# Universitat de València Departament de Física Teòrica 

# Dinámica quiral en reacciones inducidas por fotones y piones 

(Chiral dynamics in photon- and pion-induced reactions)

Tesis Doctoral
Michael Ulrich Döring
Marzo 2007
D. Eulogio Oset Báguena, Catedrático de Física Teórica de la Universidad de Valencia,

CERTIFICA: Que la presente Memoria Dinámica quiral en reacciones inducidas por fotones y piones ha sido realizada bajo mi dirección en el Departamento de Física Teórica de la Universidad de Valencia por D. Michael Ulrich Döring como Tesis para obtener el grado de Doctor en Física.

Y para que así conste presenta la referida Memoria, firmando el presente certificado.

Fdo: Eulogio Oset Báguena

D. Manuel Vicente Vacas, Profesor Titular de Física Teórica de la Universidad de Valencia,

CERTIFICA: Que la presente Memoria Dinámica quiral en reacciones inducidas por fotones y piones ha sido realizada bajo mi codirección en el Departamento de Física Teórica de la Universidad de Valencia por D. Michael Ulrich Döring como Tesis para obtener el grado de Doctor en Física.

Y para que así conste presenta la referida Memoria, firmando el presente certificado.
a María José
meinen Eltern

## Contents

1 Introduction ..... 1
1.1 Outline ..... 14
1.2 Unitary extensions of $\chi \mathrm{PT}$ ..... 24
1.2.1 Resonances in $\pi N$ interaction ..... 29
1.2.2 Dynamically generated resonances ..... 31
$2 S$-wave pion nucleon scattering lengths from $\pi N$, pionic hy- drogen and deuteron data ..... 35
2.1 Introduction ..... 35
2.2 Summary of the model for $\pi N$ interaction ..... 38
2.3 Pion deuteron scattering ..... 43
2.3.1 Faddeev approach ..... 43
2.3.2 Absorption and dispersion terms ..... 48
2.3.3 Further corrections to the real part of $a_{\pi^{-} d}$ ..... 56
2.3.4 Other Corrections ..... 60
2.4 Results ..... 66
2.4.1 The isoscalar and isovector scattering lengths ..... 68
2.5 Conclusions ..... 80
3 The $s$-wave pion nucleus optical potential ..... 83
3.1 Introduction ..... 83
3.2 Low energy pion nucleon interaction in vacuum and matter ..... 86
3.2.1 The model in nuclear matter ..... 89
3.2.2 Pion polarization in asymmetric nuclear matter ..... 94
3.3 Numerical results ..... 96
3.3.1 Self consistent treatment of the amplitude ..... 100
3.4 Higher order corrections of the isovector interaction ..... 101
3.4.1 Tadpoles and off-shell contributions ..... 102
3.4.2 Loop corrections in the $t$-channel ..... 108
3.4.3 Vertex corrections from $\boldsymbol{\pi} \boldsymbol{N} \boldsymbol{N}$ and $\boldsymbol{\pi} \boldsymbol{N} \boldsymbol{\Delta}$ related terms11
3.4.4 Triangle diagrams ..... 116
3.4.5 Isovector correction from the NLO $\boldsymbol{\pi} \boldsymbol{N}$ interaction ..... 121
3.4.6 Results for the isovector renormalization ..... 123
3.5 Renormalization of the NLO isoscalar term in $\pi N$ scattering ..... 124
3.5.1 Tadpole and off-shell contributions ..... 125
3.5.2 Loop corrections in the $t$-channel ..... 127
3.5.3 Further renormalizations of the isoscalar $\pi N$ interaction 130 ..... 30
3.5.4 Finite momentum transfer in vertex corrections ..... 131
3.6 Numerical results ..... 132
3.6.1 Theoretical uncertainties ..... 135
3.7 Dependence on the vacuum renormalization ..... 136
3.7.1 Improvement of the Ericson-Ericson approximation ..... 137
3.7.2 The size of the isoscalar contribution from the NLO Lagrangian ..... 140
3.7.3 Uncertainties from the Roper resonance in $\pi N$ scattering 142
3.8 Summary and conclusions ..... 146
4 Chiral dynamics in the $\gamma p \rightarrow \pi^{0} \eta p$ and $\gamma p \rightarrow \pi^{0} K^{0} \Sigma^{+}$reac- tions ..... 149
4.1 Introduction ..... 149
4.2 The $N^{*}(1535)$ in meson-baryon scattering ..... 152
4.2.1 $\quad \eta N$ scattering length and effective range ..... 155
4.3 Single meson photoproduction ..... 158
4.3.1 The ratio $\boldsymbol{\sigma}(\gamma \boldsymbol{n} \rightarrow \boldsymbol{\eta} \boldsymbol{n}) / \boldsymbol{\sigma}(\gamma \boldsymbol{p} \rightarrow \boldsymbol{\eta} \boldsymbol{p})$ ..... 162
4.3.2 Some remarks on gauge invariance and chiral symmetry ..... 169
4.4 Eta pion photoproduction ..... 173
4.4.1 Contact interaction and anomalous magnetic moment ..... 173
4.4.2 Kroll-Ruderman and meson pole term ..... 177
4.4.3 Baryonic resonances in $\eta \pi^{0}$ production ..... 180
4.4.4 $S U(3)$ couplings of the $\Delta^{*}(1700)$ ..... 183
4.4.5 Processes with $\Delta^{*} \eta \Delta$ and $\Delta^{*} K \Sigma^{*}$ couplings ..... 186
4.4.6 Tree level contribution from the $\Delta^{*}(1700) \rightarrow \eta \Delta$ decay ..... 189
4.5 Results ..... 191
4.5.1 Extension to higher energies ..... 197
4.5.2 The $\pi^{0} p$ invariant mass ..... 200
4.5.3 The reaction $\gamma p \rightarrow \pi^{0} K^{0} \Sigma^{+}$ ..... 203
4.6 Conclusions ..... 208
5 Clues to the nature of the $\Delta^{*}(1700)$ resonance from pion- and photon-induced reactions ..... 211
5.1 Introduction ..... 211
5.2 The model for the $\pi p \rightarrow K \pi \Lambda, K \pi \Sigma, K \bar{K} N, \eta \pi N$ reactions ..... 214
5.3 The model for the $\gamma p \rightarrow K \pi \Lambda, K \pi \Sigma, \eta \pi p$ reactions ..... 222
5.4 Results and discussion ..... 223
5.4.1 Photon-induced $\Delta^{*}(1700)$ production ..... 229
5.4.2 Other possible reaction mechanisms ..... 232
5.4.3 Photoproduction of $\Lambda$ (1405) ..... 235
5.5 Conclusions ..... 237
5.5.1 Addendum ..... 238
6 Radiative decay of the $\Lambda(1520)$ ..... 241
6.1 Introduction ..... 241
6.2 Formulation ..... 243
6.2.1 $d$-wave channels ..... 244
6.3 Radiative decay ..... 249
6.3.1 Radiative decay from $d$-wave loops ..... 253
6.4 Numerical results ..... 257
6.5 Estimates of strength of the genuine resonance component ..... 263
6.6 Conclusions ..... 265
7 Radiative decay of the $\Delta^{*}(1700)$ ..... 267
7.1 Introduction ..... 267
7.2 The model for the radiative $\boldsymbol{\Delta}^{*}(\mathbf{1 7 0 0})$ decay ..... 269
7.2.1 The $(\pi N)_{d}$ channel in the unitary coupled channel ap- proach ..... 270
7.2.2 The phototransition amplitude ..... 277
7.2.3 Gauge Invariance ..... 283
7.2.4 Photon coupling to the $\pi N$ loop in $d$-wave ..... 290
7.2.5 Effective photon coupling ..... 291
7.3 Numerical results ..... 292
7.4 Conclusions ..... 294
8 Charge fluctuations and electric mass in a hot meson gas ..... 295
8.1 Introduction ..... 296
8.2 Charge fluctuations and Susceptibilities ..... 298
8.3 Model Lagrangian and $\boldsymbol{\pi} \boldsymbol{\pi}$ interaction in the heavy $\boldsymbol{\rho}$ limit ..... 300
8.4 Charge fluctuations at low temperatures ..... 303
8.4.1 Charge fluctuations for free pions and $\rho$-mesons ..... 303
8.4.2 $\boldsymbol{\pi} \boldsymbol{\pi}$ interaction in the heavy $\boldsymbol{\rho}$ limit to order $\boldsymbol{e}^{\boldsymbol{2}} \boldsymbol{g}^{\boldsymbol{2}}$ ..... 305
8.5 The $\boldsymbol{\rho}$-meson in the heatbath ..... 307
8.6 Relativistic virial expansion ..... 312
8.6.1 Density expansion versus thermal loops ..... 315
8.6.2 Numerical results for the interacting pion gas ..... 323
8.7 Higher order corrections ..... 324
8.7.1 Extension to $\boldsymbol{S U}(\mathbf{3})$ ..... 329
8.8 Numerical results ..... 331
8.9 Summary and Conclusions ..... 333
9 Charge susceptibility in a hot pion gas with unitarized chiral interaction ..... 335
9.1 Introduction ..... 335
9.2 Unitarized chiral perturbation theory at finite temperature ..... 339
9.2.1 Extension to finite $T$ and $\mu$ ..... 341
9.2.2 Density expansion ..... 346
9.2.3 The charge susceptibility ..... 347
9.2.4 Scaling of the amplitude with the pion mass ..... 351
9.2.5 Discussion of the results and outlook ..... 352
9.3 Summary ..... 353
A The $d$-wave in the deuteron ..... 355
Index ..... V
B Evaluation of the the Pauli blocked $\pi N$ loop with pion po- larization ..... 359
C $1 / 2^{-}$meson $1 / 2^{+}$baryon $3 / 2^{+}$baryon ( $M B B^{*}$ ) interaction ..... 361
D Charge fluctuations ..... 369
D. 1 From charge fluctuations to photon selfenergy in sQED ..... 369
D. 2 Pion-pion interaction ..... 370
D.2.1 Chiral $\boldsymbol{\pi} \boldsymbol{\pi}$ interaction and vector exchange ..... 370
D.2.2 Unitarization of the $\boldsymbol{\pi} \boldsymbol{\pi}$-amplitude with the $\boldsymbol{K}$-matrix ..... 373
D. 3 The $\boldsymbol{\rho}$-meson in the heatbath ..... 373
D.3.1 Analytic results ..... 373
D.3.2 Calculation of diagram (1a2) ..... 377
D.3.3 The $\boldsymbol{\gamma} \boldsymbol{\pi} \boldsymbol{\rho}$ system at finite $\boldsymbol{\mu}$ ..... 379
D.3.4 Charge conservation ..... 380
D. 4 Solutions for the resummations ..... 381
D. 5 Extension to $\boldsymbol{S U ( 3 )}$ ..... 385

## Chapter 1

## Introduction

## Resumen y resultados obtenidos

La descripción teórica de la física hadrónica a energías bajas e intermedias sigue suponiendo un importante desafío. La riqueza de la fenomenología se refleja en la gran cantidad de resonancias contenidas en los sectores mesónmesón y mesón-barión. Arrojar nueva luz sobre cuestiones específicas en este campo es el objetivo de la presente tesis.

La interacción fuerte causante de los procesos físicos relevantes para esta tesis, ha sido descrita con éxito a energías altas por una teoría cuántica de campos, la llamada Cromodinámica Cuántica (QCD). Hoy en día, QCD se ha establecido como teoría de la interacción fuerte. A energías altas - o lo que es lo mismo - cortas distancias, el Lagrangiano de QCD se puede desarrollar perturbativamente y explica los distintos y variados fenómenos en los cuales se produce una gran transferencia de momento. No obstante, a energías más bajas, QCD manifiesta un comportamiento peculiar: debido al carácter no abeliano de los campos gauge (gluones) la constante de acoplamiento fuerte $\alpha_{s}$ se incrementa. Por eso, el desarrollo perturbativo falla a partir de cierto límite. Fenomenológicamente, la interacción débil a altas energías se conoce como "libertad asintótica". La circunstancia de que la fuerza entre cargas de color no decrezca con la distancia recibe el nombre de "confinamiento" y viene directamente conectado con el incremento de $\alpha_{s}$.

En el límite de los quarks sin masa, QCD manifiesta una simetría adicional relacionada con la conservación de quiralidad de los quarks. La invariancia de los campos de los quarks bajo transformaciones quirales se denomina "simetría quiral" de QCD, que se encuentra explicitamente rota por las pequeñas (aunque no inexistentes) masas de quark. Hoy por hoy, está aceptado que en el límite quiral exacto, la simetría quiral se halla espontáneamente rota, esto es, el estado fundamental de QCD (vacio) no comparte la simetría quiral del Lagrangiano fundamental. Para cualquier simetría espontáneamente rota, existe un estado sin masa, que llamamos "bosón de Goldstone". Este es el teorema de Goldstone. Resulta obvio identificar los ocho hadrones pseudoscalares más ligeros, $\pi^{ \pm}, \pi^{0}, K^{ \pm}, K^{0}, \bar{K}^{0}, \eta$ con los bosones de Goldstone. El hecho de que estos bosones tengan masa es consecuencia de la rotura explícita de la simetría quiral, o sea, de las masas no-nulas de los quarks.

En su realización no lineal, la simetría quiral espontáneamente rota puede utilizarse para formular teorías efectivas de campo. Los Lagrangianos correspondientes incluyen todos los términos permitidos por la simetría. En el orden más bajo $\mathcal{O}(2)$ en un desarrollo en masa y energía, el Lagrangiano quiral en la interacción $\pi \pi$ tiene sólo un parámetro libre, $f_{\pi}$, que viene determinado por la desintegración del pión. En el sector mesón-barión existen las constantes adicionales $D$ y $F$, fijadas por la fenomenología. A órdenes más altos en momentos, se permiten más estructuras derivativas en consonancia con la simetría quiral que a su vez, permite una serie de interacciones. La fuerza de estos términos no está determinada y debe determinarse desde la fenomenología.

El significado físico de estas constantes de baja energía es obvio: la física desconocida de corta distancia que tiene lugar a momentos altos, es absorbida en estas constantes; mientras que la física de larga distancia, con los hadrones ligeros como estados asintóticos, se expande perturbativamente, pudiendo entonces considerar, de forma explícita, la dinámica de estos estados. La fenomenología ayuda en este punto, dado que existe un hueco de masas entre
los ocho mesones más ligeros y el mesón $\rho$ a una masa de 770 MeV .
En cualquier caso, muy por debajo de este valor de energía, la teoría quiral de perturbaciones empieza a fallar. Desde un punto de vista más pragmático, esto se debe al número creciente de parámetros libres de términos a momentos altos en el Lagrangiano quiral. Por ejemplo, en el sector mesón-mesón, el NLO Lagrangiano a orden $\mathcal{O}(4)$ ya contiene diez constantes $L_{i}$ y otras constantes que sirven de parámetro para las estructuras $\mathcal{O}(4)$. Estas constantes subsumen nuestra ignorancia de los detalles de la dinámica fundamental QCD. En principio, las $L_{i}$ pueden determinarse a través de cálculos en el retículo, pero en la actualidad la mayor fuente de información nos la proporciona la fenomenología de bajas energías. A orden $\mathcal{O}(6)$ existen ya más de cien parámetros libres, con lo que no contamos con suficientes datos experimentales que nos avalen para seguir trabajando con esta libertad que aporta la teoría.

La segunda razón por la que colapsa dicha teoría, es la aparición de resonancias como $\rho$. Obviamente, cualquier desarrollo perturbativo no puede considerar los polos en el plano complejo de la energía invariante de colisión $\sqrt{s}$. En la región de energía donde tanto la teoría quiral de pertubaciones como la QCD perturvativa fallan (desde algunos cientos de MeV , hasta la escala de GeV ) hay que recurrir, hoy en día, a modelos.

En el futuro la respuesta definitiva vendrá de la teoría gauge en el retículo. Esta teoría, es una tentativa reciente para solucionar QCD en un retículo individualizado en espacio y tiempo. Se espera que, efectuando simulaciones en retículos más y más grandes, a la vez que reducimos más y mas la distancia de su entramado, se pueda recuperar la teoría del continuo. Pero aún resulta ingente el esfuerzo numérico. En este contexto, sería interesante mencionar que desde hace algún tiempo se viene trabajando para extrapolar los resultados del retículo a las masas físicas de los hadrones usando la teoría quiral de perturbaciones que permite expresar los resultados en función de las masas.

Mientras esperamos resultados más realistas del retículo, se ha establecido una gran variedad de modelos efectivos. Por ejemplo, en el sector pión-
pión, la inclusión explícita de resonancias permite extender la teoría quiral de perturbaciones a energías mas altas y es posible introducir el mesón $\rho$ como campo masivo en la aproximación del gauge oculto. Existe también el método de aproximación masivo de Yang-Mills basado en un modelo $\sigma$ de gauge. En la teoría quiral de resonancias, la interacción se construye acoplando fuentes externas de carácter escalar, vectorial, vectorial axial, tensorial, etc., al Lagrangiano quiral. Aparecen nuevas constantes que hay que determinar mediante la fenomenología.

La unitaridad es un elemento importante a altas energías. Los resultados de $\chi$ PT están ordenados por potencias de momento y como consecuencia, violan necesariamente la unitaridad a determinada energía de colisión. Existen varios métodos para unitarizar. El más evidente lo constituye la aproximación de la matriz $K$ donde una amplitud al orden árbol que viola la unitaridad, está iterado de manera que la amplitud final es unitaria, $S S^{\dagger}=1$, mientras en una expansión en potencias de momento, se recupera la amplitud original. La aproximación de la matriz $K$ es una simplificación de la unitarización a través de la ecuación de Bethe-Salpeter (BSE) en la cual las partes reales dispersivas de los estados intermedios se toman, también, en consideración.

La teoría quiral unitaria (UChPT) permite extender ChPT a energías más altas usando técnicas que satisfacen la unitaridad exacta en canales acoplados y encajan con los resultados de ChPT a bajas energías. El método $N / D$, o correspondientemente, la ecuación de Bethe-Salpeter, han sido las herramientas utilizadas para efectuar la citada extensión unitaria. El método de la amplitud inversa es otra alternativa, pero no la utilizaremos aquí. Estos métodos unitarios, proporcionan las propiedades básicas analíticas de la amplitud de colisión como el corte físico del lado derecho, requerido por la unitaridad, logaritmos quirales, etc. En cambio, se recuperan perturbativamente los resultados de ChPT, expandiendo la amplitud unitaria en potencias de momento y masa. En sección 1.2 ofrecemos una corta introducción a UChPT, poniendo especial énfasis en los temas tratados en la tesis.

Mientras tanto, ofrecemos una visión conjunta del contenido de la tesis [1-9]. En el sector mesón-barión la interacción $\pi N$ es la reacción mejor conocida y gracias a la exactitud de los datos, permite pruebas precisas de los modelos teóricos. En el capítulo 2 estudiaremos la interacción $\pi N$ en el umbral y cerca del mismo. Los valores exactos de la longitud de colisión isoscalar e isovectorial, $b_{0}$ y $b_{1}$, son datos de mucho interés en física hadrónica. En combinación con datos de colisión de $\pi N$ a bajas energías, determinan los parametros del Lagrangiano quiral, lo que permite hacer predicciones incluso debajo del umbral, usando la teoría quiral de perturbaciones. Los datos experimentales de los cuales se extraen $b_{0}, b_{1}$ habitualmente son el desplazamiento y la anchura del hidrógeno piónico y del deuterio piónico. Una de las cuestiones clave es si la predicción de Weinberg basada en algebra de corrientes (que $b_{0}$ desaparece) es correcta.

En ese contexto la unitarización de la amplitud $\pi N$ es interesante porque genera diagramas de colisión múltiple en los que la interacción isovector repetida, conduce a una contribución isoscalar. La novedad de este estudio es, pues, la consideración de esos términos de órdenes más altos y sus repercusiones cerca del umbral. Para aislar este efecto hay que realizar un trabajo nada trivial en relación con el problema de colisión de tres cuerpos en el deuterio piónico. En un ajuste global disponemos todos los datos a considerar y obtenemos una buena descripción del umbral hasta energías intermedias.

La parte imaginaria de la longitud de colisión de pión-deuteron $a_{\pi d}$ coincide bien con el dato experimental. La parte dispersiva de la absorción es compatible con cero. Eso, junto con diferentes correcciones como la de diagramas cruzados o la de la resonancia $\Delta(1232)$ conduce a un desplazamiento considerable de la parte real de $a_{\pi d}$ hacia valores más positivos. También hemos considerado la rotura del isospín, constatando que el ajuste a los datos explica la mitad de la rotura de dicho isospín causada por masas físicas diferentes. El modelo ha sido extendido, tomando en consideración otras fuentes
de rotura del isospín y, consecuentemente, coincide con resultados encontrados en otros trabajos al respecto. En cualquier caso, su incidencia es mínima en comparación con otras fuentes de incertidumbres teóricas [1, 2].

