \Rightarrow \left\langle \vec{F}^{+} \cdot \vec{F}^{+} \right\rangle = \left\langle \vec{F}^{-} \cdot \vec{F}^{-} \right\rangle \equiv j(j + 1)$$

- Since the following operator relation is valid,

$$A^2 = \frac{2H}{M}\left(L^2 + \hbar^2\right) + \kappa^2$$

- …the hydrogen spectrum follows:

$$\frac{1}{4}\left\langle L^2 - \frac{M}{2H} A^2 \right\rangle = j(j + 1) \Rightarrow E = -\frac{M\kappa^2}{2\hbar^2 (2j + 1)^2}, \qquad j = 0, \tfrac{1}{2}, 1, \tfrac{3}{2}, \ldots$$

Casimir or invariant operators

- Operators that commute with all generators of a Lie algebra G are called **Casimir** operators $C_n[G]$ where n is the order in the generators.
- Thus H has symmetry G if

$$H = \sum_n \kappa_n C_n[G] \Rightarrow \forall g_i \in G : \quad [H, g_i] = 0$$

- Second-order Casimir operator ($n=2$):

$$C_2[G] = \sum_{ij} \tilde{c}^{ij} g_i g_j, \quad \left(\tilde{c}_{ij} = \sum_{kl} c_{ik}^{l} c_{jl}^{k} \right) \quad \left(\sum_j \tilde{c}^{ij} \tilde{c}_{jk} = \delta_{ik} \right)$$

- Second-order Casimir operator of SO(3):

$$C_2[\mathrm{SO}(3)] = \sum_{i=1}^{3} J_i^2 \equiv \vec{J}^{\,2}$$

Dynamical symmetry

- Assume (at least) two algebras G_1 G_2 and the hamiltonian:

$$H = \sum_{n_1} \kappa_{n_1} C_{n_1}[G_1] + \sum_{n_2} \kappa'_{n_2} C_{n_2}[G_2]$$

- $\Rightarrow$ H has symmetry G_2 but *not* symmetry G_1!
- Eigenstates $|\Gamma_1 \gamma_2 \eta\rangle$ of H are independent of parameters κ_n and κ'_n in the hamiltonian.
- Dynamical symmetry (DS) breaking "*splits but does not admix eigenstates*".
- Better name: spectrum generating algebra.

Isospin symmetry in nuclei

- Empirical observations:
  - About equal masses of n(eutron) and p(roton).
  - n and p have spin 1/2.
  - Equal (to ~1%) nn, np, pp strong forces.
- This suggests introduction of isospin label and isospin symmetry of nuclear hamiltonian:

$$\mathrm{n}: \quad t = \tfrac{1}{2}, m_t = +\tfrac{1}{2}; \qquad \mathrm{p}: \quad t = \tfrac{1}{2}, m_t = -\tfrac{1}{2}$$

$$\Rightarrow \quad t_+\mathrm{n} = 0, \quad t_+\mathrm{p} = \mathrm{n}, \quad t_-\mathrm{n} = \mathrm{p}, \quad t_-\mathrm{p} = 0, \quad t_z n = \tfrac{1}{2}\mathrm{n}, \quad t_z p = -\tfrac{1}{2}\mathrm{p}$$

W. Heisenberg, Z. Phys. **77** (1932) 1
E.P. Wigner, Phys. Rev. **51** (1937) 106

Isospin SU(2) symmetry

- Isospin operators form an SU(2) algebra:

