⓪ Introduction

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① The "usual picture" + what's wrong with it. (Boltzmann Eq)

② Lagrange Multipliers + Spin-Density matrix
③ States Discussion
~~③ [illegible]~~

④ ~~Flavor basis calculation~~
④ Mass Matrix + Mixing

Main references:

M^c Elrath arXiv: 0909.3090 (Ignore mixing explanation p.3+)

M^c Elrath arXiv: 0812.2696 (2^nd section only, on Δx + wave packets) rest of paper

Fullen, Kishimoto arXiv: 0811.4370 is correct if you remove the word "super-fluid" → corrections/replacements to appear. Doing new things is hard!

ν Index of refraction:
Langacker, Liu arXiv: hep-ph/9206209
Notzold, Raffelt NPB 307(4): 924-936 October 1988

⓪

$$T_\nu = \left(\frac{4}{11}\right)^{1/3} T_\gamma = 1.95\,K = 1.68\times10^{-4}\,eV$$

$$n_{\nu_i} = n_{\bar{\nu}_i} = \frac{3}{22} N_\gamma = \frac{56}{cm^3}$$

$$\eta_\nu = \frac{n_\nu - n_{\bar\nu}}{n_\gamma} \simeq \eta_b = \frac{n_b - n_{\bar b}}{n_\gamma} \sim 10^{-10}$$

$$p_{F\nu_i} = 2.34\times10^{-4}\,eV \qquad = (3\pi^2 n)^{1/3}$$

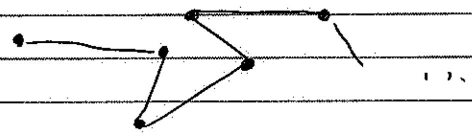
$$\sqrt{\Delta m_{12}^2} = 8.94\times10^{-3}\,eV$$

$$\sqrt{\Delta m_{23}^2} = 5.29\times10^{-2}\,eV$$

$$G_F = 2.92\times10^{12}\,eV$$

The "Usual Picture"

Particles are billiard balls $\Rightarrow$ Boltzmann equation (classical)



Point-like collisions
in between collisions, particles propagate in "vacuum"

3 parameters needed to describe any gas of particles

	Boltzmann	Neutrino gas
$k_F = (3\pi^2 n)^{1/3}$ Fermi momentum $\cong 1/\lambda$ de Broglie wavelength $\cong T$ temperature	$r \ll a$	$r \gg a$
r range of interaction	atomic (air) $r \sim 10^{-10}$ m	$\frac{1}{k_F} \sim \lambda \cong \frac{1}{T} \sim$ mm
$a = \sigma^{1/2}$ scattering length	$\lambda \sim (3mkT)^{-1/2}$	$r = (G_F)^{1/2} \sim 0.001$ fm
$l = \frac{a^2}{k_F^3} = (\sigma n v)^{-1}$ mean free path	$l \sim (\alpha n l^2 v)^{-1}$	$l = (n\sigma)^{-1} \sim 10^{56}$ m
$\Rightarrow$ RANGE is most important dynamical quantity		

$\Rightarrow$ Neutrinos are not hydrodynamic $\qquad a = T G_F \sim T r^2 \sim 10^{-38}$ m

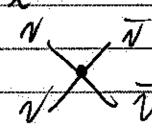
Boltzmann then does expansion in $\xi = \frac{k_F a}{k_F l} \ll 1$ — measure of # particles crossed w/o scattering between scattering events

"gas parameter"

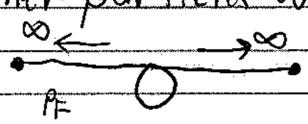
$(\overline{10}^{18})/10^{-2}$ m

So a is zero and we can remove it from the problem.
(Just as r is removed in Boltzmann approach $\Rightarrow$ local, point-like scattering operators) Equivalent to $l = \infty$

r is contact 4-Fermi operators



What this tells us is that in the absence of scattering, the propagator becomes very long. $\Rightarrow$ we need to worry about other particles which cross our path + don't scatter.



interaction is contact operator (range r)
loops know about presence of other particles.