Con esa amplitud precisa a nuestra disposición, aplicamos el modelo en el capítulo 3 en un contexto diferente. En Física Nuclear una de las cuestiones más persistente, que sigue sin ser resuelta, es la "falta de repulsión". El modelo del vacío obtenido anteriormente sirve de input para el cálculo del medio nuclear con la gran ventaja de que la colisión multiple de la interacción isovectorial genera una repulsión grande en el medio nuclear. La estrategia y ventaja es elaborar e incluir los efectos del medio en esa contribución repulsiva de Ericson-Ericson lo que deviene en correcciones de órdenes más altos, apenas accesibles en desarrollos sistemáticos en términos del acoplamiento y densidad nuclear. Más bien podemos demostrar que dichas correcciones son tan grandes, que el desarrollo converge demasiado lentamente para hacer predicciones fiables dentro de la teoría de muchos cuerpos que usamos actualmente. Por lo tanto, no es posible extraer del modelo la repulsión requerida de una manera fiable, pero se establecen algunos valores límite.

En los capítulos siguientes 4 a 7 nos ocupamos con aplicaciones más genuinas de la teoría quiral unitaria de perturbaciones. Como conescuencia de la unitarización aparecen polos en el plano complejo de la energía invariante de colisión. Dado un canal definido de espín e isospín, la existencia de polos depende de si la interacción es repulsiva o atractiva. Por ejemplo, en uno de los canales de $\pi N$ estudiado en el capítulo $2, S_{11}$, aparece un polo a una energía de colisión alrededor de $\sqrt{s} \sim 1530 \mathrm{MeV}$ a una distancia de $\sim 50$ MeV del eje real de $\sqrt{s}$ en la segunda hoja de Riemann. Se puede identificar dicho polo con la bien conocida resonancia $N^{*}(1535)$ y $U \chi P T$ nos provee de una buena descripción de la forma de la amplitud junto con proporciones de ramificación. En el capítulo 1.2.1 facilitamos una introducción más detallada del concepto de la "generación dinámica" de resonancias.

Queda claro que a energías más altas, las resonancias desempeñan un
papel relevante. También, estados de tres y más partículas, como p.ej., $\pi \pi N$ cobran mayor importancia porque estos canales están abiertos. Existe una familia de reacciones inducidas por fotones que permite probar la dinámica más complicada a energías intermedias, $\gamma p \rightarrow \pi \pi N$. En un intento por comprender estas reacciones, largo es el historial de investigación, tanto teórica como experimental, con el que contamos. En los últimos años se ha obtenido una buena descripción de los datos basada en modelos de intercambio de resonancias múltiples, como p.ej., el "Valencia Model".

Para estudiar la estructura del sabor de estos mecanismos, se debe abordar el problema con amplitud de miras; en otras palabras, hay que tomar en consideración la producción de dos mesones que no sean necesariamente piones. Como los otros miembros del octeto de mesones pseudoscalares son más pesados que el pión, hay que extender el cálculo a energías más altas que en las reacciones $\gamma p \rightarrow \pi \pi N$. Además, no es suficiente considerar sólo los acoplamientos de las resonancias en $S U(2)$; a energías más altas otros canales como $\eta N, K \Lambda, K \Sigma, \pi \Delta(1232), K \Sigma^{*}(1385)$ están abiertos, o incluso, si se hallan cerrados, contribuyen como estados virtuales intermedios. Gracias al progreso teórico en esa dirección, disponemos ahora de los requisitos exigidos por un modelo realista: muchas resonancias se describen bien como dinámicamente generadas y en virtud del formalismo de canales acoplados, se pueden predecir muchos acoplamientos en $S U(3)$.

En el capítulo 4, las reacciónes $\gamma p \rightarrow \pi \pi N$ se generalizan hasta las fotoproducciones $\gamma p \rightarrow \pi^{0} \eta p$ y $\gamma p \rightarrow \pi^{0} K^{0} \Sigma^{+}$, que en estos momentos se miden en ELSA in Bonn. En las meritadas reacciones la $N^{*}$ (1535) desempeña un papel destacado, sin embargo, como principal fuente de contribución, identificaremos otra resonancia que también se considera dinámicamente generada, la $\Delta^{*}(1700)$ y sus fuertes acoplamientos a los canales $\eta \Delta$ y $K \Sigma^{*}(1385)$. Aparte del empleo novedoso de resonancias dinámicamente generadas en este contexto, incluimos los mecanismos relevantes del Valencia Model, así como una serie de mecanismos de producción basados en $\chi$ PT.

Usando Lagrangianos quirales para la fotoproducción, la contribución al orden árbol a la producción de la $\eta$ es cero y es la unitarización de la amplitud en canales acoplados, que genera contribuciones finitas por acoplamiento del fotón a canales intermedios de mesones cargados que conduce a $\pi^{0} \eta$ y $\pi^{0} K^{0}$ en el estado final. Por eso, las reacciones estudiadas constituyen una posibilidad única para el estudio de resonancias dinámicamente generadas $y$, por otro lado, son esas resonancias las que predicen los grandes acoplamientos que resultan de una buena coincidencia con datos experimentales preliminares de ELSA presentados recientemente: secciones eficaces, secciones eficaces diferenciales y espectros de masas invariantes [3-5].

Como hemos encontrado la $\Delta^{*}(1700)$ de especial relevancia, la estudiamos en el capítulo 5 en casi una docena de reacciones diferentes inducidas por fotones y piones, puesto que ambos estados iniciales pueden ser tratados de la misma manera.

Pese a que los procesos investigados están a una energía que corresponde a la cola de la $\Delta^{*}(1700)$-media o una anchura por encima de la energía nominal de la $\Delta^{*}(1700)$ - constatamos que las secciones eficaces encontradas son considerables. Otro resultado experimental que apoya la dominancia de los mecanismos usados en este estudio es la presencia de la $\Sigma^{*}$ (1385) claramente observada y con poco espacio para fondo adicional. Experimentalmente se ha establecido, también, para algunas de las reacciones estudiadas aquí, la dominancia de la producción en onda $s$ de la $\Sigma^{*}(1385)$ lo que corresponde a los mecanismos utilizados por nosotros, comprobando al tiempo que la producción a través de resonancias con otros números cuánticos o el intercambio de mesones vectoriales en el canal $t$ es marginal a las energías más bajas.

El esquema de la generación dinámica de esa resonancia predice acoplamientos a $\eta \Delta(1232)$ y $K \Sigma^{*}(1385)$ muy diferentes de una extrapolación de $S U(3)$ del acoplamiento experimentalmente - bien conocido - a $\pi \Delta$ (1232). Hemos comprobado que el modelo quiral unitario de la $\Delta^{*}(1700)$ predice bien el acoplamiento de la $\Delta^{*}(1700)$ a este canal. El incremento hasta un factor 30
es una hipótesis que llama la atención y debe ser ratificada por resultados experimentales. Las secciones eficaces de las diferentes reacciones consideradas se distinguen entre ellas por casi dos órdenes de magnitud. Sin embargo, el modelo aporta una buena descripción global para todas las reacciones sin necesidad de introducir nuevos parámetros. Eso apoya la tesis de que la $\Delta^{*}(1700)$ se genera dinámicamente [6].

Aunque la interacción electromagnética es débil, su acoplamiento a los hadrones que muestran interacción fuerte, ofrece una oportunidad única y totalmente independiente para estudiar las propiedades de las resonancias dinámicamente generadas, como p.ej., a través de factores de forma electromagnéticos. Y eso porque la fotoproducción y la desintegración radiativa se puede predecir mediante el gauge de los componentes bariónicos y mesónicos que constituyen dichas resonancias en el panorama de la generación dinámica. Este procedimiento es muy utilizado y está bien definido porque la substitución mínima constituye una manera libre de parámetros para acoplar el campo electromagnético. Así, tras haber estudiado la fotoproducción de dos mesones, procede investigar las resonancias dinámicamente generadas por su desintegración radiativa.

En el capítulo 6, nos centramos en la citada desintegración radiativa de la $\Lambda^{*}(1520)$ según $\Lambda^{*}(1520) \rightarrow \gamma \Lambda$ y $\Lambda^{*}(1520) \rightarrow \gamma \Sigma^{0}$. Esta resonancia ha sido recientemente descrita en el esquema de generación dinámica. Para la desintegración $\Lambda^{*}(1520) \rightarrow \gamma \Lambda$ los canales dominantes de mesón-barión se cancelan exactamente en dicho esquema; mientras que en $\Lambda^{*}(1520) \rightarrow \gamma \Sigma^{0}$ se suman. De hecho, en la reacción posterior advertimos buena correspondencia con el experimento. Sin embargo, para la desintegración en $\gamma \Lambda$ el valor experimental proviene de otras fuentes, como una componente genuina de tres quarks. Por eso la desintegración radiativa puede servir para facilitar la comprensión de la mezcla de la parte dinámicamente generada de la función de onda y la componente genuina.

En correspondencia, en muchos modelos de quark, el estado final $\gamma \Lambda$ es
dominante y $\gamma \Sigma^{0}$ desaparece en el límite de mezcla de configuración nula. En segundo lugar, en algunos modelos de quarks, la desintegración a $\gamma \Lambda$ está sobreestimado; una vez permitida una componente mesón-barión en la función de onda, la fracción del estado genuino de tres quarks en la función de onda decrece, mientras que los modelos de quarks proporcionarían resultados más realistas para $\Lambda^{*}(1520) \rightarrow \gamma \Lambda[7]$.

En los capítulos 4 y 5 , hemos deducido el acoplamiento del fotón a la $\Delta^{*}(1700)$ del experimento de una forma semi-fenomenológica. Ese procedimiento está bien justificado porque la física que queremos extraer se encuentra en los largos acoplamientos de esa resonancia a $\eta \Delta$ y $K \Sigma^{*}(1385)$. No obstante, queda tarea por realizar para mejorar en este punto, ya que el modelo de canales acoplados se puede extender a fin de predecir la fuerza de la transición $\gamma N \Delta^{*}(1700)$. Lo logramos en el capítulo 7, de la misma manera en que se ha realizado en el capítulo seis para la $\Lambda^{*}(1520)$, esto es, acoplando el fotón a los componentes mesónicos y bariónicos que constituyen la resonancia.

Como el canal $\pi N$ en onda $d$ todavía no está incluido consistentemente en el esquema de generación dinámica de la $\Delta^{*}(1700)$ aparece ambivalencia de signo en el acoplamiento de ese canal a la $\Delta^{*}(1700)$. Sin embargo, uno de los signos puede excluirse porque entonces la desintegración radiativa de la $\Delta^{*}(1700)$ es prácticamente nula. Utilizando el otro signo, hemos hallado buena coincidencia con la desintegración radiativa observada, lo que viene a corroborar la $\Delta^{*}(1700)$ como resonancia dinámicamente generada.

En el estudio de átomos piónicos en el capítulo tres, ya nos encontramos con una aplicación de una teoría microscópica en el medio nuclear. Mientras que en el medio nuclear la densidad de bariones es finita y la temperatura es cero, existen varias situaciones que muestran temperatura finita y/o densidad a la misma escala que las masas de las las partículas involucradas, como p.ej., supernovas, estrellas de neutrones, enanas blancas o, como ocurre en aceleradores en la tierra, la colisión de átomos pesados.

De hecho, es común representar el estado de la materia interaccionando fuerte como una función de densidad bariónica $\rho$ y temperatura en el conocido como diagrama de fase, que exhibe una transición de fase a la temperatura crítica $T_{c}(\rho)$. Para $T<T_{c}(\rho)$ la descripción efectiva de la materia se hace en términos de grados de libertad hadrónica, mientras que para $T>T_{c}(\rho)$ hoy en día se cree que existe una nueva fase de QCD en la cual las propiedades esenciales de la materia de interacción fuerte son dominadas por los grados fundamentales de libertad - los quarks y gluones. Esa fase novedosa se conoce como Plasma de Quarks y Gluones (QGP).

Se cree que el QGP se produce durante poco tiempo cuando la materia hadrónica normal está expuesta a temperaturas y/o densidades suficientemente altas que posiblemente se alcanzan en colisiones de átomos pesados. Por eso, uno de los objetivos primordiales en el estudio de las colisiones de átomos pesados es demostrar la existencia del QGP.

La reducción de las fluctuaciones de carga, debido a la fracción de carga de los quarks en el QGP, podría evidenciar dicho QGP. En la fase hadrónica, las fluctuaciones pueden verse disminuidas o incrementadas por esta interacción hadrónica. En el presente estudio, que se encuentra en el capítulo 8, la interacción viene dada por el intercambio del mesón $\rho$. Las fluctuaciones de carga se calculan mediante la autoenergía del fotón hasta el nivel de dos loops. Demostramos, explícitamente, la conservación de la carga en este método.

Asimismo, verificamos que este cálculo es equivalente al desarrollo de la función de partición gran canónica bajo circunstancias de un potencial químico finito de carga. Otro acceso - en principio muy diferente - al cálculo del conjunto gran canónico, es el desarrollo virial. Por primera vez, podemos demostrar que los resultados obtenidos con este método equivalen exactamente a los resultados anteriormente mencionados. A fin de alcanzar la citada equivalencia, hemos extendido el formalismo del desarrollo virial hasta incluir la estadística correcta de los estados asintóticos de la colisión. Todo esto se consigue mediante el empleo de diagramas de intercambio.

A las temperaturas de la transición de fase (alrededor de 170 MeV ) cons-
tatamos que tienen lugar otros procesos más allá del segundo orden de densidad. En estos procesos deviene inviable la utilización del desarrollo virial a causa de la carencia de datos experimentales sobre la interacción de tres cuerpos. Por este motivo, recurrimos a métodos microscópicos y empleando teoría cuántica de campos a temperatura finita, así como efectuando resumaciones de diferentes tipos, logramos estimar los efectos que producen dichos órdenes más altos.

Estos cálculos son de utilidad, no sólo en el sector de extrañeza nula; sino también, en la interacción pión-kaón, y así los hemos aplicado. Nuestro trabajo, constituye hasta la fecha, el estudio más completo de los realizados sobre fluctuaciones de carga.

En conclusión, las distintas contribuciones que alteran la fluctuación de carga en la fase hadrónica, son considerables, aunque se cancelan parcialmente. El modelo simple del gas de resonancias nos sirve para describir satisfactoriamente las fluctuaciones de carga objeto de nuestro estudio [8, 9].

En el capítulo 9, como corolario a nuestra exposición, estableceremos la conexión entre las cuestiones concernientes a las fluctuaciones de carga y la teoría quiral unitaria de perturbaciones. Uno de los hallazgos interesantes de la investigación - contenido en el capítulo 8- se refiere a la importancia de la unitaridad en el cálculo de las fluctuaciones de carga. Hemos visto que interacciones que dependen fuertemente del momento, en loops térmicos, como p.ej., las del NLO Lagrangiano quiral, introducen artificios en los resultados porque los loops térmicos recogen momentos altos, muy por encima de la aplicabilidad de $\chi \mathrm{PT}$ donde esta teoría viola la unitaridad.

En este contexto, la unitaridad porporcionada por $U \chi P T$ junto con una descripción excelente de los desfasajes hasta altas energías, constituye una interesante alternativa. En segundo lugar, comprobamos en el capítulo ocho que correcciones más altas que cuadradas en densidad, contribuyen considerablemente a los resultados. Por ello, en un primer intento de combinar estos dos requerimientos, utilizamos un modelo quiral unitario y vestimos los esta-
dos intermedios usando loops térmicos. Como sólo pretendemos estudiar los efectos cualitativamente, el estudio queda restringido al canal de carga dos. Los desfasajes obtenidos del modelo se aplican en un desarrollo de densidad. Asimismo, el desarrollo virial, que implica estadística clásica, se substituye por el desarrollo en densidad introducido en el capítulo ocho con el fin de logra un tratamiento consistente de la estadística.

Todos los capítulos se inician con una breve sinopsis. Las conclusiones clave las hemos expuesto ya en esta sección, aunque al final de cada capítulo recogemos aspectos más específicos. En aras de evitar una excesiva complejidad en la lectura del texto, algunos de los detalles más técnicos se han ubicado en el Apéndice.

### 1.1 Outline

The theoretical description of hadronic physics at low and intermediate energies is still a great challenge. The rich phenomenology is reflected in a large number of resonances both in the meson-meson and meson-baryon sector. Bringing new light to this field in some specific corners is the aim of this thesis.

The strong interaction which mediates the physical processes of interest in this thesis has been successfully described at high energies by a quantum field theory, Quantum Chromodynamics (QCD). Nowadays, QCD is the established theory of strong interactions. At high energies, or short distances, the Lagrangian of QCD can be perturbatively expanded and explains a wide range of phenomena where large momentum transfers are involved. At lower energies, however, QCD exhibits a peculiar behavior: due to the non-abelian nature of the gauge fields (gluons) the strong coupling constant $\alpha_{s}$ increases. Thus, the perturbative expansion breaks down at some point. The fact that $\alpha_{s}$ decreases logarithmically with increasing energy is known as asymptotic freedom. The fact that the force between color charges does not decrease with distance is called confinement which is closely related to the rise of $\alpha_{s}$ and describes the observation that quarks do not occur isolated in nature, but only in color singlet hadronic bound states as mesons and baryons.

In the limit of massless quarks, QCD has an extra symmetry related to the conserved right- or left-handedness (chirality) of the quarks. The invariance of the quark fields under the chiral transformations is referred to as the Chiral Symmetry of QCD, which is explicitly broken by the small but non-vanishing quark masses. Nowadays it is believed, that in the exact chiral limit chiral symmetry is spontaneously broken, i.e., the QCD ground state (vacuum) does not share the chiral symmetry of the underlying Lagrangian. For any spontaneously broken global symmetry there exists a massless mode, the socalled Goldstone Boson. This is Goldstone's Theorem. It is straightforward to identify the eight lightest pseudoscalar hadrons $\pi^{ \pm}, \pi^{0}, K^{ \pm}, K^{0}, \bar{K}^{0}, \eta$ with the Goldstone bosons. The fact that these bosons are not massless is a consequence of the explicit breaking of chiral symmetry, i.e. the non-zero quark masses.

In its non-linear realization, the spontaneously broken chiral symmetry can be utilized in order to formulate effective field theories. The corresponding Lagrangians denote all terms allowed by the symmetry. At lowest order $\mathcal{O}(2)$ in an expansion in mass and energy, the chiral Lagrangian in $\pi \pi$ interaction has only one free parameter, $f_{\pi}$, which is fixed by the pion decay. In the meson-baryon sector, there are the additional constants $D$ and $F$ which are fixed from phenomenology. At higher order in momenta, there are more derivative structures allowed by chiral symmetry. In this way it is possible to construct a tower of chirally allowed interactions. The strength of these terms is not fixed and has to be determined from phenomenology. The physical meaning of these low energy constants is obvious: the unknown short-distance physics that occurs at high momenta is absorbed in these constants whereas the long-distance physics with the light hadrons as asymptotic states is perturbatively expanded, thus considering explicitly the dynamics of these states. Phenomenology helps at this point in the sense that there is a mass gap between the eight lightest meson and the $\rho$-meson at a mass of 770 MeV .

However, way below this value in energy, chiral perturbation theory breaks down. From a more practical point of view, this is simply due to the rapidly growing number of free parameters for higher momentum terms in the chiral Lagrangian. E.g., in the meson-meson sector the next-to-leading order Lagrangian at order $\mathcal{O}\left(p^{4}\right)$ contains already ten constants $L_{i}$ and other constants that parametrize the $\mathcal{O}\left(p^{4}\right)$ structures allowed by chiral symmetry. These constants parametrize our ignorance about the details of the underlying QCD dynamics. In principle, the $L_{i}$ can be determined from lattice calculations, but at the present stage the main source of information about these couplings is the low-energy phenomenology. At $\mathcal{O}\left(p^{6}\right)$ there are more than hundred free parameters and obviously there is not enough low energy data available in order to deal with this freedom provided by theory.

The second reason why chiral perturbation theory breaks down at higher momenta is the appearance of resonances. Obviously, a perturbative expansion can not account for poles in the complex plane of the invariant scattering energy $\sqrt{s}$. In the energy region where both chiral perturbation theory and perturbative QCD fail, from a few hundred MeV to the scale of GeV , one
has to resort to models at the present time. In the future, the ultimate answer will come from lattice gauge theory. Lattice gauge theory is a recent attempt to solve QCD on a space-time discretized lattice. One hopes, by performing simulations on larger and larger lattices, while making the lattice spacing smaller and smaller, that one will be able to recover the behavior of the continuum theory. Yet, the numerical effort is massive. In this context it is interesting to mention that since some time new efforts are made to extrapolate lattice results to physical masses of the hadrons using chiral perturbative results which allow for a scaling of results as a function of the masses.