$$\left[t_z, t_\pm\right] = \pm t_\pm, \quad \left[t_+, t_-\right] = 2t_z$$

- Assume the nuclear hamiltonian satisfies

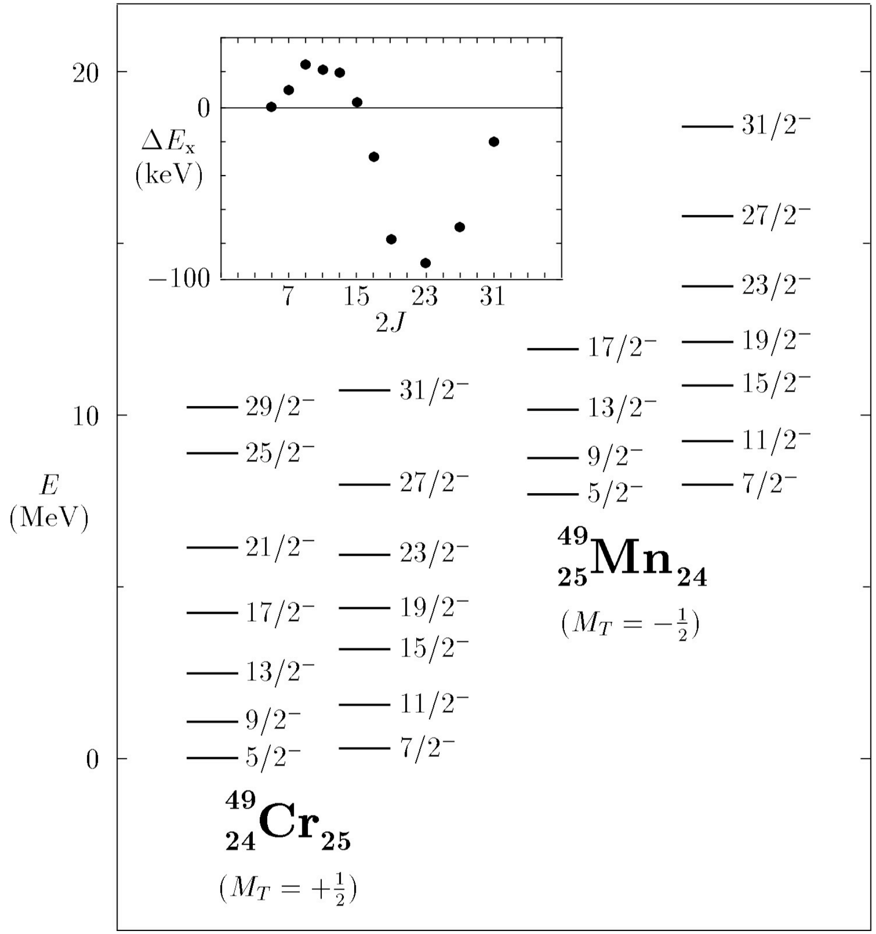
$$\left[H_{\text{nucl}}, T_\mu\right] = 0, \quad T_\mu = \sum_{k=1}^{A} t_\mu(k)$$

- $\Rightarrow$ H_{nucl} has SU(2) symmetry with degenerate states belonging to isobaric multiplets:

$$|\square T M_T\rangle, \quad M_T = -T, -T+1, \ldots, +T$$

Isospin symmetry breaking

- Empirical evidence for isospin symmetry breaking from isobaric multiplets.
- Example: *T=1/2* doublet of *A=49* nuclei.



C.D. O'Leary *et al.,* Phys. Rev. Lett. **79** (1997) 4349

Isospin SU(2) dynamical symmetry

- The Coulomb interaction can be written *approximately* as

$$H_{\text{Coul}} \;\square\; \square_0 + \square_1 T_z + \square_2 T_z^2 \;\square\; \left[H_{\text{Coul}}, T_z\right] = 0, \quad \left[H_{\text{Coul}}, T_\pm\right] \neq 0$$

- $\square$ H_{Coul} has SU(2) dynamical symmetry and SO(2) symmetry.
- M_T-degeneracy is lifted according to

$$H_{\text{Coul}} \left| \square T M_T \right\rangle = \left(\square_0 + \square_1 M_T + \square_2 M_T^2\right) \left| \square T M_T \right\rangle$$

- Summary of state labelling:

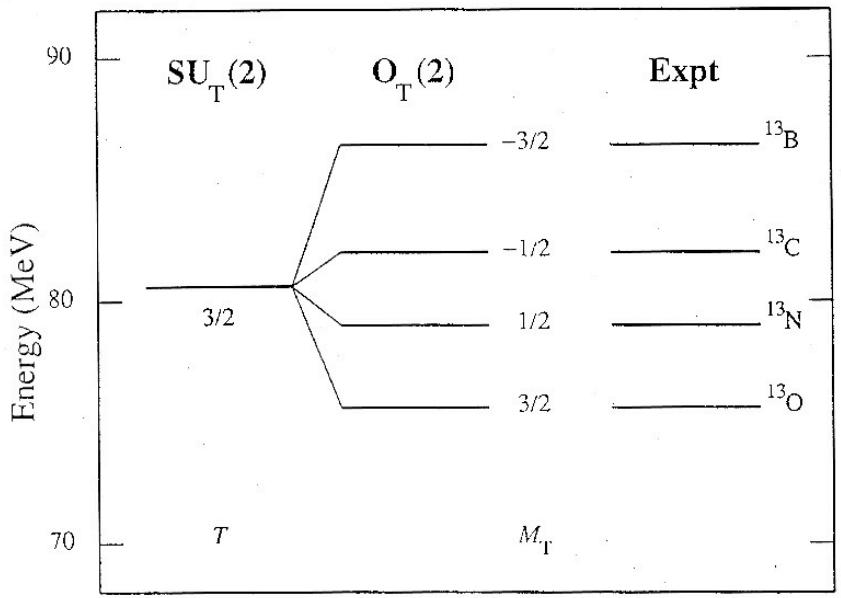
$$\begin{array}{ccc} \mathrm{SU}(2) & & \mathrm{SO}(2) \\ \square & & \square \\ T & & M_T \end{array}$$

Isobaric multiplet mass equation

- Isobaric multiplet mass equation:

$$E(\square T M_T) = \square(\square, T) + \square_1 M_T + \square_2 M_T^2$$

- Example: *T=3/2* multiplet for *A=13* nuclei.



E.P. Wigner, *Proc. Robert A. Welch Foundation Conf. On Chemical Research,* (Welch Foundation, Houston, 1958) p. 88

Flavour SU(3) dynamical symmetry

- Enlarge isospin SU(2) to SU(3) to connect more 'elementary' particles:

$$\mathrm{SU}(3) = \{T_z, T_\pm, Y, U_\pm, V_\pm\}$$

- Mass operator *M* with SU(3) symmetry:

$$\forall g_i \in \mathrm{SU}(3): \quad [M, g_i] = 0$$

- Mass operator *M* with SU(3) DS:

$$\begin{array}{cccc} \mathrm{SU}(3) & \mathrm{U}(1) & [\mathrm{SU}(2) & \mathrm{SO}(2)] \\ \downarrow & \downarrow & \downarrow & \downarrow \\ (\lambda,\mu) & Y & T & M_T \end{array}$$

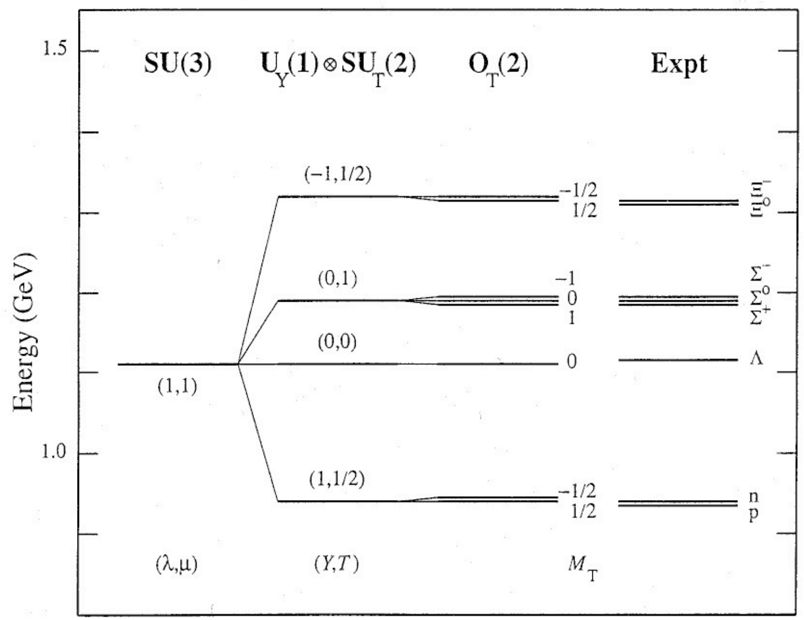
M. Gell-Mann, Phys. Rev. **125** (1962) 1067
S. Okubo, Prog. Theor. Phys. **27** (1962) 949