$(\overline{10}^{18})/1$

$$\propto G_F \int_0^{p_F} \frac{d^4p}{(2\pi)^4} \frac{\not p - m}{(p^2 - m^2 + i\varepsilon)} = p_F^3 G_F = p_F^3 r^2 \Rightarrow$$ Want an expansion for $a \simeq 0$, $p_F r \ll 1$

How many other particles are seen in loop?

Compute $\Delta x(t)^2 = \Delta x_0^2 + \Delta v^2 t^2 > n^{-2/3} \Rightarrow \sigma < \frac{1 - v^2}{n^{2/3}}$ $\rightarrow$ massive ν's.

$\Rightarrow$ today $\Delta x \sim 6$ pc or $t \sim$ ps

Proposal for a new approach: Lagrange Multipliers + Mean Field Theory (aka Self-Consistent Field Theory)

(2) Which operators have expectation values given a non-empty state $|\Psi\rangle$? $\Rightarrow$ expand theory around actual background instead of ficticious "vac

Arrange operators so annihilation is to the right so that it "measures" a property of the state. For example:

$$(a_+^\dagger a_+ - a_-^\dagger a_-)|\Psi\rangle$$

We can write this operator using fields instead: $\chi^\dagger \sigma^0 \chi$ (Weyl) or $\bar{\Psi}\gamma^0\psi$ (Majorana or Dirac)

If this operator gets a vev, we can fix it using a Lagrange Multiplier, e.g. add to Lagrangian

$\mu\bar{\Psi}\gamma^0\psi - \mu N$ is standard form of chem. pot.

where $N = |\langle\Psi|a_+^\dagger a_+ - a_-^\dagger a_-|\Psi\rangle|^2$. This is expanding around non-empty background. E.g. all eq's of motion are subject to the constraint that N is fixed. Can drop μN (it's constant → non-dynamical)

In general, given the operator $\mathcal{O}$ such that $\langle\mathcal{O}\rangle \neq 0$, we MUST add:

$$\mathcal{L} \mathrel{+}= \lambda(\mathcal{O} - \langle\mathcal{O}\rangle)$$

to the Lagrangian. (else we are expanding around the wrong vac

The leading order: put # density into problem for Weyl fermion

$$v = (a_+, a_-), \quad v^\dagger = \begin{pmatrix} a_+^\dagger \\ a_-^\dagger \end{pmatrix}, \quad \hat{\rho} = v^\dagger v = \begin{pmatrix} a_+^\dagger a_+ & a_+^\dagger a_- \\ a_-^\dagger a_+ & a_-^\dagger a_- \end{pmatrix}$$

Corresponding to operators: (p suppressed)

$$\mathrm{Tr}[\hat{\rho}\sigma_1] = a_-^\dagger a_+ + a_+^\dagger a_- = \tfrac{1}{2}[\chi^\dagger i\sigma_2 \chi^* - \chi^T i\sigma_2 \chi]$$

$$\mathrm{Tr}[\hat{\rho}\sigma_2] = i(a_+^\dagger a_- - a_-^\dagger a_+) = \tfrac{1}{2}[\chi^\dagger \sigma_2 \chi^* + \chi^T \sigma_2 \chi]$$

← rotates with lepton # symmetr $\chi \to e^{i\theta}\chi$

$$\mathrm{Tr}[\hat{\rho}\sigma_3] = a_+^\dagger a_+ - a_-^\dagger a_- = \chi^\dagger \sigma^0 \chi$$

Note $\mathrm{Tr}[\hat{\rho}\sigma^0]$

See P. Mannheim, "Nonlocal Quantum #s"

$$\chi(x) = \int \frac{d^3p}{(2\pi)^3\sqrt{2E}}\left[a_+ u_+ e^{ipx} + a_-^\dagger v_- e^{-i}\right.$$

$$u_+ = \begin{pmatrix}1\\0\end{pmatrix} \quad v_- = \begin{pmatrix}0\\1\end{pmatrix}$$

③ States Discussion

Consider the single-particle state

$$\psi(p) = Re^{i\theta} a_+^\dagger(p)|0\rangle + \sqrt{1-R^2}\, e^{i\varphi} a_-^\dagger(p)|0\rangle$$

$$\langle\hat{\rho}\rangle = \begin{pmatrix} R^2 & R\sqrt{1-R^2}\, e^{i(\theta-\varphi)} \\ R\sqrt{1-R^2}\, e^{i(\varphi-\theta)} & 1-R^2 \end{pmatrix}\frac{1}{V}$$