While awaiting more realistic lattice results, there has been established a large variety of effective models in the meantime. E.g. in the $\pi \pi$ sector, the explicit inclusion of resonances allows for an extension of chiral perturbation theory towards higher energy. For example, it is possible to introduce the $\rho$-meson as a heavy gauge field in the hidden gauge approach. There is also the massive Yang-Mills approach which is based on a gauged $\sigma$-model. In resonance chiral perturbation theory the interaction is constructed by coupling external scalar, vector, axial vector, tensor etc. sources to the chiral Lagrangian; new constants appear that again have to be determined by phenomenology.

Unitarity is an important issue at higher energies. The results from $\chi \mathrm{PT}$ are ordered by powers of momenta and thus, necessarily, will violate unitarity at some scattering energy. There are several methods of unitarization. The most straightforward one is the $K$-matrix approach in which a unitarity violating tree level amplitude is iterated in a way such that the final amplitude is unitary, $S S^{\dagger}=1$, whereas in an expansion in powers of the c.m. scattering momentum the original amplitude is recovered. The $K$-matrix approach is a simplification of the unitarization via the Bethe-Salpeter equation (BSE) in which the real, dispersive, parts of the intermediate states are also taken into account.

Unitarized Chiral Perturbation Theory ( $\mathrm{U} \chi \mathrm{PT}$ ) allows one to extend $\chi \mathrm{PT}$ at higher energies by using techniques that satisfy exact unitarity in coupled channels and match the results of the $\chi \mathrm{PT}$ amplitude at lower energies. The $N / D$ method, or analogously the Bethe-Salpeter equation, have been the
tools used to make this unitary extension. The Inverse Amplitude Method has also been a method used to unitarize but we will not use it here. These unitary methods provide automatically the basic analytic properties of the scattering amplitude such as the physical, right hand cut required by unitarity, chiral logs, etc. In turn, the results of $\chi \mathrm{PT}$ are perturbatively recovered by expanding the unitarized amplitude in powers of momenta and masses. In Sec. 1.2 we will give a brief introduction to $\mathrm{U} \chi \mathrm{PT}$ with special emphasis on the topics treated in this thesis.

Meanwhile we give an overview of the topics treated in this thesis [1-9]. In the meson-baryon sector, the $\pi N$ interaction is the experimentally best known reaction and, due to the accuracy of the data, allows for precise tests of theoretical models. In chapter 2 we will study the $\pi N$ interaction at threshold and close above it. The precise values of the isoscalar and isovector $\pi N$ scattering lengths $b_{0}$ and $b_{1}$ are one of the important issues in hadronic physics. Together with low energy $\pi N$ scattering data they determine parameters of the chiral Lagrangian which allows one to make predictions even below the $\pi N$ threshold using chiral perturbation theory. The experimental data from where $\left(b_{0}, b_{1}\right)$ are usually extracted are the shift and width of pionic hydrogen and deuterium atoms. One of the key questions is whether Weinberg's prediction from current algebra, that $b_{0}$ vanishes, holds. In this context the unitarization of the $\pi N$ amplitude is interesting, as it generates rescattering diagrams where the repeated isovector interaction results in an isoscalar contribution. The novelty of this study is, thus, the consideration of these higher order terms and their effect in the threshold behavior. In order to isolate this effect, a non-trivial work for the three-body scattering problem of pionic deuterium is necessary. In a global fit all available data is taken into account and a good description from threshold up to intermediate energies is obtained

The dispersive part from absorption has been found to be compatible with zero. This, together with corrections from crossed diagrams and the $\Delta$ (1232) resonance, and with other corrections taken from the literature, leads to a substantial shift of the real part of $a_{\pi^{-} d}$ towards positive values. We have also addressed the isospin violation issue and found that our fit to data accounts for about half the isospin breaking only from mass splittings. The model
has been extended to account for other sources of isospin breaking and then can match results of isospin breaking found in other works. We find that the effect of this breaking in the $b_{0}, b_{1}$ parameters is well within uncertainties from other sources $[1,2]$.

With the precision amplitude at hand, we apply the model in chapter 3 in a different context. In nuclear physics, the so-called "problem of the missing repulsion" is one of unresolved and most persistent issues. The vacuum model serves here as a new input in the in-medium calculation with the most welcome feature, that the multiple rescattering of the isovector interaction generates a large repulsion in nuclear matter. The strategy and novelty is to elaborate the various in-medium effects around this large Ericson-Ericson rescattering piece which leads to relevant higher order corrections that are hardly accessible in a systematic expansion in the coupling and density. In fact, we show that these higher order corrections are so large that the expansion converges too slowly to make reliable predictions in the framework of many-body theory that is used; due to this finding it is not possible to extract the needed repulsion from the model in a reliable way, but some boundaries are provided.

In the following chapters 4 to 7 we will be concerned with more genuine applications of unitarized chiral perturbation theory. One of the consequences of unitarization is the appearance of poles in the complex plane of the invariant scattering energy. For a given spin-isopin channel the occurrence of a pole depends on whether the interaction is attractive or repulsive. For example, for one of the $\pi N$ channels studied in chapter $2, S_{11}$, a pole appears at a scattering energy of around $\sqrt{s}=1530 \mathrm{MeV}$ and around 50 MeV away from the real $\sqrt{s}$-axis in the second Riemann sheet of the complex plane. This pole is identified with the well-known $N^{*}(1535)$ and the $\mathrm{U} \chi \mathrm{PT}$ result delivers a good description of the shape of the amplitude together with branching ratios. In Sec. 1.2.1 we will give a more comprehensive introduction on the concept of the so-called "Dynamical Generation" of resonances.

It is clear that at higher energies resonances play an important role. Also, 3 - and more particle states such as $\pi \pi N$ become more and more important as these channels are open. There is a family of photon-induced reactions that allows for a test of the more complicated dynamics at intermediate
energies, $\gamma p \rightarrow \pi \pi N$. These reactions enjoy a long history of experimental and theoretical efforts to understand them. In the last years good data descriptions have been obtained based on complex models of multi-resonance interaction such as the Valencia Model. In order to study the flavor structure of the mechanisms, one should see the photoproduction in a broader context, i.e. the production of two mesons that are not necessarily pions. As the other members of the octet of pseudoscalar mesons are heavier than the pion, one has to extend the effort towards even higher energies than in the $\gamma p \rightarrow \pi \pi N$ reactions. Furthermore, it is not enough to employ only the $S U(2)$ couplings of the resonances; at high energies other channels such as $\eta N, K \Lambda, K \Sigma, \pi \Delta(1232), K \Sigma^{*}(1385)$ etc. are open, or, even if they are closed, they contribute as virtual intermediate states. With the advent of theoretical progress in this direction these requirements on a realistic model are nowadays given: many resonances can be well described as dynamically generated and due to the coupled channel formalism many $S U(3)$ couplings can be predicted.

In chapter 4 the $\gamma p \rightarrow \pi \pi N$ reaction is generalized to the photoproductions $\gamma p \rightarrow \pi^{0} \eta p$ and $\gamma p \rightarrow \pi^{0} K^{0} \Sigma^{+}$which are currently measured at ELSA in Bonn. In these reactions the $N^{*}$ (1535) plays an important role although we will identify another resonance that qualifies as dynamically generated as the main source of contribution, the $\Delta^{*}(1700)$ together with its strong couplings to $\eta \Delta(1232)$ and $K \Sigma^{*}(1385)$. A virtue of this approach, concerning the $\eta p$ spectrum around the $N^{*}(1535)$, and a test of the nature of this resonance as a dynamically generated object, is that one can make predictions about cross sections for the production of the resonance without introducing the resonance explicitly into the formalism; only its components in the $\left(0^{-}, 1 / 2^{+}\right)$ and $\left(0^{-}, 3 / 2^{+}\right)$meson-baryon base are what matters, together with the wellknown coupling of the photons to these components and their interaction in a coupled channel formalism.

Besides the novel use of dynamically generated resonances in this context, we also include the relevant mechanisms of the Valencia model as well as a series of production mechanisms from $\chi \mathrm{PT}$.

Using chiral Lagrangians for the photoproduction of the $\eta$, the contribution vanishes at tree level; it is the unitarization in coupled channels which
renders the amplitude finite by coupling the photon to intermediate charged states which lead to the final state $\pi^{0} \eta$ or $\pi^{0} K^{0}$. Thus, the reactions under study offer a unique opportunity to study dynamically generated resonances; in turn, precisely these resonances predict the large couplings that result in good experimental agreement with preliminary data from ELSA presented recently: total and differential cross sections and invariant mass spectra [3-5].

Since the $\Delta^{*}(1700)$ has been found to be so important, we study this resonance in chapter 5 in almost a dozen of different photon- and pion-induced reactions, as both initial states can be treated on the same footing.

We find that in spite of exploiting the tail of the resonance, around half of the width or one width above the nominal energy of the $\Delta^{*}(1700)$, the cross sections obtained are sizable. Another experimental fact that supports the dominance of the mechanisms used here is the presence of the $\Sigma^{*}(1535)$ clearly visible in the data and with little room for additional background. Furthermore, in some of the reactions differential cross sections have been measured which show an $s$-wave dominance for the production of the $\Sigma^{*}(1385)$. This is in the line of the present investigation and at the same time discards strong contributions from potential production mechanisms via resonances with different quantum numbers than those of the $\Delta^{*}(1700)$. Furthermore, the flat angular distribution shows that $t$-channel vector meson exchange plays no important role at the very lowest energies where we claim validity of the present model.

The scheme of dynamical generation of this resonance predicts couplings to $\eta \Delta(1232)$ and $K \Sigma^{*}(1385)$ very different from the $S U(3)$ extrapolation of the experimentally known $\pi \Delta$ (1232) coupling. We have checked that the coupling to this channel within the unitary coupled channel approach for the dynamical generation of the $\Delta^{*}(1700)$ agrees well with experiment. The enhancement of up to factor of 30 is a strong claim that should be visible in experiment. The experimental cross sections of the various reactions differ by almost two orders of magnitude; however, the model delivers a good global agreement for all reactions and without introducing new parameters. This gives support to the thesis that the $\Delta^{*}(1700)$ is dynamically generated [6].

Although the electromagnetic interaction is weak, its coupling to the strongly interacting hadrons offers a unique and totally independent oppor-
tunity to study the properties of dynamically generated resonances such as electromagnetic form factors. This is because the photoproduction and radiative decay can be predicted by gauging the baryonic and mesonic components that build up the resonances in the picture of dynamical generation. This is a well-known and well-defined procedure as minimal substitution provides a parameter-free way to couple the electromagnetic field. Thus, having studied photoproduction of two mesons, it is straightforward to test dynamically generated resonances for their radiative decay.

In chapter 6 the radiative decay of the $\Lambda^{*}(1520)$ according to $\Lambda^{*}(1520) \rightarrow$ $\gamma \Lambda$ and $\Lambda^{*}(1520) \rightarrow \gamma \Sigma^{0}$ is investigated. This resonance has been recently described within the scheme of dynamical generation. For the $\Lambda^{*}(1520) \rightarrow$ $\gamma \Lambda$ decay, the dominant meson-baryon channels cancel exactly in the unitary coupled channel approach whereas for $\Lambda^{*}(1520) \rightarrow \gamma \Sigma^{0}$ they add. Indeed, for the latter reaction we find good agreement with experiment. However, for the decay into $\gamma \Lambda$ the experimental value comes from a different source such as a genuine three-quark component. Thus, the radiative decay can serve to get further insight of the admixture of the dynamically generated part of the wave function and the genuine component. Indeed, in many quark models, the $\gamma \Lambda$ final state is dominant and $\gamma \Sigma^{0}$ goes to zero in the limit of vanishing configuration mixing. Second, in some quark models the radiative decay into $\gamma \Lambda$ is overpredicted; once one allows for a meson-baryon component in the wave function, the fraction of the genuine three-quark state in the wave function decreases and quark model results would give more realistic results for $\Lambda^{*}(1520) \rightarrow \gamma \Lambda[7]$.

In chapter 4 and 5 the coupling of the photon to the $\Delta^{*}(1700)$ has been taken from experiment in a semi-phenomenological way. This is well-justified as the physics that we want to extract lies in the large couplings of this resonance to $\eta \Delta$ and $K \Sigma^{*}(1385)$. However, it is straightforward to improve at this point as the coupled channel model can be easily extended to predict the $\gamma N \Delta^{*}(1700)$ transition strength. This is achieved in chapter 7 by coupling the photon to the mesonic and baryonic components that build up the resonance, in the same way as it is carried out in chapter 6 for the $\Lambda^{*}(1520)$. As a novelty, the $\pi N$ channel in $d$-wave has been included in the coupled channel scheme. Furthermore, the phototransition via the dominant $s$-wave loops
has been treated in a fully gauge invariant way. As a result, we find good agreement with the observed radiative decay width giving further support to the $\Delta^{*}(1700)$ being a dynamically generated resonance.

In the study of pionic atoms in chapter 3 we have already faced an inmedium application of a microscopical theory. While in the nuclear medium the baryon density is finite and the temperature is zero, there are several physical environments which exhibit a finite temperature and/or density at the same scale as the particles involved, such as supernovae, neutron stars, white dwarfs, or, in earth-based colliders, the collision of heavy atoms.

In fact, it is common to display the state of strongly interacting matter as a function of baryon density and temperature in a so-called phase diagram which shows a phase transition at some critical temperature $T_{c}(\rho)$. For $T<$ $T_{c}(\rho)$ the effective description of matter is in terms of hadronic degrees of freedom whereas for $T>T_{c}(\rho)$ it is nowadays believed that there exists a new phase of QCD where the bulk properties of strongly interacting matter are governed by the fundamental degrees of freedom - the quarks and gluons. This novel phase is known as Quark Gluon Plasma (QGP).

It is believed that the QGP will be transiently produced if normal hadronic matter is subjected to sufficiently high temperatures and /or densities which potentially are reached in heavy ion collisions. Thus, one of the primary goals in the study of heavy ion collisions is to show the existence of the QGP. The reduction of charge fluctuations (CF) in the deconfined phase of heavy ion collisions, due to the fractional charge of the quarks, may serve as a signature of the QGP. In the hadronic phase, fluctuations are altered by particle interactions. It is, thus, of interest to study the residual correlations of the hadrons and this will be our main concern in chapter 8 .

In the present study we start with an approximation of the interaction mediated via the $\rho$-meson introduced as a heavy gauge boson. The charge fluctuations are evaluated via two-loop photon selfenergy diagrams and charge conservation is shown. This method is found equivalent to the loop expansion of the grand canonical partition function at finite chemical potential which facilitates the calculations at some points. Another approach - in principle
quite different - to the evaluation of the grand canonical partition function is the virial expansion. For the first time, we can show that this method is exactly equivalent to the thermal loop expansion which has been possible by the inclusion of so-called "exchange diagrams" in the virial expansion which restores the Bose-Einstein statistics of the interacting particles.

At temperature around the phase transition ( $\sim 170 \mathrm{MeV}$ ), corrections of higher order in density and interaction play an important role. Virial expansions are not feasible any more and one has to resort to other methods such as resummations. Besides considering such schemes we also extend the study to $S U(3)$. Although the various corrections are sizable they partially cancel and the final results are well described by the free resonance gas $[8,9]$.

In chapter 9 we link the questions concerning the charge fluctuations to the theoretical framework of unitarized chiral perturbation theory. One of the findings from chapter 8 is the importance of unitarity in the calculation of the CF. We have seen that strongly momentum dependent interactions in thermal loops, such as from the NLO chiral Lagrangian, introduce artifacts in the results because thermal loops pick up high c.m. momenta far beyond the applicability of $\chi \mathrm{PT}$ and where this theory violates unitarity. In this context, the unitarity provided by $U \chi P T$, together with an excellent description of phase shifts up to high energies, is an interesting alternative. We have also seen in chapter 8 that corrections higher than quadratic in density contribute considerably to the results. Thus, in a first attempt to combine these two requirements, we utilize a chiral unitary model and dress the intermediate states using thermal loops; as a novelty, finite chemical potential is included in the intermediate states. As we only study the qualitative effects at this point, the study is restricted to the charge two channel. The phase shifts obtained from the model are employed in a density expansion. Furthermore, the virial expansion, which assumes classical statistics, is replaced by the density expansion proposed in chapter 8 in order to provide a consistent treatment of the statistics.

All chapters start with a short abstract. The key conclusions have been outlined in this section and more specific aspects can be found at the end of each chapter. Some of the rather technical details of the thesis are summarized in the Appendix.

### 1.2 Unitary extensions of $\chi \mathrm{PT}$

In this section we outline the principles of Unitarized Chiral Perturbation Theory, $U \chi P T$. For standard chiral perturbation theory [10], there are several reviews on the topic as, e.g., Refs. [11-13] or the pedagogical introduction in Ref. [14]. A concise introduction to resonant chiral perturbation theory is given in Ref. [15]. In this thesis we will mainly discuss extensions and applications of methods summarized, e.g., in Ref. [16].

The unitary extensions of chiral perturbation theory from Ref. [16] come from a slightly different point of view than $\chi \mathrm{PT}$. In $U \chi P T$, the philosophy is to start from the analytic properties of the most general scattering amplitude [17]. Unitarity is implemented via the right-hand, physical cut. Expressing the amplitude in a dispersion relation, the unitary amplitude is obtained by matching the amplitude to the chiral perturbative results at low energies, hence determining the unknown subtraction constants. This will be outlined in the following. A more comprehensive discussion can be found in Ref. [17]. In the following we will also give some results for the case of coupled channels. All amplitudes have to be understood as partial wave amplitudes.

The two-body scattering amplitude in the complex $\sqrt{s}$ plane has several cuts or discontinuities. There is a right-hand, physical cut along the real axis that starts at threshold, and a left hand, unphysical cut for $s<s_{\text {left }}$. The physical cut is determined from unitarity which in coupled channels reads

$$
\begin{equation*}
\operatorname{Im} T_{i, j}=T_{i, l}(s) \sigma_{l}(s) T_{l, j}^{*}(s) \tag{1.1}
\end{equation*}
$$

where $\sigma_{i} \equiv 2 M_{l} q_{i} /(8 \pi \sqrt{s})$ for meson baryon scattering or $\sigma_{i} \equiv q_{i} /(8 \pi \sqrt{s})$ for meson meson scattering, with $q_{i}$ the modulus of the c.m. three-momentum, and the subscripts $i$ and $j$ refer to the physical channels. An alternative formulation of unitarity follows immediately,

$$
\begin{equation*}
\operatorname{Im} T^{-1}(s)=-\sigma(s) \tag{1.2}
\end{equation*}
$$

The scattering amplitude $T$ is expressed according to the $N / D$ method [18] as a ratio of two functions

$$
\begin{equation*}
T(s)=\frac{N(s)}{D(s)} \tag{1.3}
\end{equation*}
$$

The denominator $D$ contains the right-hand, physical cut and the numerator $N$ accounts for contributions from the unphysical cut such as crossed contributions. In the following, this cut is omitted. Yet, this is an approximation which is kept under control. In [17] a test is done of the contributions from the left hand cut in meson-meson scattering concluding that the contribution is small; but more important: It is weakly energy dependent in the region of physical energies. This is the key to the success of the method exposed here, since any constant contribution in a certain range of energies can be accommodated in terms of the subtraction constant as we will see (see also a detailed discussion of the contribution of the left hand cut in $\pi N$ scattering in [19]).

For higher partial waves $L=1,2, \cdots$, one can explicitly take into account the behavior of a partial wave amplitude at threshold, which vanishes like $p^{2 L} \equiv \nu^{L}$, and consider the new quantity $T_{L}^{\prime}(s)=T(\sqrt{s}) / \nu^{L} \equiv N^{\prime} / D^{\prime}$. The quantity $\nu=\nu(p)$ is a smooth function and vanishes linearly in the c.m. momentum $p$ at threshold. The new amplitude $T^{\prime}$ satisfies the same unitarity condition as before from Eq. (1.1). In this thesis we will encounter the case $L=2$ or $d$-wave scattering when considering the dynamically generated resonance $\Lambda^{*}(1520)$. The important point for the present purposes is that the higher partial waves exhibit the same analytical properties as the $s$-wave, in particular the on-shell factorization outlined below.

By omitting the left hand cut the numerator function is just a smooth polynomial in $\sqrt{s}$. One can divide $N^{\prime}$ and $D^{\prime}$ by this polynomial and, thus, cast the amplitude in the form [17]

$$
\begin{align*}
T_{L}^{\prime}(s) & =\frac{1}{D_{L}^{\prime}(s)^{\prime}}, \quad N_{L}^{\prime}(s)=1 \\
D_{L}^{\prime}(s) & =-\frac{\left(s-s_{0}\right)^{L+1}}{\pi} \int_{s_{\mathrm{th}}}^{\infty} d s^{\prime} \frac{\nu\left(s^{\prime}\right)^{L} \sigma\left(s^{\prime}\right)}{\left(s^{\prime}-s\right)\left(s^{\prime}-s_{0}\right)^{L+1}} \\
& +\sum_{m=0}^{L} a_{m} s^{m}+\frac{R_{i}}{s-s_{i}} . \tag{1.4}
\end{align*}
$$

Here, the inverse amplitude, or $D^{\prime}$, has been expressed by an $L$-subtracted dispersion relation. This gives rise to the constants $a_{m}$ which in principle are undetermined. In the general form of the amplitude in Eq. (1.4) the zeros
of the amplitude $T$ have been taken explicitly into account by the Castillejo-Dalitz-Dyson (CDD) poles in the denominator (terms with $R$ in Eq. (1.4)).