Gell-Mann-Okubo mass formula

- Gell-Mann-Okubo mass formula:

$$E(\lambda\mu YTM_T) = \square(\lambda,\mu) + \square_1 Y + \square_2\left(T(T+1) - \tfrac{1}{4}Y^2\right) + \square_3 M_T + \square_4 M_T^2$$

- Example: (1,1) octet.



Selection rules

- The most important consequence of a (dynamical) symmetry is the existence of *conserved quantum numbers*. These lead to selection rules in radiative-transition and particle-transfer processes.
- Assume that eigenstates and transition/transfer operators (tensors) carry labels *□* and *□*:

G_1	G_2
□	□
□	*□*

Selection rules

- Assume a transition/transfer process:

$$|\square_{\rm i}\square_{\rm i}\rangle \;\square\; \overset{\square\square}{\square\square\square} \;\square\; |\square_{\rm f}\square_{\rm f}\rangle$$

- The process is allowed if and only if

$$\square_{\rm f} \;\square\; \square_{\rm i} \;\square\; \square$$

- The *intensity* of the process is governed by the generalized **Wigner-Eckart** theorem:

$$\langle\square_{\rm f}\square_{\rm f}|T^{\square}_{\square}|\square_{\rm i}\square_{\rm i}\rangle = \langle\square_{\rm i}\square_{\rm i}\;\square\square|\square_{\rm f}\square_{\rm f}\rangle\langle\square_{\rm f}\|T^{\square}\|\square_{\rm i}\rangle$$

$\langle\square_{\rm i}\square_{\rm i}\;\square\square|\square_{\rm f}\square_{\rm f}\rangle:$ generalized **Clebsch** □ **Gordan** coefficients

$\langle\square_{\rm f}\|T^{\square}\|\square_{\rm i}\rangle:$ reduced matrix element (no $\square$ dependence)

Angular momentum selection rules

- SO(3) symmetry implies eigenstates $|JM_J\rangle$
- Consider *e.g.* the *E2* transition operator:

$$T^{E2}_{\mu} = \sum_{k=1}^{A} e_k r^2(k) Y_{2\mu}(\theta_k, \varphi_k)$$

- Wigner-Eckart theorem:

$$\langle J_\mathrm{f} M_{J\mathrm{f}} | T^{E2}_{\mu} | J_\mathrm{i} M_{J\mathrm{i}} \rangle = \langle J_\mathrm{i} M_{J\mathrm{i}}\, 2\mu | J_\mathrm{f} M_{J\mathrm{f}} \rangle \langle J_\mathrm{f} \| T^{E2} \| J_\mathrm{i} \rangle$$

- Selection rule:

$$J_\mathrm{f} \in J_\mathrm{i} \otimes 2 \Rightarrow J_\mathrm{f} = |J_\mathrm{i} - 2|, |J_\mathrm{i} - 1|, J_\mathrm{i}, J_\mathrm{i} + 1, J_\mathrm{i} + 2$$

Isospin selection rules

- SU(2) symmetry implies eigenstates $|TM_T\rangle$.
- Internal *E1* transition operator is isovector:

$$T^{E1}_{\mu} = \sum_{k=1}^{A} e_k\, r_{\mu}(k) = \frac{e}{2}\left[\sum_{k=1}^{A} r_{\mu}(k) + 2\sum_{k=1}^{A} t_z(k)\, r_{\mu}(k)\right]$$