	Pure ν:	Pure $\bar{\nu}$:	Mixed
$\langle\hat\rho\rangle$	$\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\frac{1}{V}$	$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\frac{1}{V}$	$\frac{1}{V\sqrt{2}}\begin{pmatrix} 1 & e \\ e^{i(\varphi\theta)} & \end{pmatrix}$
$Tr[\langle\hat\rho\rangle\sigma_1]$	0	0	$\frac{1}{V}\cos(\theta-$
$Tr[\langle\hat\rho\rangle\sigma_2]$	0	0	$\frac{1}{V}\sin(\theta-$
$Tr[\langle\hat\rho\rangle\sigma_3]$	$-\frac{1}{V}$	$\frac{1}{V}$	0

⇒ Background states with Majorana masses are those in a particle-antiparticle superposition.

Notation: $|\{p\};\{k\}\rangle$ = Slater d — $\{p\}$: N_+ anti-neutrinos, $\{k\}$: N_- neutrinos

Consider 2-particle state:

$$|\psi_2\rangle = \alpha_1|p,k;\emptyset\rangle + \alpha_2|p;k\rangle + \alpha_3|k;p\rangle + \alpha_4|\emptyset;p,k\rangle$$

Again when $\alpha_i = \frac{1}{2}$ we find Majorana mass $Tr[\langle\hat\rho\rangle\sigma_1] = \frac{2}{V}$

Consider N-particle state:

$$|\psi\rangle = \alpha|\{p\};\emptyset\rangle + \sum_i^N \beta_i|\{p\}-\{p_i\};\{p_i\}\rangle + \sum_{i\neq j}^N \gamma_{ij}|\{p\}-\{p_i,p_j\};\{p_i,p_j\}\rangle + \dots$$
$$+\alpha'|\emptyset;\{p\}\rangle + \sum_i^N \beta_i'|\{p_i\};\{p\}-\{p_i\}\rangle + \sum_{i\neq j}^N \gamma_{ij}'|\{p_i,p_j\};\{p\}-\{p_i,p_j\}\rangle + \dots$$

Again when $\alpha = \beta_i = \gamma_{ij} = \dots = \frac{1}{\sqrt{2^N}}$, $Tr[\langle\hat\rho\rangle\sigma_1] = \frac{N}{V}$

This choice (prob. for each state being $\frac{1}{2^N}$) maximizes the Von Neumann entropy $S = -Tr\rho\ln\rho = -\sum_i \lambda_i \ln\lambda_i = N\ln 2$ $\quad(\lambda_i = \frac{1}{2}$
A thermal bath must have max. entropy. $\quad\lambda_i = |\alpha|^2, |\beta$

We assume mixed states behave like the Harmonic Oscillator:

$$a|\psi_N\rangle = \sqrt{N}|\psi_{N-1}\rangle \qquad a^\dagger|\psi_N\rangle = \sqrt{N+1}|\psi_{N+1}\rangle$$

④ Mass Matrix and Mixing

We can write the mass matrix as:

$$\langle \hat{\rho}_{ij} \rangle = \mathrm{Tr}[v_i^\dagger v_j \sigma_1] \qquad V_i = (a_{i+}, a_{i-})$$

If $a, a^\dagger$ behave like the H.O., this is an outer product of vect[ors] in large N limi[t]

$$\langle \hat{\rho}_{ij} \rangle = \begin{pmatrix} \sqrt{N_1} \\ \sqrt{N_2} \\ \sqrt{N_3} \end{pmatrix} (\sqrt{N_1}, \sqrt{N_2}, \sqrt{N_3})$$

The corresponding Lagrange multipliers (at zero temp) are the Fermi Momenta, so:

$$M_{ij} = \left(\frac{3\pi^2}{V}\right)^{1/3} \begin{pmatrix} N_1^{1/6} \\ N_2^{1/6} \\ N_3^{1/6} \end{pmatrix} (N_1^{1/6}, N_2^{1/6}, N_3^{1/6}) \qquad \text{with eigenvalues } \lambda = \left(\frac{3\pi^2}{V}\right)^{1/3} (N_1^{1/3} + N_2^{1/3} + N_3^{1/3}), \underbrace{0, 0}$$

two zero eigenva[lues] ⇒ one mixing a[ngle] not fixed yet.