In the following, we will drop the extension to arbitrary partial waves and concentrate on $s$-wave scattering only. The full formalism can be found in Ref. [17]. The dispersion relation from Eq. (1.4), considering also coupled channels, is now rewritten as
$T^{-1}(\sqrt{s})_{i j}=-\delta_{i j}\left\{\widetilde{a}_{i}\left(s_{0}\right)+\frac{s-s_{0}}{\pi} \int_{s_{i}}^{\infty} d s^{\prime} \frac{\sigma\left(s^{\prime}\right)_{i}}{\left(s^{\prime}-s-i \epsilon\right)\left(s^{\prime}-s_{0}\right)}\right\}+V^{-1}(\sqrt{s})_{i j}$,
where $s_{i}$ is the value of the $s$ variable at the threshold of channel $i$, and $s_{i}=s_{0}$ in the notation from Eq. (1.4) for one channel. The subtraction constants from Eq. (1.4) can be formally separated in a part leading in $N_{c}$, the number of colors, and a part sub-leading [17], $a_{m}=a_{m}^{\text {lead. }}+a_{m}^{\text {sub. }-1 .}\left(s_{0}\right)$. The sub-leading part is associated with the integral which is also sub-leading in large $N_{C}$. This means that in Eq. (1.5) for the one channel case

$$
\begin{equation*}
\widetilde{a}_{i}\left(s_{0}\right) \equiv a_{0}^{\text {sub. }-1 .}, V^{-1}(\sqrt{s}) \equiv a_{m}^{\text {lead. }}\left(s_{0}\right) \tag{1.6}
\end{equation*}
$$

Thus, $V^{-1}(\sqrt{s})_{i j}$ indicates other contributions coming from local and pole terms, as well as crossed channel dynamics but without right-hand cut. These extra terms can be taken directly from $\chi P T$ after requiring the matching of the general result to the $\chi P T$ expressions. Note also that

$$
\begin{equation*}
g(s)_{i}=\widetilde{a}_{i}\left(s_{0}\right)+\frac{s-s_{0}}{\pi} \int_{s_{i}}^{\infty} d s^{\prime} \frac{\sigma\left(s^{\prime}\right)_{i}}{\left(s^{\prime}-s-i \epsilon\right)\left(s^{\prime}-s_{0}\right)} \tag{1.7}
\end{equation*}
$$

is the familiar scalar loop integral that for $\pi \pi$ scattering reads

$$
\begin{equation*}
g^{(\Lambda)}=\int_{0}^{\Lambda} \frac{q^{2} d q}{(2 \pi)^{2}} \frac{\omega_{1}+\omega_{2}}{\omega_{1} \omega_{2}\left(s-\left(\omega_{1}+\omega_{2}\right)^{2}+i \epsilon\right)} \tag{1.8}
\end{equation*}
$$

in the cut-off regularization which is an alternatively regularization scheme.
One can further simplify the notation by employing a matrix formalism. Introducing the matrices $g(s)=\operatorname{diag}\left(g(s)_{i}\right), T$ and $V$, the latter defined in terms of the matrix elements $T_{i j}$ and $V_{i j}$, the $T$-matrix can be written as:

$$
\begin{equation*}
T(\sqrt{s})=[\mathbb{I}-V(\sqrt{s}) \cdot g(s)]^{-1} \cdot V(\sqrt{s}) \tag{1.9}
\end{equation*}
$$

which can be recast in a more familiar form as

$$
\begin{equation*}
T(\sqrt{s})=V(\sqrt{s})+V(\sqrt{s}) g(s) T(\sqrt{s}) \tag{1.10}
\end{equation*}
$$

The former equation has the formal appearance of the Bethe-Salpeter equation (BSE). There is a peculiar feature worth noting: the term $V g T$ of the equation is a product of functions $V(\sqrt{s}), g(s)$, and $T(\sqrt{s})$ while in the BSE using an ordinary $\vec{r}$ dependent potential, this term has an explicit $d^{4} q$ integration involving $V$ and $T$ half off-shell. The appearance of $V$ and $T$ on shell in Eq. (1.10) is a simple consequence of the dispersion relation of Eq. (1.5).

Note that $g(s)$ of Eq. (1.7) is nothing but the $d^{4} q$ integral of a mesonmeson propagator (the check of the imaginary part is immediate), hence in simple words we can say that the dispersion relation justifies a BSE in which the $V$ and $T$ are factorized on shell outside the integral of the $V g T$ term.

At this point we would like to clarify the differences between this version of the BSE based on the $N / D$ method and dispersion relations, from other versions of the BSE used in the literature. A discussion of different approaches to the BSE and its most often used three dimensional reductions is done in [20]. There is a main difference of the approach used here with respect to all of them. In all of these approaches the off shell dependence of the potential plays a key role since it is responsible for the convergence of the integral involved in the $V g T$ term of the BSE. In the present approach this is not needed since only the on shell information of the potential enters the formalism. The fourth-fold integral mentioned above only involves the product of the meson and baryon propagators and is regularized with dimensional regularization, which is equivalent to using a dispersion relation with a subtraction. This procedure has the welcome feature that the results are invariant with respect to unitary transformations of the fields in the Lagrangian, and thus respect the equivalence theorem, something which does not occur in other variants of the BSE as noted in [21]. The invariance of the results under these transformations is also satisfied by the $K$ matrix approach, which shares with our approach the use of the potential on shell. The difference is that only the imaginary part of the integral of the two hadron propagators is kept in the $K$ matrix approach, while we also include the real part evaluated with dimensional regularization, or with a cut off of natural
size. This accounts for basic properties of the scattering matrix, such as analyticity, chiral logs, etc.

The techniques discussed in this section have been applied successfully to $\bar{K} N$ interaction in $s$-wave [22] and $p$-waves [23]. In this latter work, the kernel, $V$, has contact terms and pole terms corresponding to the $\Lambda, \Sigma$, and $\Sigma^{*}(1385)$ particles. A similar procedure is carried out in [24] also for $\bar{K} N$ scattering and in [19] in the $\pi N$ scattering case. The quality of the results and the sophistication of that latter model is equivalent to that of other successful relativistic approaches to $\pi N$ like [25, 26], and fewer parameters are needed. In the case of the $\bar{K} N$ interaction of [22] and [23] a quite good description of the data was obtained with only one parameter.

Summarizing with regard to this thesis, let us record the following points: Starting from the unitarity condition for the scattering amplitude, the BetheSalpeter equation can be derived from a dispersion relation. The kernel is determined from matching to some interaction; in the thesis we will encounter the lowest order (LO) chiral Lagrangian in the meson-meson ( $M M$ ) and meson-baryon ( $M B$ ) sector; the next-to leading (NLO) chiral Lagrangian in the treatment of $\pi N$ scattering; the lowest order chiral Lagrangian that drives the interaction of the $0^{-}$octet of pseudoscalar mesons $(M)$ with the $1 / 2^{+}$ decuplet of excited baryons $\left(B^{*}\right)$. Second, we have found that for $s$-wave and $d$-wave scattering, the loop function and the kernel factorize and the kernel takes the on-shell value of the particles. The loop function $G$ provides the unitarity cut for the scattering amplitude and contains one (three) free subtraction constants for $s$-wave ( $d$-wave) scattering which have to be fixed from phenomenology. The latter procedure, together with undetermined low energy constants from the interaction, will be one of the main concerns of this thesis: $U \chi P T$ provides quite some freedom which in many cases leads to a good fit to the data. On the other hand, once the free parameters are fixed, one should test the model in as many as possible different reactions in order to obtain information of the consistency and predictive power of the mechanisms.

### 1.2.1 Resonances in $\pi N$ interaction

The chiral unitary approach in coupled channels, briefly outlined in the last section, finds application in many aspects of meson-baryon scattering. One of the advantages of this method in the $\pi N$ sector is that resonances and background can be treated on the same footing.

One of the key questions in $\pi N$ scattering is the treatment of background. In phenomenological analyses the background has to be subtracted from the amplitude in order to extract resonance parameters, and this is where model dependence enters. Yet, there are more problems as in case of the $N^{*}(1535)$ resonance which appears as a structure in the $S_{11}$ channel on top of a background that is larger than the resonance structure itself and which has a strong influence from the lower tail of the nearby $N^{*}(1650)$; also, that resonance shows a large branching ratio into the $\eta N$ channel. There are three broad categories of coupled-channel approaches that try to take into account resonance and background properties in a more systematical way:

The K-Matrix approach has been used most extensively to derive resonance parameters from the partial wave amplitudes. In the $K$-matrix approximation, only on-shell intermediate states are taken into account when solving the Bethe-Salpeter equation for two-body scattering - the real parts of the loop functions are neglected. Within this approximation, unitarity is satisfied. The approximation leads to a tremendous technical simplification because the solution of the Bethe-Salpeter equation is reduced to an algebraic equation. The price one has to pay is a truncation of the strength of multiple scattering contributions. As a consequence, bound states cannot be generated in this approximation. In concrete applications of the $K$-matrix method, all resonances have to be put explicitly. A phenomenological background can be added. This allows to extract resonance parameters from data. The extracted parameters contain some model dependence which is quite mild in the case of isolated resonances.

The various groups differ by their treatment of the background. The Kent State group parameterizes the background $K$-matrix by a polynomial of the invariant collision energy so that unitarity is manifestly retained [27-29].

The Gießen group has suggested a K-matrix approach which fits elastic and inelastic scattering data with asymptotic two-body channels including an additional channel to account for some missing components of the total inelastic cross section [30]. The background is parametrized by $t$-channel exchanges.

An extension of the $K$-matrix approach is provided by the PittsburghArgonne approach which is an update of the Carnegie-Mellon-Berkeley analysis which relies on dispersion relations to guarantee unitarity and therefore can generate complex resonance poles from analytic continuations. In that approach, the background is represented by the tails of subthreshold resonances. In some cases, such as the $P_{11}$ partial wave, the background turns out to have a considerable energy dependence. [31].

Unitarized meson-exchange models for pion-nucleon scattering have been developed by several groups [26,32-35]. Here one iterates the $t$-channel and $u$ channel exchange diagrams in addition to the actual $s$-channel resonances. In principle, meson-exchange models can be and should be matched to ChPT in the vicinity of the threshold. In practice, however, chiral symmetry has been used as a qualitative argument only, e.g., to choose a derivative coupling for the pion-nucleon vertex. Some of the low-energy constants of chiral perturbation theory have been estimated assuming resonance saturation [36,37]. This shows the link between effective field theory and meson-baryon phenomenology. A feature common to chiral unitary approaches and meson-exchange models is the possibility to generate dynamical resonances and to derive the background scattering amplitude from an underlying theory. However, in the meson-exchange models, due to the finite mass of the exchanged particles, the $t$-channel and $u$-channel exchange diagrams correlate the background in different partial waves and thus give predictive power to the meson-exchange approach. One of the most sophisticated meson-exchange models is the Jülich model [33-35].

The chiral unitary approach pioneered by Dobado, Pelaez, Kaiser, Siegel, Weise, Oller, Oset, Meißner and others [19, 22, 24, 38-42] provides a method to unitarize Chiral Perturbation Theory (ChPT). It is based on a solution of the Bethe-Salpeter equation employing a scattering kernel that is derived
within ChPT. Since those scattering kernels are contact interactions, the solution of the Bethe-Salpeter equation is considerably simplified and in fact is performed by summing the one loop diagrams describing the intermediate two-body states. Because of the energy dependence of the chiral interactions, bound states or quasibound states can get generated, as, e.g., in the scalarisoscalar $s$-wave channel of meson-meson scattering. In the baryon sector with strangeness zero, which is of interest here, the $N^{*}(1535), \Delta^{*}(1700)$ and other resonances have been claimed to be dynamically generated [41,43-45].

### 1.2.2 Dynamically generated resonances

The history of dynamically generated resonances, which appear in the solution of the meson-meson or meson-baryon coupled channel LippmannSchwinger equation (LSE) with some interaction potential, is quite old. One of the typical examples is the $\Lambda(1405)$ resonance which appears naturally in coupled channels containing the $\pi \Sigma$ and $\bar{K} N$ channels [46, 47]. The advent of unitary extensions of chiral perturbation theory has brought more systematics into this approach with chiral Lagrangians providing the kernel, or potential, for the LSE or its relativistic counterpart, the Bethe Salpeter equation (BSE) which is more often used. In this sense the $\Lambda(1405)$ has been revisited from this new perspective and at the same time new resonances like the $N^{*}(1535), \Lambda(1670)$, etc. have been claimed to be also dynamically generated [22, 39-41, 45, 48-50]. Actually, one of the surprises along these lines was the realization that the chiral theory predicted the presence of two nearby poles in isospin $I=0$, strangeness $S=-1$ close to the nominal $\Lambda(1405)$ mass, such that the physical resonance would be a superposition of the two states [51,52]. Recent work including also the effect of higher order Lagrangians in the kernel of the BSE [53-55] also find two poles in that channel, one of them with a large width [53]. Interestingly, recent measurements of the $K^{-} p \rightarrow \pi^{0} \pi^{0} \Sigma^{0}$ reaction [56] show the excitation of the $\Lambda(1405)$ in the $\pi^{0} \Sigma^{0}$ invariant mass, peaking around 1420 MeV and with a smaller width than the nominal one of the PDG [57]. An analysis of this reaction and comparison with the data of [58] from the $\pi^{-} p \rightarrow K^{0} \pi \Sigma$ reaction led the authors of Ref. [59] to conclude that the combined experimental information of these
two reactions provided evidence of the existence of two $\Lambda$ (1405) states.
More recent work has extended the number of dynamically generated resonances to the low lying $3 / 2^{-}$resonances which appear from the interaction of the octet of pseudoscalar mesons with the decuplet of baryons [43, 44]. One of the resonances that appears qualitatively as dynamically generated is the $\Lambda(1520)$, built up from $\pi \Sigma^{*}(1385)$ and $K \Xi^{*}(1530)$ channels, although the necessary coupling to the $\pi \Sigma$ and $\bar{K} N$ channels makes the picture more complex [60,61].

More systematic studies have shown that there are two octets and one singlet of resonances from the interaction of the octet of pseudoscalar mesons with the octet of stable baryons $[51,52]$. The $N^{*}(1535) S_{11}$ belongs to one of these two octets and plays an important role in the $\pi N$ interaction with its coupled channels $\eta N, K \Lambda$ and $K \Sigma$ [45]. In spite of the success of the chiral unitary approach in dealing with the meson-baryon interaction in these channels, the fact that the quantum numbers of the $N^{*}(1535) S_{11}$ are compatible with a standard three constituent quark structure and that its mass is roughly obtained in many standard quark models [62,63], or recent lattice gauge calculations [64], has as a consequence that the case for the $N^{*}(1535) S_{11}$ to be described as a dynamically generated resonance appears less clean than that of the $\Lambda(1405) S_{01}$ where both quark models and lattice calculations have shown systematic difficulties [65]. Those states that do not have a prominent quark core although their quantum numbers are allowed by quark models are also known as cryptoexotics. Sometimes dynamically generated resonances are called hadronic molecules or loose composites but one should keep in mind that for almost all dynamically generated resonances it is the coupled channel formalism with its transitions in $S U(3)$ that provides the necessary strength for the formation of a pole in the complex plane of the invariant scattering energy $\sqrt{s}$.

Ultimately, it will be the ability of the models to describe different experiments in which the resonances are produced that will settle the issue of what represents Nature better at a certain energy scale. A detailed description of many such experiments has been discussed in [16].

The concept of dynamical generation of resonances has been applied in several different contexts and provides a successful description of phe-
nomenology. However, there are some caveats which will be outlined in the following. First, the method of dynamically generating resonances is not a tool to describe all resonances of the particle data group (PDG) [57]. Restricting ourselves to the baryonic resonances, thus far, only the low lying $1 / 2^{-}$and $3 / 2^{-}$resonances qualify as such. The quantum numbers of these resonances are such that they can also be in principle interpreted as ordinary three constituent quark states with one quark in a $p$-wave which means that one should be ready to accept some three constituent quark components in the wave function. Conversely, the coupling of meson-baryon components to a seed of three constituent quarks is also unavoidable, as given for instance from the existence of meson-baryon decay channels. Nature will make this meson-baryon cloud more important in some cases than others, and those where the dress of the meson cloud overcomes the original three constituent quark seed are candidates to be well described in the chiral unitary approach and appear as what we call dynamically generated resonances where the three constituent quark components are implicitly assumed to be negligible.

Then the question arises, which are the mesons and baryons that are used as building blocks in the chiral unitary approach and which can be dynamically generated. The answer to this is provided by exploiting the chiral theories in the large $N_{c}$ limit. The dynamically generated resonances appear as a solution of the Bethe-Salpeter equation and hence it is the iteration of the kernel through loop diagrams that will lead to the appearance of these resonances. But these are sub-leading terms in the large $N_{c}$ counting that vanish in the limit of $N_{c} \rightarrow \infty$. Hence, the dynamically generated resonances disappear in a theoretical scheme when $N_{c} \rightarrow \infty$ and the resonances that remain are what we call genuine ones. In this sense, the $\Delta(1232)$ (and other baryons of the decuplet) is a genuine resonance which appears degenerate with the nucleon in the large $N_{c}$ limit [66]. This statement might seem to clash with a well-known historical fact, the dynamical generation of the $\Delta(1232)$ from the iteration of the crossed nucleon pole term in the Chew and Low theory [67]. However, attractive as the idea has always been, the input used in this approach, in particular the simplified $\pi N N$ coupling, is at odds with present chiral Lagrangians and hence that old idea is no longer supported in present chiral approaches. A more modern and updated for-
mulation of the problem, according with requirements of chiral dynamics is given in [19]. There, the $\Delta$, which qualifies as a genuine resonance, appears through a Castillejo, Dalitz, Dyson pole [68] in the $N / D$ formulation of [17].

An interesting work on the meaning of the large $N_{c}$ limit and the classification of states into dynamically generated or genuine resonances is Ref. [69], where the author shows what large $N_{c}$ means in practice, with some subtleties about the strict $N_{c}=\infty$ limit. At the same time one shows that the $\rho$ meson qualifies as a genuine resonance while the $\sigma, f_{0}(980)$, and $a_{0}(980)$ qualify as dynamically generated.

## Chapter 2

## $S$-wave pion nucleon scattering lengths from $\pi N$, pionic hydrogen and deuteron data

The isoscalar and isovector scattering lengths $\left(b_{0}, b_{1}\right)$ are determined using a unitarized coupled channel approach based on chiral Lagrangians. Using experimental values of pionic hydrogen and deuterium as well as low energy $\pi N$ scattering data, the free parameters of the model are calculated. Isospin violation is incorporated to a certain extent by working with physical particle masses. For the deuterium scattering length $a_{\pi^{-} d}$ new significant corrections concerning real and imaginary parts are evaluated, putting new constraints from $\pi^{-} d$ scattering on the values of $\left(b_{0}, b_{1}\right)$. In particular, dispersion corrections, the influence of the $\Delta(1232)$ resonance, crossed terms and multiple scattering in a Faddeev approach are considered.

### 2.1 Introduction

The precise values of the isoscalar and isovector $\pi N$ scattering lengths are one of the important issues in hadronic physics. Together with low energy $\pi N$ scattering data they determine parameters of the chiral Lagrangian which allows to make predictions even below $\pi N$ threshold using chiral perturbation theory. The experimental data from where $\left(b_{0}, b_{1}\right)$ are usually extracted
are the shift and width of pionic hydrogen and deuterium atoms. From the recent measurements at PSI one deduces the elastic $\pi^{-} p$ plus the $\pi^{-} p \rightarrow \pi^{0} n$ transition scattering lengths [70-73]. Using in addition the measured $a_{\pi^{-} d}$ amplitude $[73,74]$ in order to determine the isospin even and odd combinations

$$
\begin{align*}
& a_{+}=\frac{1}{2}\left(a_{\pi^{-} p}+a_{\pi^{-} n}\right) \\
& a_{-}=\frac{1}{2}\left(a_{\pi^{-} p}-a_{\pi^{-} n}\right), \tag{2.1}
\end{align*}
$$

or, correspondingly, the isoscalar and isovector scattering lengths $b_{0}=a_{+}$, $b_{1}=-a_{-}$, requires, however, a non-trivial work on the $\pi^{-} d$ system. This is because the impulse approximation (IA) vanishes in the limit $b_{0}=0$ and the extraction of $a_{\pi^{-} n}$ from $a_{\pi^{-} d}$ calls for a multiple scattering treatment with the double scattering as the leading contribution. Also, higher order corrections as absorption and dispersion have an important effect, as has been extensively discussed for instance in Ref. [75]. It is in particular the $\pi^{-} d$ scattering length that narrows down the value of $\left(b_{0}, b_{1}\right)$. Although the error of $a_{\pi^{-} d}$ is dominated by large theoretical uncertainties, the corrections on $a_{\pi^{-} d}$ directly affect the values of $\left(b_{0}, b_{1}\right)$.