- Wigner-Eckart theorem:

$$\langle T_\mathrm{f} M_{T\mathrm{f}} | T^{E1}_{0} | T_\mathrm{i} M_{T\mathrm{i}} \rangle = \langle T_\mathrm{i} M_{T\mathrm{i}}\, 10 | T_\mathrm{f} M_{T\mathrm{f}} \rangle \langle T_\mathrm{f} \| T^{E1} \| T_\mathrm{i} \rangle$$

- Selection rule for *N*=*Z* nuclei:

$$M_{T\mathrm{i}} = M_{T\mathrm{f}} = 0: \quad \langle T_\mathrm{i} 0\; 10 | T_\mathrm{f} 0 \rangle = 0 \text{ for } T_\mathrm{i} = T_\mathrm{f}$$

- *No E1 transitions are allowed between N=Z states with the same isospin.*

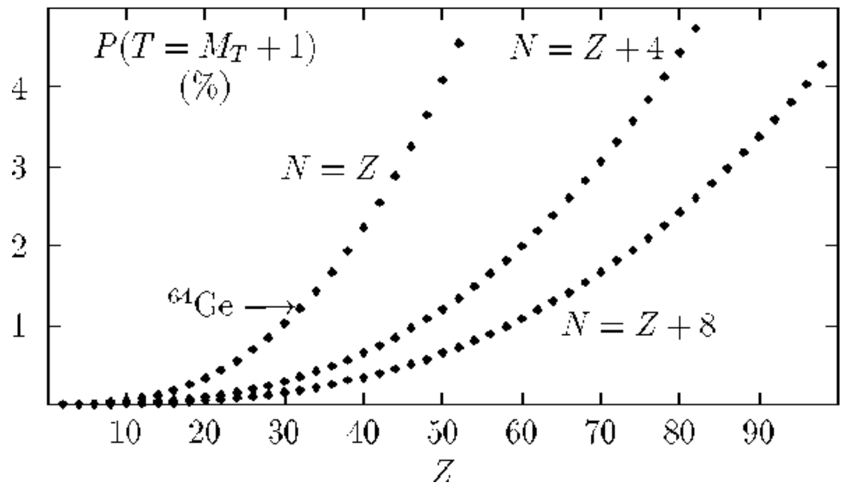
L.E.H. Trainor, Phys. Rev. **85** (1952) 962
L.A. Radicati, Phys. Rev. **87** (1952) 521

Coulomb isospin mixing

- Coulomb interaction has isoscalar, isovector and isotensor parts:

$$H_{\mathrm{Coul}} = \sum_{k<l} e_k\, e_l / r_{kl} = e^2 \sum_{k<l} \left(\tfrac{1}{2} + t_z(k)\right)\left(\tfrac{1}{2} + t_z(l)\right) / r_{kl}$$

- Isovector part is main responsible of mixing.
- Many approaches to calculate mixing. All give maximum for *N*=*Z*.