The Rotation Matrix is:

$$U_{MNS} = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & -2/\sqrt{6} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & 1/\sqrt{3} \end{pmatrix} \qquad \text{if } \cos\theta = 0 \text{ is TBM}$$

Where the remaining angle paramaterizes the unbroken $SO(2)$ symmetry among the 2 zero eigenvalues.

The mean field theory breaks the original $U(3)$ family symmetry down to $SO(2)$

Smaller Δm^2 and remaining mixing angle must come from a perturbation.

If we identify $v^2 = \Delta m^2_{23} \Rightarrow 6.4 \times 10^8\ \nu/cm^3$
$v^2 = \Delta m^2_{12} \Rightarrow 3.2 \times 10^6\ \nu/cm^3$
(pretending 1 flavor) ↓

Compare $330/cm^3$ which should be the aver[age] value in the universe

Not a precise prediction! But is order of magn[itude]

Conclusions:

Majorana masses are caused by the relic density of massless νs in the SM

Majorana phases are absent, because the theory fundamentally conserves lepton #

CP phases are absent ($\theta_{13}=0$, and the parity operator doesn't exist)

We can predict m_ν and N_ν from mixing experiments

Lepton # is conserved. All apparent lepton # violation is due to exchange interactions with background.

Intro to Neutrino Mixing

3 types, ν_e, ν_μ, ν_τ

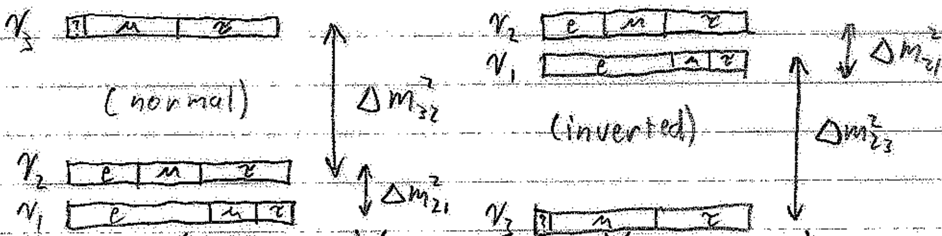
Observed to "mix".

Ray Davis + John Bahcall found that the # of solar ν's was about 1/3 that expected from the solar model (late 1960's)
Now understood to be caused dominantly by matter effects in solar medium (MSW)

Atmospheric ν problem! Kamiokande II ~1988
where from

$\Rightarrow$ Mass eigenstates are not same as flavor (interaction) eigenstat

$$\underset{\text{(flavor)}}{\nu_f} = \underset{\text{(PMNS)}}{U}\ \underset{\text{(mass)}}{\nu_m} \qquad U_{PMNS} = U_{23} I_\delta U_{13} I_{-\delta} U_{12}$$



$$U_{PMNS} = \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & c_{23} & s_{23} \\ 0 & -s_{23} & c_{23} \end{pmatrix}}_{\text{atmospheric}} \begin{pmatrix} c_{13} & 0 & s_{13}e^{-i\delta} \\ 0 & 1 & 0 \\ -s_{13}e^{i\delta} & 0 & c_{13} \end{pmatrix} \underbrace{\begin{pmatrix} c_{12} & s_{12} & 0 \\ -s_{12} & c_{12} & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{\text{solar}}$$

$$P(\nu_e \to \nu_\mu) = |\langle \nu_\mu | \nu(t) \rangle|^2 = \sin^2 2\theta \sin^2 \frac{\Delta m^2 L}{4E}$$ simplified 2-flavor model

with $|\nu(0)\rangle = |\nu_e\rangle = \cos\theta |\nu_1\rangle + \sin\theta |\nu_2\rangle$

$$|\nu(t)\rangle = \cos\theta\, e^{-iE_1 t} |\nu_1\rangle + \sin\theta\, e^{-iE_2 t} |\nu_2\rangle$$

Can this be explained in SM?

No, if in vacuum.

What about finite density $\rightarrow$ #2