The determination of the pion deuteron scattering length from the elementary ones is one of the problems which has attracted much attention [76-80] in the past, but has also stimulated more recent studies [75, 81-85]. In Ref. [78], $a_{\pi^{-d}}$ is calculated in a Faddeev approach incorporating several processes in the multiple scattering series as nucleon-nucleon correlations, absorption and the corresponding dispersion. It delivers, together with [77], a very complete description of multiple scattering in the deuteron. Here, instead, we use the fixed center approximation (FCA) to the Faddeev equations. Other contributions, like absorption and dispersion correction, are evaluated separately, and fully dynamically, in a Feynman diagrammatic approach. This is feasible because the multiple scattering series is rapidly converging since the scattering lengths are small compared to the deuteron radius.

In Ref. [86] the values of $\left(b_{0}, b_{1}\right)$ have been calculated from the pion deuteron scattering length up to NNLO in chiral perturbation theory including the $\mathcal{L}^{(2)}$ and $\mathcal{L}^{(3)} \pi N$ Lagrangians. Using realistic deuteron wave
functions, and other modifications, the authors reobtain the double and triple scattering formulas in the isospin limit.

Typical results for the isoscalar and isovector scattering lengths, obtained recently in Ref. [75] and Ref. [86], are:

$$
\begin{array}{cccc}
\left(b_{0}, b_{1}\right)=(-12 \pm 2 \text { stat. } \pm 8 \text { syst., }-895 \pm 3 \text { stat. } \pm 13 \text { syst. }) & \cdot 10^{-4} m_{\pi^{-}}^{-1} & {[75]} \\
\left(b_{0}, b_{1}\right)= & (-34 \pm 7,-918 \pm 13) & \cdot 10^{-4} m_{\pi^{-}}^{-1} & {[86] .} \tag{2.2}
\end{array}
$$

Here, and in the subsequent results, the unit of inverse pion mass refers to the charged pion. The results of Eq. (2.2) are not in agreement with each other. The problem with $a_{+}$is that it becomes a small quantity from a cancellation of terms of the order of $a_{-}$, hence $a_{+}$is difficult to determine, and the discrepancy between the two results in Eq. (2.2) indicates that the uncertainties in $a_{+}$are larger than shown in Eq. (2.2). Actually, in Ref. [86] larger uncertainties are advocated from isospin violation, since the analysis is made by assuming isospin symmetry. The present study is formulated in the particle base and thus, isospin breaking effects from different physical masses are incorporated. This provides a part of the isospin violation [87-92] that has already been observed in a similar context in $\bar{K} N$ scattering [22].

Our first purpose in the present study is to carry out further calculations in the problem of $\pi^{-} d$ scattering at threshold incorporating novel terms. We start with the absorption of the $\pi^{-}$in the deuteron and the dispersion tied to it. The latter contributes to the real part of $a_{\pi^{-} d}$, and in the literature a quite large correction originates from this source. Since high precision deuteron wave functions are at hand nowadays, and the analysis is carried out fully dynamically, a revision of the results from [77-79] is appropriate. After calculating the effects of the $\Delta(1232)$ excitation in the dispersion, other contributions as crossed terms are considered. Together with corrections of different nature from the literature, a final correction to $a_{\pi^{-} d}$ is given. This enables us to parametrize the pion deuteron scattering length in terms of the elementary $s$-wave $\pi N$ scattering lengths $a_{\pi N}$ via the use of the Faddeev equations. We also test the model dependence of the results on the deuteron wave functions.

The second purpose is then the application of the unitary coupled channel model from Ref. [45] to $\pi N$ scattering at low energies. The model provides
the $\pi N$ scattering lengths for the Faddeev equations for $a_{\pi^{-} d}$ and the $\pi N$ scattering amplitudes at low energies and threshold. First, it is tested if the model can explain threshold data and low energy $\pi N$ scattering consistently. Then, a precise parametrization of the $\pi N$ amplitude at low energies, including threshold, is achieved.

This can be used to extrapolate to the negative energies felt by pionic atoms, the study of which has been one of the stimulating factors in performing the present work.

### 2.2 Summary of the model for $\pi N$ interaction

We follow here the approach of Ref. [45] where the $N / D$ method adapted to the chiral context of [17] is applied. Developed for the case of meson meson interactions, the method of [17] was extended to the meson baryon interaction in [19, 24], and Ref. [45] follows closely the formalism of these latter works. In the CM energy region of interest, from threshold up to around 1250 MeV , pions and nucleons play the predominant role compared to the influence of the heavier members of the meson and baryon octet. We have carried out the $S U(3)$ study as in Ref. [45], which involves the $K \Sigma, K \Lambda$ and $\eta n$ channels in addition to the $\pi N$ ones. At low energies, these channels are far off shell and we have seen that the fit to the data improves only slightly at the cost of three new additional subtraction constants. Therefore, we restrict the coupled channel formalism to $\pi^{-} p, \pi^{0} n$ in the charge zero sector, and $\pi^{+} p$ in the double charge sector. The scattering amplitudes are described by the Bethe-Salpeter equation

$$
\begin{equation*}
T(\sqrt{s})^{-1}=V^{-1}(\sqrt{s})-G(\sqrt{s}) \tag{2.3}
\end{equation*}
$$

where the kernel $V$ is obtained from the lowest order meson baryon Lagrangian [11-13]. which is given by

$$
\begin{align*}
\mathcal{L}_{1}^{(B)}= & \left\langle\bar{B} i \gamma^{\mu} \nabla_{\mu} B\right\rangle-M_{B}\langle\bar{B} B\rangle \\
& +\frac{1}{2} D\left\langle\bar{B} \gamma^{\mu} \gamma_{5}\left\{u_{\mu}, B\right\}\right\rangle+\frac{1}{2} F\left\langle\bar{B} \gamma^{\mu} \gamma_{5}\left[u_{\mu}, B\right]\right\rangle \tag{2.4}
\end{align*}
$$

where the symbol $\rangle$ denotes the trace of $\mathrm{SU}(3)$ matrices and

$$
\begin{align*}
\nabla_{\mu} B & =\partial_{\mu} B+\left[\Gamma_{\mu}, B\right] \\
\Gamma_{\mu} & =\frac{1}{2}\left(u^{+} \partial_{\mu} u+u \partial_{\mu} u^{+}\right) \\
U & =u^{2}=\exp (i \sqrt{2} \Phi / f) \\
u_{\mu} & =i u^{+} \partial_{\mu} U u^{+} \tag{2.5}
\end{align*}
$$

with $\Phi$ and $B$ the usual $3 \times 3 S U(3)$ matrices of the fields for the meson octet of the pion and the baryon octet of the nucleon, respectively [11, 12]. The term with the covariant derivative $\nabla_{\mu}$ in Eq. (2.4) generates the WeinbergTomozawa interaction and leads to the lowest order transition amplitude

$$
\begin{equation*}
V_{i j}=-C_{i j} \frac{1}{4 f_{i} f_{j}} \bar{u}\left(p^{\prime}\right) \gamma^{\mu} u(p)\left(k_{\mu}+k_{\mu}^{\prime}\right) \tag{2.6}
\end{equation*}
$$

where $p, p^{\prime}\left(k, k^{\prime}\right)$ are the initial and final momenta of the baryons (mesons). The coefficients $C_{i j}$ are $S U(3)$ factors which one obtains from the Lagrangian, and the $f_{i}$ are the $\pi, \eta, K$ decay constants [10].

In the present case we need only the transitions for $\pi N$. Furthermore, only the zero-component of the interaction from Eq. (2.6) is needed, or in other words only $s$-wave $\pi N$ scattering which results in

$$
\begin{align*}
V_{i j}(\sqrt{s}) & =-C_{i j} \frac{1}{4 f_{\pi}^{2}}\left(2 \sqrt{s}-M_{i}-M_{j}\right) \\
& \times \sqrt{\frac{M_{i}+E_{i}(\sqrt{s})}{2 M_{i}}} \sqrt{\frac{M_{j}+E_{j}(\sqrt{s})}{2 M_{j}}} \tag{2.7}
\end{align*}
$$

In Eq. (2.3), $G$ is the loop function of the pion nucleon propagator, which in dimensional regularization reads:

$$
\begin{align*}
G_{i}(\sqrt{s})= & \frac{2 M_{i}}{(4 \pi)^{2}}\left(\alpha(\mu)+\log \frac{m_{i}^{2}}{\mu^{2}}+\frac{M_{i}^{2}-m_{i}^{2}+s}{2 s} \log \frac{M_{i}^{2}}{m_{i}^{2}}\right. \\
+ & \frac{Q_{i}(\sqrt{s})}{\sqrt{s}}\left[\log \left(s-\left(M_{i}^{2}-m_{i}^{2}\right)+2 \sqrt{s} Q_{i}(\sqrt{s})\right)\right. \\
& +\log \left(s+\left(M_{i}^{2}-m_{i}^{2}\right)+2 \sqrt{s} Q_{i}(\sqrt{s})\right) \\
& -\log \left(-s+\left(M_{i}^{2}-m_{i}^{2}\right)+2 \sqrt{s} Q_{i}(\sqrt{s})\right) \\
& \left.\left.-\log \left(-s-\left(M_{i}^{2}-m_{i}^{2}\right)+2 \sqrt{s} Q_{i}(\sqrt{s})\right)\right]\right) \tag{2.8}
\end{align*}
$$

where $Q_{i}(\sqrt{s})$ is the on shell center of mass momentum of the $i$-th pion nucleon system and $M_{i}\left(m_{i}\right)$ are the nucleon (pion) masses. The parameter $\mu$ sets the scale of regularization $(\mu=1200 \mathrm{MeV})$ and the subtraction parameter $\alpha$ is fitted to the data. In order to ensure isospin conservation in the case of equal masses, the subtraction constants $\alpha_{\pi^{-} p}, \alpha_{\pi^{0} n}$, and $\alpha_{\pi^{+} p}$ are taken to be equal for states of the same isospin multiplet. Isospin breaking effects from other sources than mass differences are discussed later (by allowing the $\alpha_{\pi N}$ to be different).

The work of [45] concentrated mostly in the region around the $N^{\star}$ (1535) resonance, which is dynamically generated in the scheme. The data around this region were well reproduced, although the description of the $I=3 / 2$ sector required the introduction of the extra $\pi \pi N$ channel. The low energy data was somewhat overestimated in [45] although qualitatively reproduced. Here, however, our interest is to concentrate around threshold in order to obtain an as accurate as possible description of the data in this region and determine, together with the pionic atom data on hydrogen and deuterium, the values of the isoscalar and isovector scattering lengths, with a realistic estimate of the error.

## Isoscalar Piece

The chiral Lagrangian at lowest order that we use contributes only to the isovector $\pi N$ amplitude at tree level, but isoscalar contributions are generated from rescattering. Additional isoscalar terms emerge in the expansion in momenta of the chiral Lagrangian [19, 40, 87], and we take the relevant terms from $\mathcal{L}_{\pi N}^{(2)}$ into account following Ref. [93]. In particular, there is a term independent of $q^{0}$ with $q$ the pion momentum, and one quadratic in $q^{0}$, which enter into the potential $V$ from Eq. (2.7) as

$$
\begin{equation*}
V_{i j} \rightarrow V_{i j}+\delta_{i j}\left(\frac{4 c_{1}-2 c_{3}}{f_{\pi}^{2}} m_{\pi}^{2}-2 c_{2} \frac{\left(q^{0}\right)^{2}}{f_{\pi}^{2}}\right) \frac{M_{i}+E_{i}(\sqrt{s})}{2 M_{i}} . \tag{2.9}
\end{equation*}
$$

The on shell value of the $c_{3}$ term ( $c_{3} q^{2}$ in Ref. [93]) has been taken, consistently with the approach of refs. [24,45] which uses the on shell values for the vertices in the scattering equations. The $c_{i}$-combinations, in the notation of Ref. [93], are fitted to the experiment. For a construction of a $\pi N$ potential
up to higher energies, one has to regulate the quadratic term with $c_{2}$, which we do by multiplying it by a damping factor,

$$
\begin{equation*}
e^{-\beta^{2}\left[\left(q^{0}\right)^{2}-m_{\pi}^{2}\right]} \tag{2.10}
\end{equation*}
$$

Both cases, with the damping factor, and without are studied.

## The $\pi N \rightarrow \pi \pi N$ channel

The $\pi \pi N$ channel opens up at CM energies around 1215 MeV . In the fits, we include energies higher than that, and therefore the 2-loop diagrams from this source should be taken into account as described in Ref. [45]. In Ref. [45], various functional forms for the real part of the $\pi \pi N$ propagator have been tested, and setting it identically to zero resulted in good data agreement. Here we approximate it as a function constant in energy $\sqrt{s}$ and parametrize it in terms of the quantity $\gamma$. The imaginary part of the $\pi \pi N$ propagator has been calculated explicitly in Ref. [45]. At the energies of interest, it is small, and becomes only important at higher energies.

The energy dependence of the $\pi N \rightarrow \pi \pi N$ vertices has also been determined in Ref. [45] from a fit to $\pi N \rightarrow \pi \pi N$ data. Fig. 12 of that reference shows that at low energies they can be well represented by constants, namely $a_{11}=2.6 m_{\pi}^{-3}$ and $a_{31}=5.0 m_{\pi}^{-3}$, which are the values we use. In the present approach, the $\pi \pi N$ propagator with its two adjacent $\pi \pi N$ vertices provide $\pi N \rightarrow \pi N$ amplitudes which are added directly to the kernel of the BetheSalpeter equation (2.3). With the notation of Ref. [45], we obtain for the $\pi N \rightarrow \pi N$ channels:

$$
\begin{aligned}
\pi^{-} p \rightarrow \pi^{-} p: \delta V= & {\left[\left(\frac{\sqrt{2}}{3} a_{11}+\frac{\sqrt{2}}{6} a_{31}\right)^{2}+\left(\frac{1}{3} a_{11}-\frac{1}{3} a_{31}\right)^{2}\right] \gamma } \\
\pi^{-} p \rightarrow \pi^{0} n: \delta V= & {\left[\left(\frac{\sqrt{2}}{3} a_{11}+\frac{\sqrt{2}}{6} a_{31}\right)\left(-\frac{1}{3} a_{11}+\frac{1}{3} a_{31}\right)\right.} \\
& \left.+\left(\frac{1}{3} a_{11}-\frac{1}{3} a_{31}\right)\left(-\frac{\sqrt{2}}{6} a_{11}-\frac{\sqrt{2}}{3} a_{31}\right)\right] \gamma
\end{aligned}
$$

$$
\begin{align*}
& \pi^{0} n \rightarrow \pi^{0} n: \delta V=\left[\left(\frac{\sqrt{2}}{6} a_{11}+\frac{\sqrt{2}}{3} a_{31}\right)^{2}+\left(\frac{1}{3} a_{11}-\frac{1}{3} a_{31}\right)^{2}\right] \gamma \\
& \pi^{-} n \rightarrow \pi^{-} n: \delta V=\left(-\sqrt{\frac{1}{2}} a_{31}\right)^{2} \gamma \tag{2.11}
\end{align*}
$$

where all possible $\pi \pi N$ intermediate states in the loop are considered. The $\delta V$ of Eq. (2.11) are then added to the kernel $V_{i j}$ together with the isoscalar piece. In section 2.4 the constraints of $\gamma$ are discussed: The contribution from the $\pi \pi N$ channels should not exceed a small percentage of the corresponding $V_{i j}$ at $\pi N$ threshold.

## Further refinements of the coupled channel approach

Following the outline of Ref. [45] we take into account the Vector Meson Dominance (VMD) hypothesis and let the $\rho$ meson mediate the meson baryon interaction in the $t$ channel. This is justified by the identical coupling structure of the $\rho N N$ coupling within VMD and the kernel $V$ from Eq. (2.7), thus revealing the lowest order chiral Lagrangian as an effective manifestation of VMD. The $\rho$ meson exchange is incorporated in the formalism via a modification of the coefficients $C_{i j}$ in (2.7) - for details see [45]. The explicit consideration of the $\rho$ exchange helps to obtain a better energy dependence, reducing the strength of the amplitudes as the energy increases.

One of the conclusions in [86] was that the uncertainties of the $\left(b_{0}, b_{1}\right)$ values should be bigger than quoted in the paper due to the neglect of isospin violation in the analysis. In the present work we introduce a certain amount of isospin violation by working in coupled channels keeping the exact masses of the particles. Although this is not the only origin of isospin violation [87, 89-92] it gives us an idea of the size of uncertainties from this source. Since there are threshold effects in the amplitudes, and the thresholds are different in different $\pi N$ channels, this leads to non negligible isospin breaking effects as was shown in the case of $\bar{K} N$ interaction in [22].

Thus, the parameters for the fit of the $s$-wave amplitude in $\pi N$ scattering are the subtraction constant $\alpha$ from the $\pi N$ loop, two parameters from the isoscalar terms of the $\mathcal{L}_{\pi N}^{(2)}$ chiral Lagrangian, and $\gamma$ from the $\pi N \rightarrow \pi \pi N$
loop. For the parametrization of the $\pi N$ potential up to higher energies, a damping factor, introducing another parameter for the quadratic isoscalar term, is studied. In order to account for isospin breaking from other sources than mass splitting, different $\alpha_{i}$ for the three $\pi N$ channels will be investigated.

### 2.3 Pion deuteron scattering

The traditional approach to $\pi^{-} d$ scattering is the use of Faddeev equations [77-79], although the fast convergence of the multiple scattering series makes the use of the first few terms accurate enough. On top of this there are other contributions coming from pion absorption, and the dispersion contribution tied to it, crossed terms and the $\Delta(1232)$ resonance, plus extra corrections which are discussed in detail in Ref. [75].

### 2.3.1 Faddeev approach

We follow here the fixed center approximation (FCA) to the Faddeev equations which was found to be very accurate in the study of $K^{-} d$ scattering [94] by comparing it to a full Faddeev calculation [95,96]. See also the recent discussion endorsing the validity of the static approximation in Ref. [97]. The FCA accounts for the multiple scattering of the pions with the nucleons assuming these to be distributed in space according to their wave function in the deuteron. The Faddeev equations in the FCA are given in terms of the Faddeev partitions

$$
\begin{equation*}
T_{\pi^{-d}}=T_{p}+T_{n} \tag{2.12}
\end{equation*}
$$

where $T_{p}$ and $T_{n}$ describe the interaction of the $\pi^{-}$with the deuteron starting with a collision on a proton and a neutron respectively. The partitions at threshold satisfy

$$
\begin{align*}
T_{p} & =t_{p}+t_{p} G T_{n}+t^{x} G T^{x} \\
T_{n} & =t_{n}+t_{n} G T_{p} \\
T^{x} & =t^{x}+t_{n}^{0} G T^{x}+t^{x} G T_{n} \tag{2.13}
\end{align*}
$$

Here, $G$ is the pion propagator and $t_{p}, t_{n}, t_{n}^{0}, t^{x}$ the elementary $s$-wave scattering $T$-matrices of $\pi^{-}$on proton and neutron, $\pi^{0}$ on the neutron, and the charge exchange $\pi^{-} p \leftrightarrow \pi^{0} n$, in this order. In Fig. 2.1 the different partitions from Eq. (2.13) are displayed. It is illuminative to expand the Faddeev equations in powers of elementary scattering events $t$ which shows that indeed all possible multiple scattering diagrams of all lengths are included in the compact notation of Eq. (2.13).