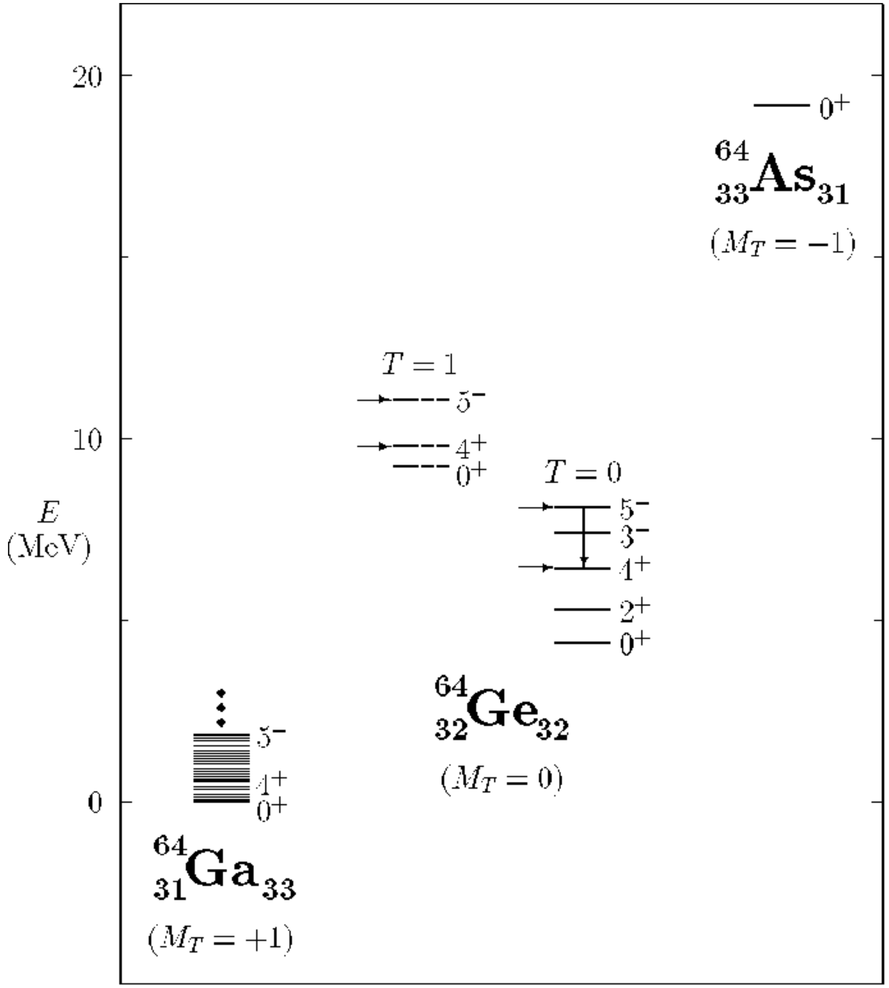


G. Colò *et al.*, Phys. Rev. C **52** (1995) R1175

E1 transitions and isospin mixing

- *B(E1;5⁻→ 4⁺)* in ^{64}Ge from:
  - lifetime of *5⁻* level;
  - *δ(E1/M2)* mixing ratio of *5⁻→ 4⁺* transition;
  - relative intensities of transitions deexciting *5⁻*.
- Estimate of minimum isospin mixing:

$$P(T=1,5^-) + P(T=1,4^+) \geq 2.5\%$$



E.Farnea *et al.*, Phys. Lett. B **551** (2003) 56

Quantal many-body systems

- Generic many-body hamiltonian:

$$H = \sum_i \varepsilon_i c_i^+ c_i + \sum_{ijkl} \upsilon_{ijkl} c_i^+ c_j^+ c_k c_l + \cdots$$

- Rewrite H as (bosons: $q=0$; fermions: $q=1$)

$$H = \sum_{il} \left(\varepsilon_i \delta_{il} - (-)^q \sum_j \upsilon_{ijjl} \right) u_{il} + (-)^q \sum_{ijkl} \upsilon_{ijkl} u_{ik} u_{jl} + \cdots$$

- Operators $u_{ij} \equiv c_i^+ c_j$ generate U(n) for $q=0,1$:

$$\left[u_{ij}, u_{kl}\right] = u_{il}\delta_{jk} - u_{kj}\delta_{il} - \underbrace{\left(1 - (-)^q\right)\left[c_i^+ c_k^+ c_l c_j + c_i^+ c_k^+ c_j c_l\right]}_{0}$$

Quantal many-body systems

- Given a chain of nested algebras:

$$\mathrm{U}(n) = G_{\mathrm{SGA}} = G_1 \supset G_2 \supset \cdots \supset G_{\mathrm{sym}}$$

- A *particular* class of many-body hamiltonians is of the form:

$$H = \sum_{n_1} \kappa_{n_1} C_{n_1}[G_1] + \sum_{n_2} \kappa_{n_2} C_{n_2}[G_2] + \cdots$$

- *H* is a sum of commuting operators,

$$\forall n_a, n_b, a, b: \quad \left[C_{n_a}[G_a], C_{n_b}[G_b]\right] = 0$$

- …and is thus integrable and solvable!