While the full Faddeev approach involves integrations over the pion momentum, the FCA factorizes the pion propagator to $G \sim 1 / r$ and Eqs. (2.13) become a coupled set of algebraic equations. In Ref. [94] it is shown in detail how the pion propagator factorizes. For this, one can consider the triple rescattering of the pion with the first scattering event in coordinate space at $\mathbf{x}$, the second at $\mathbf{x}^{\prime}$ and the third at $\mathbf{x}^{\prime \prime}$. Replacing in the intermediate baryon propagator $E(\mathbf{p}) \rightarrow M$, i.e., taking the heavy baryon limit, the integration over $\mathbf{p}$ leads to $\delta\left(\mathbf{x}-\mathbf{x}^{\prime \prime}\right)$ which brings together the $\mathbf{x}$ and $\mathbf{x}^{\prime \prime}$ coordinates. Additionally, for the low energy pions which are considered here, one takes the static limit for the pion energy, $\omega_{\pi} \rightarrow m_{\pi}$ which means for the integration over the pion propagator

$$
\begin{equation*}
\int \frac{d \mathbf{q}}{(2 \pi)^{3}} \frac{e^{-i \mathbf{q} \cdot \mathbf{r}}}{\omega_{\pi}^{2}-m_{\pi}^{2}-\mathbf{q}^{2}+1 \epsilon} \rightarrow-\frac{1}{4 \pi r} \tag{2.14}
\end{equation*}
$$

and indeed $G \sim 1 / r$ factorizes.
The Faddeev equations are at the level of operators. At any place where charge is transferred from one nucleon to the other, the sign has to be changed due to the exchanged final state: The deuteron has isospin 0 which means for the wave function $\Psi \sim \frac{1}{\sqrt{2}}(|p n\rangle-|n p\rangle)$. The sign change applies for at two places in Eq. (2.13), for the term $t^{x} G T^{x}$ and for the term $t_{n}^{0} G T^{x}$. Following Ref. [94] we find for the $\pi^{-} d$ amplitude density

$$
\begin{align*}
& \hat{A}_{\pi^{-} d}(r)=\frac{\tilde{a}_{p}+\tilde{a}_{n}+\left(2 \tilde{a}_{p} \tilde{a}_{n}-b_{x}^{2}\right) / r-2 b_{x}^{2} \tilde{a}_{n} / r^{2}}{1-\tilde{a}_{p} \tilde{a}_{n} / r^{2}+b_{x}^{2} \tilde{a}_{n} / r^{3}} \\
& b_{x}=\tilde{a}_{x} / \sqrt{1+\tilde{a}_{n}^{0} / r} \tag{2.15}
\end{align*}
$$

with $\tilde{a}_{i}$ being related to the scattering lengths $a_{i}$ and the elementary $t_{i}$ by

$$
\begin{equation*}
\tilde{a}_{i}=\left(1+\frac{m_{\pi}}{m_{N}}\right) a_{i}=-\frac{1}{4 \pi} t_{i} . \tag{2.16}
\end{equation*}
$$



Figure 2.1: Graphical illustration of the Faddeev partitions in pion-deuteron scattering.

Table 2.1: Contributions to the multiple scattering series for Re $a_{\pi^{-} d}$.

|  | from $[73]\left[m_{\pi}^{-1}\right]$ | Phenom. Ham. $[99]\left[m_{\pi}^{-1}\right]$ |
| :--- | :--- | :--- |
| $\left(b_{0}, b_{1}\right)$ | $(-0.0001,-0.0885)$ | $(-0.0131,-0.0924)$ |
| Impulse Approximation | $-2.14 \cdot 10^{-4}$ | -0.02793 |
| Double Scattering | -0.02527 | -0.02725 |
| Triple Scattering | 0.002697 | 0.003489 |
| $4-$ and higher scattering | $1.06 \cdot 10^{-4}$ | $5.4 \cdot 10^{-5}$ |
| Solution Faddeev | -0.02268 | -0.05163 |

The masses in Eq. (2.16) have to be understood as the physical ones in each channel.

The final $\pi^{-} d$ scattering amplitude is then obtained by folding the amplitude density with the deuteron wave function as

$$
\begin{equation*}
a_{\pi^{-} d}=\frac{M_{d}}{m_{\pi^{-}}+M_{d}} \int d \mathbf{r}\left|\varphi_{d}(\mathbf{r})\right|^{2} \hat{A}_{\pi^{-} d}(r) \tag{2.17}
\end{equation*}
$$

This is the real part of $a_{\pi^{-} d}$ that has to be modified by the corrections of the following sections. The latter will also provide the correct imaginary part of the pion-deuteron scattering length.

If we keep up to the $(1 / r)^{2}$ terms in Eq. (2.15) and assume isospin symmetry, the resulting formula coincides with the triple scattering result of [86] up to $\mathcal{O}\left(p^{4}\right)$ in their modified power counting. In order to show the convergence of the multiple scattering series in the the $\pi^{-} d$ collision we show in Table 2.1 the different contributions for two cases: First, for the experimental values from Ref. [73] with $\left(b_{0}, b_{1}\right)=(-0.0001,-0.0885) m_{\pi^{-}}^{-1}$, and second for $\left(b_{0}, b_{1}\right)=(-0.0131,-0.0924) m_{\pi^{-}}^{-1}$ from the phenomenological Lagrangian of Ref. [99] in Eq. (2.21). We can see that in both cases the double scattering is very important and in the case of [73] where $b_{0}$ is quite small, the double scattering is the leading contribution.

## Faddeev equations including $\pi N$ scattering via $\Delta$ excitation

The $s$-wave interactions from the elementary building blocks of the Faddeev equations (2.13) are not the only relevant processes in $\pi d$ scattering. At low energies the $\Delta(1232)$ plays an important role in the $p$-wave channel. However, in the multiple rescattering series that has been discussed before, the $\Delta$ excitation can only occur in intermediate scattering events as the global process is in $s$-wave.

In this section the Faddeev approach is presented which takes into account the $\Delta$ excitation and ensures at the same time that external pions only couple in $s$-wave to the nucleons. In Sec. 2.3 .2 the $\Delta$ is taken into account in a Feynman approach which means an alternative to the formulation given in this section. The $\pi^{-} d \rightarrow \pi^{-} d$ transition in $s$-wave, $T_{D}$, is now given by

$$
\begin{align*}
T_{D} & =T_{p}+T_{n} \\
T_{p} & =t_{p}+t_{p} G T_{n i}+t^{x} G T_{i}^{x} \\
T_{n} & =t_{n}+t_{n} G T_{p i} \\
T_{p i} & =t_{p}+\left(t_{p}+t_{p \Delta}\right) G T_{n i}+\left(t^{x}+t_{\Delta}^{x}\right) G T_{i}^{x} \\
T_{n i} & =t_{n}+\left(t_{n}+t_{n \Delta}\right) G T_{p i} \\
T_{i}^{x} & =t^{x}+\left(t^{x}+t_{\Delta}^{x}\right) G T_{n i}+\left(t_{n}^{0}+t_{n \Delta}^{0}\right) G T_{i}^{x} \tag{2.18}
\end{align*}
$$

where $t_{p}, t_{n}, t_{n}^{0}$, and $t^{x}$ are defined as in Sec. 2.3.1. The additional index $\Delta$ indicates tree level processes of $\pi N$ scattering via direct and crossed $\Delta$ excitation. The charge states of these terms are labeled in the same way as the normal $t$ 's from the $s$-wave interaction. The last three lines of Eq. (2.18) resemble Eq. (2.13) and describe internal rescattering of the pion off the two nucleons. The additional structures in the first two lines of Eq. (2.18) ensure that the external pions do not couple to the $\Delta$ directly. Eq. (2.18) is again at the level of operators and one has to change sign in some Faddeev partitions taking into account the antisymmetric wave function of $p n$ in isospin zero. In a similar way as for Eq. (2.13), the sign changes apply to the terms $t^{x} G T_{i}^{x}$, $\left(t^{x}+t_{\Delta}^{x}\right) G T_{i}^{x}$, and ( $\left.t_{n}^{0}+t_{n \Delta}^{0}\right) G T_{i}^{x}$ in Eq. (2.18). The solution is similar to

Eq. (2.15),

$$
\begin{align*}
& \hat{A}_{p}^{\Delta}(r)=\tilde{a}_{p}+\frac{\left(\tilde{a}_{p} \tilde{a}_{n} / r+\tilde{a}_{p}^{2}\left(\tilde{a}_{n}+\tilde{a}_{n \Delta}\right) / r^{2}+f_{0} \tilde{a}_{n} \tilde{a}^{x} \tilde{a}_{\Delta}^{x} / r^{2}\right.}{\left.+f_{0}\left(\tilde{a}_{x}\right)^{2} / r\left[1+\tilde{a}_{n} / r-\left(\tilde{a}_{n}+\tilde{a}_{n \Delta}\right)\left(\tilde{a}_{p}+\tilde{a}_{p \Delta}\right) / r^{2}\right]\right)} \\
& 1-\left(\tilde{a}_{n}+\tilde{a}_{n \Delta}\right)\left(\tilde{a}_{p}+\tilde{a}_{p \Delta}-f_{0}\left(\tilde{a}^{x}+\tilde{a}_{\Delta}^{x}\right)^{2} / r\right) / r^{2} \\
& \hat{A}_{n}^{\Delta}(r)=\tilde{a}_{n}+\frac{\tilde{a}_{n}}{r} \frac{\tilde{a}_{p}+\tilde{a}_{n}\left(\tilde{a}_{p}+\tilde{a}_{p \Delta}\right) / r-f_{0}\left(\tilde{a}^{x}+\tilde{a}_{\Delta}^{x}\right)\left[\tilde{a}^{x}+t_{n}\left(t^{x}+t_{\Delta}^{x}\right) / r\right] / r}{1-\left(\tilde{a}_{n}+\tilde{a}_{n \Delta}\right)\left(\tilde{a}_{p}+\tilde{a}_{p \Delta}-f_{0}\left(\tilde{a}^{x}+\tilde{a}_{\Delta}^{x}\right)^{2} / r\right) / r^{2}} \\
& f_{0}=\frac{1}{1-\left(\tilde{a}_{n}^{0}+\tilde{a}_{n \Delta}^{0}\right) / r} \tag{2.19}
\end{align*}
$$

in the notation of Eq. (2.16). The amplitude density $\hat{A}_{\pi^{-} d}(r)$ from Eq. (2.17) is now given by

$$
\begin{equation*}
\hat{A}_{\pi^{-} d}(r)=\hat{A}_{p}^{\Delta}(r)+\hat{A}_{n}^{\Delta}(r) \tag{2.20}
\end{equation*}
$$

Omitting all processes with $\Delta$, indeed Eq. (2.15) turns out.
In the following, we will make no further use of Eq. (2.19) because the influence of the $\Delta$ will be calculated in Feynman diagrammatic way at one loop with exactly one $\Delta$ insertion. By doing so, one goes beyond the fixed center approximation at the cost of omitting diagrams with multiple appearance of the $\Delta$ as Eq. (2.19) provides. However, one should take into account that multiple interactions with $\Delta$ occur at higher order only and that the rescattering series converges rapidly. Therefore it more advantageous to consider one $\Delta$ excitation in a Feynman approach, which goes beyond the FCA, than considering multiple $\Delta$ excitations using Eq. (2.19) in the fixed center approximation.

Thus, in the following we will use the reduced form of the Faddeev equations from Eq. (2.13) for the $s$-wave rescattering and treat the $p$-wave separately in the next section.

### 2.3.2 Absorption and dispersion terms

Pion absorption in deuterium has been studied in [98] using Feynman diagrammatic techniques. The absorption contribution reflects into the imaginary part of the (elastic) $\pi^{-} d$ scattering length. Its diagrammatic evaluation leads at the same time to a dispersive real contribution to the $\pi^{-} d$ scattering


Figure 2.2: Absorption plus dispersion terms in $\pi^{-} d$ scattering.
length. The evaluation of this latter contribution has been done using Faddeev approaches [77,78,80]. We shall evaluate it here including extra terms from the $\Delta$ excitation, going beyond the non-relativistic treatment of the pions in $[77,78]$, and testing various approximations for the dispersive part.

In order to evaluate the absorption and dispersion terms, a Feynman diagrammatic approach is used which offers flexibility to account for different mechanisms. We shall evaluate the contribution of the diagrams of Fig. 2.2, where Type B contributes only to the real part, including permutations of the scattering vertices on different nucleons and different time orderings as shown in Figs. 2.3 and 2.4.

On the first hand we consider the diagrams of type A and find the possibilities shown in Fig. 2.3. For the purpose of evaluating the absorption and dispersion corrections we shall use the effective Hamiltonian [99, 100]

$$
\begin{equation*}
H_{I}=4 \pi\left[\frac{\lambda_{1}}{m_{\pi}} \bar{\Psi} \vec{\phi} \vec{\phi} \Psi+\frac{\lambda_{2}}{m_{\pi}^{2}} \bar{\Psi} \vec{\tau}\left(\vec{\phi} \times \partial^{0} \vec{\phi}\right) \Psi\right] \tag{2.21}
\end{equation*}
$$

with $\lambda_{1}=0.0075, \lambda_{2}=0.053$, which shows the dominance of the isovector part with $\lambda_{2}$. For the $\pi N N$ vertex the usual Yukawa $\left(f_{\pi N N} / m_{\pi}\right) \sigma \cdot \mathbf{q} \tau^{\lambda}$ vertex is taken. The value of $\lambda_{2}$ corresponds very closely to the final isovector term that we find, while the value of $\lambda_{1}$ is about twice as large. Yet, using the new values that come from our analysis in a first step of a selfconsistent


Figure 2.3: Charge States in Absorption.
procedure only leads to changes in the final results that are much smaller than the uncertainties found.

The normalization of our $T$ amplitude is such that the scattering matrix $S$ is given by

$$
\begin{equation*}
S=1-i \frac{1}{V^{2}} \frac{1}{\sqrt{2 \omega}} \frac{1}{\sqrt{2 \omega^{\prime}}} \sqrt{\frac{M_{d}}{E_{d}}} \sqrt{\frac{M_{d}}{E_{d}^{\prime}}} T(2 \pi)^{4} \delta^{4}\left(p_{\pi^{-}}+p_{d}-p_{\pi^{-}}^{\prime}-p_{d}^{\prime}\right) \tag{2.22}
\end{equation*}
$$

From the diagram A of Fig. 2.2 we obtain in the $\pi^{-} d$ center of mass frame after performing the $q^{0}$ and $q^{\prime 0}$ integrations

$$
\begin{align*}
T & =i \int \frac{d^{4} l}{(2 \pi)^{4}} \int \frac{d^{3} \mathbf{q}}{(2 \pi)^{3}} \int \frac{d^{3} \mathbf{q}^{\prime}}{(2 \pi)^{3}} F_{d}(\mathbf{q}+\mathbf{l}) F_{d}\left(\mathbf{q}^{\prime}+\mathbf{l}\right) \\
& \times \frac{1}{q^{2}-m_{\pi}^{2}+i \epsilon} \frac{1}{q^{\prime 2}-m_{\pi}^{2}+i \epsilon} \frac{1}{l^{0}-\epsilon(\mathbf{l})+i \epsilon} \frac{1}{l^{\prime 0}-\epsilon\left(\mathbf{l}^{\prime}\right)+i \epsilon} \\
& \times \Sigma t_{1} t_{2} t_{1^{\prime}} t_{2^{\prime}}\left(\mathbf{q} \cdot \mathbf{q}^{\prime}\right) \tag{2.23}
\end{align*}
$$

with $q=\left(m_{\pi}-l^{0}, \mathbf{q}\right), q^{\prime}=\left(m_{\pi}-l^{0}, \mathbf{q}^{\prime}\right), \mathbf{l}^{\prime}=-\mathbf{l}$, and $\epsilon(\mathbf{l})$ refers to the nucleon kinetic energy. In Eq. (2.23), $F_{d}$ is the deuteron wave function in momentum space including $s$ - and $d$-wave (see Appendix), and the amplitude from the sum of diagrams of Fig. 2.3 is given by

$$
\begin{equation*}
\Sigma t_{1} t_{2} t_{1^{\prime}} t_{2^{\prime}}=2(4 \pi)^{2} \frac{1}{m_{\pi}^{2}}\left(2 \lambda_{1}+3 \lambda_{2}\right)^{2}\left(\frac{f_{\pi N N}}{m_{\pi}}\right)^{2} \simeq 40.0 \mathrm{fm}^{4} \tag{2.24}
\end{equation*}
$$



Figure 2.4: Additional diagrams in absorption.
where we have made the usual approximation that $q^{0}$ and $q^{\prime 0}$ in the $\pi N \rightarrow \pi N$ amplitude are taken as $m_{\pi} / 2$ which is exact for $\operatorname{Im} T$.

The $\mathbf{q} \cdot \mathbf{q}^{\prime}$ term in Eq. (2.23) comes from the $\pi N N$ p-wave vertices $\vec{\sigma} \mathbf{q}^{\prime} \vec{\sigma} \mathbf{q}=\mathbf{q}^{\prime} \cdot \mathbf{q}+i\left(\mathbf{q}^{\prime} \times \mathbf{q}\right) \vec{\sigma}$ after neglecting the crossed product term which does not contribute when using the $s$-wave part of the deuteron wave function.

A different topological structure for the absorption terms is possible and given by the diagrams shown in Fig 2.4. The evaluation of these diagrams involves now the spin of both the nucleons 1 and 2 and one obtains the combination

$$
\begin{equation*}
\sigma_{1 i} \sigma_{2 j} q_{i} q_{j}^{\prime} \tag{2.25}
\end{equation*}
$$

which upon integration over $\mathbf{q}, \mathbf{q}^{\prime}$ leads to a structure of the type

$$
\begin{equation*}
\sigma_{1 i} \sigma_{2 j} l_{i} l_{j} \tag{2.26}
\end{equation*}
$$

for the $s$-wave part of the wave function. The extra $l$ integration, involving $l_{i} l_{j}$ and terms with even powers of $\mathbf{l}$ allows one to write

$$
\begin{equation*}
\sigma_{1 i} \sigma_{2 j} l_{i} l_{j} \longrightarrow \frac{1}{3} \sigma_{1 i} \sigma_{2 j} l^{2} \delta_{i j}=\frac{1}{3} \vec{\sigma}_{1} \vec{\sigma}_{2} \mathbf{l}^{2} \equiv \frac{1}{3} \mathbf{l}^{2} \tag{2.27}
\end{equation*}
$$

where in the last step we have used that $\vec{\sigma}_{1} \vec{\sigma}_{2}=1$ for the deuteron. The final result leads to $1 / 3$ of the former contribution from the diagrams of Fig. 2.3 for the imaginary part.

The integration over the energy variable $l^{0}$ in Eq. (2.23) has been performed in three different ways. While for all three calculations the imaginary part stays the same as expected, the real part varies significantly, as we will see in the following. The result for the dispersive part for the diagrams of Figs. 2.3 and 2.4 depends much on the treatment of the pion poles and the pion propagator. The choice of the wave function will have only moderate influence on the results.

In a first approximation (App1), only the nucleon pole is picked up in the $l^{0}$-integration of Eq. (2.23). Furthermore, the energy components $q^{0}$ and $q^{10}$ of the pion momenta in the propagators are replaced by the on-shell value of $l^{0}$ which is $m_{\pi} / 2$. This is exact for the imaginary part of the elastic $\pi^{-} d$ scattering length. The imaginary part is given by cutting the two internal nucleon lines in the diagrams of Fig. 2.3 and Fig. 2.4, and then putting the two nucleons on-shell. The pion mass in this picture is shared between the two nucleons that obtain a kinetic energy of $m_{\pi} / 2$ each after the absorption of the virtual pion on the second nucleon.

In a second approach (App2), the pion poles of negative energy in the lower $l^{0}$ half plane are still neglected, but for $q^{0}$ and $q^{10}$ we substitute now the residue of the nucleon pole

$$
\begin{equation*}
\frac{1}{\left(\frac{m_{\pi}}{2}\right)^{2}-\mathbf{q}^{2}-m_{\pi}^{2}} \longmapsto \frac{1}{\left[m_{\pi}-\epsilon(\mathbf{l})\right]^{2}-\mathbf{q}^{2}-m_{\pi}^{2}} \tag{2.28}
\end{equation*}
$$

This leads to new poles in the integration of Eq. (2.23) which correspond to cuts that affect one pion and one nucleon line of the loops in Figs. 2.3 and 2.4. From kinematical reasons the particles cannot go on-shell for these cuts. Indeed, if also the pion poles of negative energy are taken into account (App3), these poles cancel.

Approach 3 (App3) makes no simplifications in the $l^{0}$-integration any more, except the substitution of $q^{0}=q^{\prime 0}=m_{\pi} / 2$ in the elementary scattering length as in Eq. (2.24). The 9-dimensional integral of the amplitude (2.23) for the diagrams in Fig. 2.3 in the formulation of approach 3 (App3) is:

Table 2.2: Real and imaginary contributions from absorption to $a_{\pi^{-} d}$ for three different approaches. All values in $10^{-4} \cdot m_{\pi^{-}}^{-1}$.

|  | (App1) | (App2) | $(\mathrm{App} 3)$ |
| :--- | :--- | :--- | :--- |
| $\operatorname{Im} a_{\pi^{-} d}, s$-wave | $57.4 \pm 5.7$ | idem | idem |
| $\operatorname{Im} a_{\pi^{-} d}, d$-wave | $2.21 \pm 0.33$ | idem | idem |
| $\operatorname{Im} a_{\pi^{-} d} s+d$-wave | $59.6 \pm 5.3$ | idem | idem |
| $\operatorname{Im} a_{\pi^{-} d}$ experimental | $63 \pm 7$ | idem | idem |
| $\Delta \operatorname{Re} a_{\pi^{-} d}, s$-wave | $19.3 \pm 8.2$ | $13.6 \pm 8.9$ | $2.4 \pm 4.3$ |

$$
\begin{align*}
& T=\frac{1}{(2 \pi)^{9}} \int d^{3} \mathbf{1} d^{3} \mathbf{q}^{\prime} d^{3} \mathbf{q} F_{d}(\mathbf{q}+\mathbf{l}) F_{d}\left(\mathbf{q}^{\prime}+\mathbf{l}\right) \mathbf{q} \mathbf{q}^{\prime} A \Sigma t_{1} t_{2} t_{1^{\prime}} t_{2^{\prime}}, \\
& A=-\frac{1}{2 \epsilon(l)-m_{\pi}-i \epsilon} \\
& \times \frac{\left(2 \epsilon(l)-m_{\pi}\right)(\epsilon(l)+\omega)\left(\epsilon(l)-m_{\pi}+\omega\right)+\left(2 \epsilon(l)-m_{\pi}+2 \omega\right)\left(\omega^{\prime 2}+\omega^{\prime}\left(2 \epsilon(l)-m_{\pi}+\omega\right)\right)}{2 \omega \omega^{\prime}\left(\omega+\omega^{\prime}\right)(\epsilon(l)+\omega)\left(\epsilon(l)+\omega^{\prime}\right)\left(\epsilon(l)-m_{\pi}+\omega\right)\left(\epsilon(l)-m_{\pi}+\omega^{\prime}\right)} . \tag{2.29}
\end{align*}
$$

It can be factorized to integrals of lower dimension by writing the sum of pion energies in the denominator of (2.29) as

$$
\begin{equation*}
\frac{1}{\omega(q)+\omega\left(q^{\prime}\right)}=\int_{0}^{\infty} d x e^{-\omega(q) x} e^{-\omega\left(q^{\prime}\right) x} \tag{2.30}
\end{equation*}
$$

thus simplifying the numerical evaluation. The amplitude in Eq. (2.29), divided by 3 , provides the imaginary part of the diagrams in Fig. 2.4, whereas the real part of the diagrams in Fig. 2.4 has a different analytical structure. The diagrams of Fig. 2.4 contribute with $36 \%$ to the real part with respect to the diagrams of Fig. 2.3.

Table 2.2 shows the result of all three approaches for the sum of the diagrams from Fig. 2.3 and Fig. 2.4. We have used two refined wave functions, the CD-Bonn potential in the recent version from Ref. [101], and the Paris potential from Ref. [102]. We take the average of the results obtained with
either wave function. The difference of the results gives the error in Table 2.2. The statistical error from Monte-Carlo integrations has been kept below $0.1 \cdot 10^{-4} \cdot m_{\pi^{-}}^{-1}$. The dispersive contribution from the $d$-wave has been only calculated for the CD-Bonn potential from Ref. [101], for the amplitude of approach 3, Eq. (2.29). The numerical value is

$$
\Delta \operatorname{Re} a_{\pi^{-} d}, d-\text { wave, absorption }=0.18 \cdot 10^{-4} \cdot m_{\pi^{-}}^{-1}
$$

We also show the influence of the $\mathbf{q}^{\prime} \times \mathbf{q}$ term, that stems from the $\vec{\sigma} \mathbf{q}^{\prime} \vec{\sigma} \mathbf{q}$ structure of the absorption diagrams of Fig. 2.3. It had been omitted in Eq. (2.23), since it contributes only in the $d$-wave $\rightarrow d$-wave transition. The contribution from this source has been calculated for the amplitude of approach 3, in Eq. (2.29), for the CD-Bonn potential. We obtain:

$$
\Delta a_{\pi^{-} d, \mathbf{q}^{\prime} \times \mathbf{q}}=(0.13-i 0.38) \cdot 10^{-4} \cdot m_{\pi^{-}}^{-1} .
$$

We have also tested the relevant contributions of the absorption process with a Hulthen wave function in two different parameterizations taken from [104]. The results for both parameterizations would lead to large errors of the order of $40 \%$ for the value of $\operatorname{Im} a_{\pi^{-} d}$ for the $s$-wave in Table 2.2, and even larger ones for the imaginary part from the $d$-wave. This is, because the two $p$-wave vertices make the absorption in the diagrams of Figs. 2.3 and 2.4 sensitive to the derivative of the used wave function, and the Hulthen wave function is known to be less accurate, as has also been pointed out in Ref. [75]. Therefore we do not use this simplified wave function in this study.

Whereas the imaginary part in Table 2.2 remains the same, the dispersive contribution from the $s$-wave decreases when going from approach 1 to 3 . The fact that in (App3) it even changes sign for the Paris potential compared to the CD-Bonn potential is due to cancellations between terms, which by themselves are of larger magnitude.

In all calculations a monopole form factor with cut-off $\Lambda$ has been applied to the $\pi N N$ vertices of the absorption diagrams of Fig. 2.3 and Fig. 2.4. Since $\Lambda$ is the only free parameter involved in the calculation, we plot the dependence on $\Lambda$ of the imaginary part, $s-$ and $d$-wave, and the real part, $s$-wave in Fig. 2.5. The values correspond to approach 3, Eq. (2.29).


Figure 2.5: Real and imaginary contribution to $a_{\pi^{-} d}$ from absorption as a function of $\Lambda$ from the monopole form factor.

The real part from the $d$-wave is not plotted separately, since it is even one order of magnitude smaller than the imaginary part from the $d$-wave. Also plotted is the experimental value of $\operatorname{Im} a_{\pi^{-} d}=0.0063 \pm 0.0007 m_{\pi^{-}}^{-1}$ taken from Ref. [74]. The imaginary part from the $d$-state has been amplified ten times in the figure.

The results for the imaginary part depend very moderately on $\Lambda$. In Table 2.2, the values correspond to $\Lambda=1.72 \mathrm{GeV}$ that has been used in the construction of the CD-Bonn potential [101]. ${ }^{1}$ We do not observe

[^0]an amplification of the $d$-wave in absorption, relative to its weight in the deuteron wave function, as claimed in Ref. [78]. From the $\mathbf{q} \cdot \mathbf{q}^{\prime}$-structure of the absorption amplitude (2.29) one would expect an amplification of the $d$-wave, which has more weight at higher momenta than the $s$-wave (despite its small contribution to the norm of the deuteron wave function of $4-6 \%$ ). However, the correct combination of the angular momentum $l=2$ of the $d$-wave, together with the spin of the nucleons in order to give a total spin of 1 , leads to a very effective suppression of this enhancement.

Also, the effect of rescattering in absorption has been investigated. In order to avoid double counting we consider all rescattering diagrams that have exactly one absorption insertion of the form of the diagrams in Figs. 2.3 and 2.4. In practice, this means a replacement of the two $s$-wave $\pi N$ vertices in the diagrams of Figs. 2.3 and 2.4 by a Faddeev-like rescattering series similar to Eq. (2.13). By doing so, there are no pion exchanges between the nucleons, which are unconnected to the external pions. Thus, effects, that are already contained in the deuteron wave function, are not double counted. The explicit evaluation of this class of diagrams results in negligible changes of the values in Table 2.2 of the order of $1 \%$ or less.

No interference between $s$ and $d$-wave is observed for absorption and dispersion. In the next section and in the Appendix this issue is discussed further.

### 2.3.3 Further corrections to the real part of $a_{\pi^{-} d}$

The diagram of Fig. 2.2 where the nucleon pole is substituted by the $\Delta$ pole (Type B) is evaluated in a similar fashion as Type A. The sum of all possible charge configurations provides now

$$
\begin{equation*}
\Sigma t_{1} t_{2} t_{1^{\prime}} t_{2^{\prime}}=\frac{32 f_{\pi N \Delta}^{2} \pi^{2}\left(4 \lambda_{1}-3 \lambda_{2}\right)^{2}}{9 m_{\pi}^{4}} \simeq 10.8 \mathrm{fm}^{4} \tag{2.31}
\end{equation*}
$$

where $f_{\pi N \Delta}=2.01 f_{\pi N N}$ is the $\pi N \Delta p$-wave coupling. There is no imaginary part from this diagram as the $\Delta$ cannot go on shell. The numerical value
the direct and crossed diagram with a $\Delta$ intermediate state (Fig. 2.6) are slightly changed, but this is only a tiny fraction compared to the theoretical uncertainties from other sources, see Eq. (2.42).

Table 2.3: Values of further contributions to the real part of $a_{\pi^{-} d}$.

|  | Diagram | Value in $10^{-4} \cdot m_{\pi^{-}}^{-1}$ |
| :--- | :--- | :--- |
| $\Delta$ excitation | Fig. 2.2, Type B | 6.4 |
| Crossed pions | Fig. 2.6, $1^{\text {st }}$ | $-1.3 \pm 0.1$ (stat.) |
| Crossed $\Delta$ excitation | Fig. 2.6, $4^{\text {th }}$ | $9.5 \pm 1.1$ (stat.) |
| Wave function correction (WFC), $s$-wave | Fig. 2.6, $5^{\text {th }}$ | $-16.2 \pm 0.1$ (stat.) |
| $\mathrm{WFC}, s-d$ interference, $\mu=0, \mu^{\prime}=0$ | Fig. 2.6, $5^{\text {th }}$ | $14.4 \pm 0.1$ (stat.) |
| $\mathrm{WFC}, s-d$ interference, $\mu=0, \mu^{\prime}= \pm 1$ | Fig. 2.6, $5^{\text {th }}$ | $21.9 \pm 0.1$ (stat.) |



Figure 2.6: Crossed Diagrams.
can be found as ' $\Delta$ excitation' in Table 2.3. It is relatively small since the effect of the strong $\pi N \Delta$ coupling is suppressed partly by a cancellation of the isovector part $\sim \lambda_{2}$ from different charge states for the diagram.

Another source of contribution for the real part of the $\pi^{-} d$ scattering length is given by the crossed pion diagram displayed in Fig. 2.6, first diagram. The $T$ matrix for this process is given by

$$
\begin{align*}
T & =\int \frac{d^{3} \mathbf{l}}{(2 \pi)^{3}} \frac{d^{3} \mathbf{q}}{(2 \pi)^{3}} \frac{d^{3} \mathbf{q}^{\prime}}{(2 \pi)^{3}} F_{d}(\mathbf{q}+\mathbf{l}) F_{d}\left(\mathbf{q}^{\prime}+\mathbf{l}\right) \Sigma\left(t_{1} t_{2} t_{1}^{\prime} t_{2}^{\prime}\right)\left(\vec{\sigma} \cdot \mathbf{q} \vec{\sigma} \cdot \mathbf{q}^{\prime}\right) \\
& \times \frac{(\epsilon+\omega)\left(\epsilon^{\prime}+\omega\right)+\left(\epsilon+\epsilon^{\prime}+\omega\right) \omega^{\prime}+\omega^{\prime 2}-m_{\pi}\left(\epsilon^{\prime}+\omega+\omega^{\prime}\right)}{2 \omega \omega^{\prime}\left(\epsilon^{\prime}+\omega\right)\left(\epsilon^{\prime}+\omega^{\prime}\right)\left(\omega+\omega^{\prime}\right)\left(m_{\pi}-\epsilon-\omega\right)\left(m_{\pi}-\epsilon-\omega^{\prime}\right)} \tag{2.32}
\end{align*}
$$

where the spin-isospin factor

$$
\begin{equation*}
\Sigma\left(t_{1} t_{2} t_{1}^{\prime} t_{2}^{\prime}\right)=8(4 \pi)^{2} \frac{1}{m_{\pi}^{2}}\left(\lambda_{1}^{2}-3 \lambda_{1} \lambda_{2}\right)\left(\frac{f_{\pi N N}}{m_{\pi}}\right)^{2} \simeq-6.0 \mathrm{fm}^{4} \tag{2.33}
\end{equation*}
$$

is proportional to the isoscalar scattering length $b_{0} \sim \lambda_{1}$, and therefore the contribution of the diagram is very small. In order to show how a nonvanishing $b_{0}$ influences the crossed diagram, we have evaluated the amplitude in Eq. (2.32), and the numerical result at the relatively large value of $\lambda_{1}=$ 0.0075 leads to the small contribution displayed in Table 2.3 as 'Crossed pions'. The second and third diagram of Fig. 2.6 force the nucleons to be far off shell and are equally negligible.

The crossed contribution with the $\Delta$ resonance in the fourth diagram of Fig. 2.6 is similar to Eq. (2.32), the only difference being the nucleon kinetic energy $\epsilon^{\prime}$ which is substituted by $M_{\Delta}-M_{N}+\mathbf{l}^{\prime 2} /\left(2 M_{\Delta}\right)$, and the spin-isospin factor changing to

$$
\begin{equation*}
\Sigma\left(t_{1} t_{2} t_{1}^{\prime} t_{2}^{\prime}\right)=(4 \pi)^{2}\left[6 \lambda_{2}^{2}+\frac{16}{3} \lambda_{1} \lambda_{2}+\frac{32}{9} \lambda_{1}^{2}\right] \frac{1}{m_{\pi}^{2}}\left(\frac{f_{\pi N \Delta}^{\star}}{m_{\pi}}\right)^{2} \simeq 56 \mathrm{fm}^{4} \tag{2.34}
\end{equation*}
$$

The diagram provides a larger contribution than the former ones, since it has a large weight in the isovector part $\lambda_{2}$. This explains why the contribution of this diagram, displayed in Table 2.3 ('Crossed $\Delta$ excitation'), is relatively large compared to the $\Delta$ excitation in Fig. 2.2, Type B, and the crossed pion diagram in Fig. 2.6, first diagram.

Another type of correction is displayed in the last, fifth diagram of Fig. 2.6. The difference to the other corrections discussed so far concerns a pion that is not connected to the external pion line. Nevertheless, we are not double counting effects of the deuteron wave function and effects of the diagram. On the contrary, the last diagram of Fig. 2.6 can be understood as a correction of the nucleon-nucleon interaction to double scattering, usually called wave function correction (WFC). The nucleon-nucleon interaction is, of course, modeled by a richer structure in terms of meson exchange as it is considered in the construction of the CD-Bonn potential of Ref. [101]. Yet, with the nucleons in the deuteron being relatively far away from each other,
one pion exchange should give the right size of this correction in $\pi^{-} d$ scattering. As one sees in Table 2.3 ('Wave function correction (WFC), $s$-wave'), the contribution of the $s$-wave of this diagram is less than $1 / 10$ of the one of double scattering, with the same sign (see Table 2.1). To conclude, the correction induced by the fifth diagram of Fig. 2.6 results in minor changes, that are of the size of $1 / 3$ to $1 / 2$ of triple scattering (see Table 2.1). However, we do not include the contribution of this diagram in the determination of the corrections of $a_{\pi^{-} d}$. This is because it represents part of the non-static effects discussed in the next section.

Besides the contributions of $s$ and $d$-wave to the various corrections discussed so far, the interference between $s$ and $d$-wave should be carefully analyzed. In Ref. [78] sizable contributions from this source were found. The spin structure of the two nucleons together with the angular momentum of the nucleons in the $d$-state prohibits any kind of interference for the diagrams of Fig. 2.3 and 2.4, as an explicit calculation shows. Some explicit formulas for the angular structure of the interference can be found in the Appendix.

The situation is different for the fifth diagram in Fig. 2.6. There, the pion acts similarly as in the one-pion-exchange, that mixes the small amount of $d$-wave to the $s$-wave of the deuteron wave function. From the $p$-wave character of the $\pi N N$ coupling we expect even an amplification of the interference of $s$ and $d$-wave. This is indeed the case, as the numerical results in Table 2.3 show. There, we distinguish between interference that leaves the third component of the angular momentum, $\mu$ and $\mu^{\prime}$ (for incoming and outgoing state), unchanged ('WFC, $s-d$ interference, $\mu=0, \mu^{\prime}=0$ '), and the interference that involves different values of $\mu$ and $\mu^{\prime}$ ('WFC, $s-d$ interference, $\mu=0$, $\mu^{\prime}= \pm 1^{\prime}$ ). The latter implies a spin flip of one or both nucleons. There is some cancellation between the $s$-wave and the $s$-wave $d$-wave interference and the net effect is similar to what has been customarily taken as non-static effects in other works, as we mention in the next section.

In addition to the diagrams considered so far we could add others of the type

which also contribute only to the real part. The approximate contribution $\lambda_{2}\left(m_{\pi}+m_{\pi} / 2\right)$ in the $\pi N \rightarrow \pi N s$-wave vertices becomes now $\lambda_{2}\left(-m_{\pi}+m_{\pi} / 2\right)$ and this gives a factor 9 reduction with respect to the other terms plus an extra reduction from the intermediate nucleon propagator and we disregard them.

### 2.3.4 Other Corrections

Besides the dispersive contribution to the real part, the various crossed terms, and the $\Delta$ contribution, all of them discussed in the last section, corrections of a different nature occur for the real part of the $\pi^{-} d$ scattering length. They have been summarized in Ref. [75], and we follow here this work in order to discuss the incorporation of these effects in the present study.

## Fermi motion / Boost Correction

The single scattering term for the $\pi N p$-wave interaction gives a contribution when the finite momentum of the nucleons in the deuteron is taken into account. A value of $61(7) \cdot 10^{-4} m_{\pi}^{-1}$ arises from the study of Ref. [75] tied to the $c_{0}$ coefficient of the isoscalar $p$-wave $\pi N$ amplitude. For the $s$-wave such contributions cancel with other binding terms according to Ref. [105]. In a different analysis in Ref. [86] a finite correction arises from the $s$-wave interaction which is tied to the $c_{2}$ coefficient in the chiral expansion. The full $c_{2}$ term in Ref. [12] provides a momentum dependence which accounts for $p$-wave plus also effective range corrections of the $s$-wave. A large fraction of the $c_{2}$ coefficient in [12] is accounted for by the explicit consideration of the $\Delta$ in $[37,93]$. The contribution to the $\pi^{-} d$ scattering length in Ref. [86] depends much on the prescription taken for the expansion and in the NNLO expansion it gives one order of magnitude bigger contribution than the NNLO*
expansion. The value of the NNLO* expansion is considered more realistic in Ref. [86]. We adopt here the scheme followed so far, looking into different mechanisms and using the Bonn and Paris wave functions to have an idea of uncertainties. For this purpose we evaluate the contribution due to Fermi motion in $s$ and $p$-waves.

By using the formalism of Ref. [106] we have up to $p$-waves for the $\pi N$ scattering amplitude

$$
\begin{equation*}
\mathcal{F}=b_{0}+b_{1}(\tilde{\mathbf{t}} \cdot \vec{\tau})+\left[c_{0}+c_{1}(\tilde{\mathbf{t}} \cdot \vec{\tau})\right] \mathbf{q}^{\prime} \cdot \mathbf{q} . \tag{2.36}
\end{equation*}
$$

In addition a range dependence of the $s$-wave amplitude is used in [75] where

$$
\begin{align*}
& a_{\pi^{-} p}(\omega)=a^{+}+a^{-}+\left(b^{+}+b^{-}\right) \mathrm{q}^{2} \\
& a_{\pi^{-} n}(\omega)=a^{+}-a^{-}+\left(b^{+}-b^{-}\right) \mathrm{q}^{2} \tag{2.37}
\end{align*}
$$

By performing a boost to the $\pi N$ CM frame where Eqs. $(2.36,2.37)$ hold, we obtain the impulse approximation for the $p$-wave contribution, including the correction for the range of the $s$-wave part as:
$a_{\pi^{-} d}=2\left(c_{0}+b^{+}\right) \frac{1+m_{\pi} / m_{N}}{1+m_{\pi} / m_{d}}\left(\frac{m_{\pi}}{m_{\pi}+m_{N}}\right)^{2}\left\langle\left[1+\left(\frac{\mathbf{p}}{m_{\pi}+m_{N}}\right)^{2}\right] \mathbf{p}^{2}\right\rangle$
which coincides with Ref. [75] up to small corrections of $\mathcal{O}\left(p^{4}\right)$ and the introduction of the $s$-wave range parameter correction which amounts to a $25 \%$ decrease of the term ${ }^{2}$. We obtain from Eq. (2.38) a contribution of $57 \pm 9 \cdot 10^{-4} m_{\pi}^{-1}$ with $9 \%$ of this value coming from the $\mathbf{p}^{4}$ term.

Next, we also take into account corrections due to double scattering with one $s$-wave and another $p$-wave, or two $p$-waves. This leads to small corrections but with large uncertainties which are genuine and should be taken into account as we show below. Another reason to explicitly evaluate these corrections is that they are included in what is called dispersion corrections taken in Ref. [75] from Ref. [77, 78].

We consider double scattering with one $\pi N$ vertex in $s$-wave and the other one in $p$-wave and vice versa, or the two vertices in $p$-wave. After

[^1]Table 2.4: Fermi corrections to $a_{\pi^{-} d}$ in double scattering.

| Contribution | Bonn from Ref. [101] |  | Paris from Ref. [102] |  |
| :--- | :--- | :--- | :--- | :--- |
| $\left[10^{-4} \cdot m_{\pi}^{-1}\right]$ | $s$ | $d$ | $s$ | $d$ |
| $\mathrm{q}^{2}$-term of Eq. $(2.39)$ | -30 | 0 | -2 | 0 |
| Rest of Eq. $(2.39)$ | 13 | 9 | 11 | 13 |
| From Eq. $(2.40)$ | -3 | 2 | -4 | 2 |
| Sum |  |  | $5 \pm 15$ |  |

performing the boosts to the CM frames we find

$$
\begin{align*}
a_{\pi^{-} d}^{(s-p)}= & 4 \pi \frac{\left(1+m_{\pi} / m_{N}\right)^{2}}{1+m \pi / m_{d}}\left(2 b_{0} c_{0}-4 b_{1} c_{1}\right) \int \frac{d^{3} \mathbf{q}}{(2 \pi)^{3}} \frac{1}{\mathbf{q}^{2}} \int d^{3} \mathbf{p} \\
& \tilde{\varphi}(\mathbf{p}) \tilde{\varphi}(\mathbf{p}+\mathbf{q})^{\star}\left[\left(\frac{m_{\pi}}{m_{\pi}+m_{N}}\right)^{2}\left(\mathbf{p}^{2}+(\mathbf{p}+\mathbf{q})^{2}\right)-\frac{m_{\pi}}{m_{\pi}+m_{N}} \mathbf{q}^{2}\right] \tag{2.39}
\end{align*}
$$

with $\tilde{\varphi}(\mathbf{p})$ the Fourier transform of the deuteron wave function $\varphi(\mathbf{r})$, normalized such that $\int d^{3} \mathbf{p}|\tilde{\varphi}(\mathbf{p})|^{2}=1$. The large term in Eq. (2.39) comes from the $\mathbf{q}^{2}$ of the square bracket in the equation. This term is easily evaluated since it becomes proportional to $|\varphi(\mathbf{r}=0)|^{2}$. The precise value of $\varphi(0)$ is not well known since it depends on the short range forces assumed in the model. Thus one can imagine that there will be large uncertainties in this term. For the $d$-wave part of the wave function it vanishes, but the $s$-wave part provides a contribution, as can be seen in Tab. 2.4, which depends much on the model. It is interesting to notice that in the case of the Paris potential where there is a stronger repulsion at short distances, the value of the correction is much smaller than that for the Bonn potential. We do not consider here the range of the $s$-wave part. It does not come as in the impulse approximation but it produces a small correction to another small correction.

As for the $p$-wave in both vertices the contribution is

$$
\begin{align*}
a_{\pi-d}^{(p-p)}= & 4 \pi \frac{\left(1+m_{\pi} / m_{N}\right)^{2}}{1+m \pi / m_{d}}\left(2 c_{0}^{2}-4 c_{1}^{2}\right) \int \frac{d^{3} \mathbf{q}}{(2 \pi)^{3}} \frac{1}{\mathbf{q}^{2}} \int d^{3} \mathbf{p} \tilde{\varphi}(\mathbf{p}) \tilde{\varphi}(\mathbf{p}+\mathbf{q})^{\star} \\
& {\left[\left(\frac{m_{\pi}}{m_{\pi}+m_{N}}\right)^{2}(\mathbf{p}+\mathbf{q})^{2}-\frac{m_{\pi}}{m_{\pi}+m_{N}}(\mathbf{p}+\mathbf{q}) \mathbf{q}\right] } \\
& {\left[\left(\frac{m_{\pi}}{m_{\pi}+m_{N}}\right)^{2} \mathbf{p}^{2}+\frac{m_{\pi}}{m_{\pi}+m_{N}} \mathbf{p q}\right] } \tag{2.40}
\end{align*}
$$

This contribution should be smaller than the former one by comparing the strength of double scattering in $s$-wave with double scattering with $s$ and $p$-waves. A straightforward evaluation with a monopole form factor in each vertex with $\Lambda=1 \mathrm{GeV}$, since beyond that momentum, the wave function is certainly unreliable, gives indeed a small contribution compared to the typical size of the corrections discussed. The results obtained with Eq. (2.40) for the double scattering with $p$-wave, shown in Tab. 2.4, agree with the value of -3 in the same units quoted in refs. [75, 84].

## Isospin violation

The topic is thoroughly investigated in [88] but not considered in [86] for the evaluation of the $\pi N$ scattering lengths from pionic atom data. Their non consideration reverts into admittedly smaller errors in $\left(b_{0}, b_{1}\right)$ than the given ones according to Ref. [86]. The effects from this source are estimated relatively small in the $\pi^{-} d$ scattering length, of the order of $3.5 \cdot 10^{-4} m_{\pi}^{-1}$ according to $[75,84,85]$. We shall take a different attitude. Our approach allows for isospin violation since the masses of the particles are taken different. This is not the only source of isospin violation [87-92], but it gives the right order of magnitude. In order to consider breaking from other sources than mass splitting, we shall allow the subtraction constants $\alpha_{i}$ in the three $\pi N$ channels to be different. The global fit from the section 2.4 takes into account the isospin breaking in the $\pi^{-} d$ scattering length.

## Non-localities of the $\pi N s$-wave interaction

These are corrections to the assumption of point-like interaction in the $\pi N$ vertices. They are considered for single and double scattering in Ref. [75] leading to a modification of the pion propagator $G \sim 1 / r$. The non-locality
of $\pi N$ interaction affects mainly the isovector part of $s$-wave $\pi N$ scattering [75]. This is closely associated with the VMD assumption, which states that the $\pi N$ interaction is predominantly mediated by the $\rho$-meson. In the work of Ref. [45] the $\rho$ meson is explicitly taken into account, modifying the Weinberg-Tomozawa term at intermediate energies, but close to pion threshold the $\rho$ does not play a role in the approach of [45] and in the present work. Therefore, we adopt the corrections from refs. [75, 84, 85] where values of $17(9) \cdot 10^{-4} m_{\pi}^{-1}$ and $29(7) \cdot 10^{-4} m_{\pi}^{-1}$ are obtained, respectively. We shall take a value of $23 \cdot 10^{-4} m_{\pi}^{-1}$, and a larger uncertainty of 15 in the same units, in order to account for the discrepancy of these two results.

## Non-static effects

These are corrections that go beyond the assumptions made from fixed centers. The accuracy of the static approximation is further supported by the study of [97] where, due to the dominance of the isovector $\pi N$ amplitudes, recoil corrections are shown to be small. Estimates of these non-static effects are done in $[84,85,105]$ and, as quoted in [75], they lead to a correction of $11(6) \cdot 10^{-4} m_{\pi}^{-1}$. Since the 5 th diagram in Fig. 2.6 is part of the non-static effects, we do not include its contribution in the corrections, but adopt the value from $[84,85,105]$.

Dispersion corrections and other real parts of the amplitude
The dispersion corrections tied to the absorption of Figs. 2.3 and 2.4 have been calculated, e. g., in refs. [77, 78, 80, 85]. In Ref. [77], a repulsive contribution of $-50 \pm 3 \cdot 10^{-4} m_{\pi}^{-1}$ is found (the error taken from the precision of displayed decimal digits in Ref. [77]). The authors of Ref. [78] consider any absorption contribution to the real part from reactions of the type $\pi^{-} d \rightarrow N N \rightarrow \pi^{-} d$ within their non-relativistic treatment of the pion. In the formulation of the present study, this would correspond to the sum of the absorption diagrams of Figs. 2.3 and 2.4, plus diagrams that contain external pions which couple directly via $p$-wave to a nucleon. The latter contribute only when Fermi motion is considered (see above). Their diagram $D$ would correspond to the diagrams of Fig. 2.3 in the present work, and $D^{\prime}$ would correspond to the diagrams in Fig. 2.4. Using different models and wave functions, the authors of Ref. [78] obtain values for the sum of the
diagrams $D$ and $D^{\prime}$ of $+14.9,-19.5$, and -9.28 in the units of $10^{-4} m_{\pi}^{-1}$. This tells us that there are large intrinsic uncertainties in this calculation.

In Ref. [80], a value for $\Delta a_{\pi^{-} d}$ of $-56 \pm 14 \cdot 10^{-4} m_{\pi}^{-1}$ was deduced from the two publications [77,78]. The value of Ref. [80] is then adopted by the recent work of Ref. [75]. The other important diagram included in Ref. [78], called $C$, which leads to this final number, can be interpreted in a Feynman diagrammatic way as double scattering involving one $\pi N$ vertex and a second scattering involving only the $p$-wave mediated by the nucleon pole. According to the work done here this should be complemented also with the $\Delta$ pole, which is dominant, and the crossed nucleon pole term. Altogether, this would be the Fermi correction to double scattering with $s$ and $p$-waves evaluated before, and in some case (see Tab. 2.4) we found a correction of $-30 \cdot 10^{-4} m_{\pi}^{-1}$, which follows the trend of the result of Ref. [80], but we also discussed that this contribution is very uncertain since it depends on the unknown value of $\varphi(0)$ which is highly model dependent.

We, hence, follow our Feynman diagrammatic technique accounting for the mechanisms implicit in refs. [77,78] and substitute their numbers by our dispersion correction plus the rescattering terms involving $p$-waves discussed above in the subsection of Fermi corrections.

Other possible contributions related to pion interaction with the pion cloud are shown to cancel with related vertex contributions in Ref. [86], something also found in $K^{+}$and $\pi$ nucleus scattering in refs. [107] and [108], respectively.

In addition, there are other minor contributions from the literature to the real part of the pion deuteron scattering length, and we show them in Table 2.5 without further comments.

Table 2.5: Corrections to Re $a_{\pi^{-} d}$.

| Contribution | Value in $10^{-4} \cdot m_{\pi^{-}}^{-1}$ | Source |
| :--- | :--- | :--- |
| $\left(\pi^{-} p, \gamma n\right)$ double scattering | -2 | $[75]$ |
| Form factor/ Non-locality | $23 \pm 15$ | $[75,84,85]$ |
| Non-static | $11 \pm 6$ | $[75]([84,85,105])$ |
| virtual pion scattering | $-7.1 \pm 1.4$ | $[109]$ |
| Dispersion | $2.4 \pm 4.3$ | Present study |
| Crossed $\pi$ and $\Delta(1232)$ | 14.6 | Present study |
| Fermi motion, IA | $57 \pm 9$ | Present study |
| Fermi m. double scatt. $(\mathrm{s-p}, \mathrm{p}-\mathrm{p})$ | $5 \pm 15$ | Present study |
| Sum | $104 \pm 24$ |  |

### 2.4 Results

The threshold data from pionic hydrogen and deuterium are obtained from the PSI experiments of [70-73]. According to preliminary results from PSI experiments [111], these results could change in the near future with a consequence in the extracted values of $b_{0}, b_{1}$. Systematic uncertainties were also considered in [75]. In addition, the Coulomb corrections on the pionic hydrogen have been recently revised in Ref. [112]. This shifts the hydrogen data by the order of one $\sigma$. We take the newer values from Ref. [112] but keep the more conservative error estimates from refs. [71, 73-75].

$$
\begin{align*}
a_{\pi^{-} p \rightarrow \pi^{-} p} & =(870 \pm 2 \text { stat. } \pm 10 \text { syst. }) \cdot 10^{-4} m_{\pi}^{-1} \\
a_{\pi^{-} p \rightarrow \pi^{0} n} & =-1250(60) \cdot 10^{-4} m_{\pi}^{-1} \\
a_{\pi^{-} d} & =(-252 \pm 5 \text { stat. } \pm 5 \text { syst. }+i 63(7)) \cdot 10^{-4} m_{\pi}^{-1} \tag{2.41}
\end{align*}
$$

The sum of the corrections to the real part of the pion-deuteron scattering length from the sections 2.3.2, 2.3.3, and 2.3.4 is

$$
\begin{equation*}
(104 \pm 24) \cdot 10^{-4} m_{\pi}^{-1} \tag{2.42}
\end{equation*}
$$

The corrections in (2.42) are positive, which means that this additional attraction of the pion must be compensated for by a larger contribution
from the multiple scattering series in order to give the experimental value in (2.41).

We proceed now to determine the isoscalar and isovector scattering length $\left(b_{0}, b_{1}\right)$ from the experimental data in (2.41), and from low energy $\pi N$ scattering data [113]. As Table 2.1 shows, the deuteron scattering length is particularly sensitive to $b_{0}$. This is due to the impulse approximation that cancels in the limit $b_{0} \rightarrow 0$, but contributes significantly for $b_{0} \neq 0$ as the results for the phenomenological Hamiltonian (2.21) in Table 2.1 demonstrate. The data from pionic hydrogen, $a_{\pi^{-} p \rightarrow \pi^{-} p}$ and $a_{\pi^{-} p \rightarrow \pi^{0} n}$, on the other hand, provide exact restrictions on the isovector scattering length $b_{1}$. Finally, the low energy $\pi N$ scattering data, together with the threshold values, determine the free parameters of the coupled channel approach of section 2.2.

The authors of refs. [114, 115] use a different approach, based on effective field theory (EFT) in the framework of QCD and QED, which gives somewhat different values for the $\pi^{-} p$ scattering length, with larger errors tied basically to the poorly known $f_{1}$ parameter. That means, for instance, that these authors explicitly take into account corrections that go into the generation of mass splittings, while the empirical analyses that lead to the data in Eq. (2.41) and scattering data at finite energies, that we shall use later on, only deal with electromagnetic corrections through the Coulomb potential and use physical masses. Hence, the simultaneous use of data extracted through these different methods for the global analysis that we use here, should be avoided. A reanalysis of raw $\pi N$ and $\pi d$ data through the EFT techniques is possible, and steps in this direction are already given in [114-116]. In any case, we shall show later on how our final results change with different values of this $\pi^{-} p$ scattering length. The global analysis of the bulk of threshold and scattering data that we do in the present work leads us to use the phenomenological multiple scattering method which has been used to extract the amplitudes, which implies that the results we obtain should only be used within this framework.

### 2.4.1 The isoscalar and isovector scattering lengths

Previous Results

In the literature, different values for $\left(b_{0}, b_{1}\right)$ have been extracted from pionic atoms and low energy $\pi N$ scattering extrapolated to threshold. In Ref. [117], a value of $a_{+}=-80 \pm 20 \cdot 10^{-4} m_{\pi}^{-1}$ has been extracted from the data by performing a partial-wave analysis. Note, however, the comments in Ref. [71] on this value, concerning the outdated database and other uncertainties.

Extrapolations of low energy $\pi N$ scattering to threshold have been updated over the years $[113,118,119]$, and the value for the isospin even scattering length in the SM95 partial wave analysis is $a_{+}=-30 \cdot 10^{-4} m_{\pi}^{-1}$. In an earlier publication in Ref. [119], the same group found a deep minimum in their global fit for $a_{+}=-100 \cdot 10^{-4} m_{\pi}^{-1}$. The current value is $a_{+}=-10(12) \cdot 10^{-4} m_{\pi}^{-1}$ in the most recent analysis, FA02, of Ref. [113].

From the constraints of the strong shifts in hydrogen and deuterium pionic atoms, the authors of refs. [71,72] deduce small and positive values for $a_{+}$of 0 to $50 \cdot 10^{-4} m_{\pi}^{-1}$. The findings in Ref. [71] are still compatible with isospin symmetry, although the two bands (constraints) from the shift at a $\chi^{2}$ of 1 do barely intersect with the constraint from the hydrogen width. As pointed out in Ref. [71], this would be evidence of isospin violation. The value in Ref. [71] of $\left(b_{0}, b_{1}\right)$ relies on the corrections to the real part of $a_{\pi^{-} d}$ from the analysis of Ref. [80]. In a recent publication on new measurements of $a_{\pi^{-} d}$, Ref. [73], one finds an extensive discussion on updated corrections, including refs. $[75,84,85,109]$. The authors in [73] find a value of $\left(b_{0}, b_{1}\right)=-1_{-21}^{+9} \cdot 10^{-4} m_{\pi}^{-1}$. See the discussion in Ref. [75] on this value. The recent theoretical approach in Ref. [96] provides a value of around $b_{0}=(-30 \pm 40) \cdot 10^{-4} m_{\pi}^{-1}$.

To conclude, the experiments and subsequent analyses on pionic atoms lead to a value of $b_{0}$ being compatible with zero, or slightly negative, with errors of the same size or much larger than $b_{0}$ itself. In the various extrapolations of low energy $\pi N$ scattering data to threshold, more negative values of $b_{0}$ are favored.


Figure 2.7: Constraints on $\left(b_{0}, b_{1}\right)$ from $\operatorname{Re} a_{\pi^{-} d}$ and pionic hydrogen.

## Constraints on $\left(b_{0}, b_{1}\right)$ from threshold data

The restrictions on $\left(b_{0}, b_{1}\right)$ can be separately analyzed for the three threshold data points of Eq. (2.41), and for low energy $\pi N$ scattering. The separation of threshold and finite energy allows for a consistency test of the data, and, on the other hand, for a test of the freedom that the theoretical model has, when only one threshold point is fitted. The influence of the data in Eq. (2.41) corresponds to bands in the $\left(b_{0}, b_{1}\right)$ plane, whose width is determined by the experimental and theoretical errors.

We begin with the influence of $a_{\pi^{-d}}$ on $\left(b_{0}, b_{1}\right)$. For the real part of the deuterium scattering length $a_{\pi^{-} d \rightarrow \pi^{-} d}$ the band in the $\left(b_{0}, b_{1}\right)$ plane is calculated in four different ways, in order to determine the effect of isospin breaking by using physical masses instead of averaged ones. In all approaches, the large correction from Eq. (2.42) is taken into account, and in the approaches one to three only the experimental error is considered. In the fourth approach, the large theoretical error from Eq. (2.42) is added, widening the band by a factor of 3. In Fig. 2.7, the results for the deuterium are plotted.

- First, the four $\pi N$ scattering lengths in the solution of the Faddeev equations in Eq. (2.15) are expressed in terms of $b_{0}$ and $b_{1}$. This, of course, implies the assumption of isospin symmetry. Then, random values of $\left(b_{0}, b_{1}\right)$ are generated and $a_{\pi^{-} d}$ is calculated with the help of Eqs. $(2.15,2.17)$. The pairs $\left(b_{0}, b_{1}\right)$, that lead to a $\chi^{2} \leq 1$ with the experimental value $a_{\pi^{-} d, \exp }$. of Eq. (2.41), are kept, and plotted in Fig. 2.7 (dotted line).
- In a second approach, we generate the four scattering lengths with the coupled channel (CC) approach of section 2.2. For that, we take random values for the five free parameters of the theory $\left(\alpha, \beta, \gamma, c_{i}\right)$, and also use averaged masses for pions and nucleons, an assumption that we will drop in the third approach. Then, from the scattering lengths, $\left(b_{0}, b_{1}\right)$ are calculated with the help of Eq. (2.1). At the same time, $a_{\pi^{-} d}$ is calculated with the help of the Faddeev equations $(2.15,2.17)$. The same selection rule of $\chi^{2} \leq 1$ as in the first approach sorts out the $\left(b_{0}, b_{1}\right)$ pairs that are plotted in Fig. 2.7 with the dashed line. The result coincides exactly with the first approach. This is indeed expected since we take the same subtraction constant $\alpha_{\pi N}$ for all $\pi N$ channels.
- In a third approach, we use physical masses instead of averaged ones, and proceed otherwise exactly as in approach 2. The result, plotted with the thin solid line in Fig. 2.7, shows a nearly identical result compared to the first two approaches. This gives us a measure to which extend isospin breaking effects from different masses can affect the result. The maximal effect of isospin breaking from this source cannot exceed a small fraction of the experimental error.
- The fourth approach takes into account the large theoretical error from Eq. (2.42) of $24 \cdot 10^{-4} m_{\pi}^{-1}$ and follows otherwise the third approach. The band is widened significantly (thick solid line). In the following calculations we use this approach.

The constraints from pionic hydrogen are also plotted in Fig. 2.7. For each of the two bands from pionic hydrogen, marked as ' $\pi^{-} p \rightarrow \pi^{-} p$ ' and ' $\pi^{-} p \rightarrow \pi^{0} n$ ' in Fig. 2.7, we have followed the approaches 1 to 4 as for the
deuteron. The four approaches give identical results for each band, as before. Therefore, in Fig. 2.7 one finds only one band from $a_{\pi^{-} p \rightarrow \pi^{-} p}$ and one from $a_{\pi^{-} p \rightarrow \pi^{0} n}$.

The horizontal band shows the constraint from the experimental $a_{\pi^{-} p \rightarrow \pi^{0} n}$ that is directly related to the hydrogen width. We have the isospin relation

$$
\begin{equation*}
b_{1}=1 / \sqrt{2}\left(a_{\pi^{-} p \rightarrow \pi^{0} n} \pm \Delta a_{\pi^{-} p \rightarrow \pi^{0} n}\right) \tag{2.43}
\end{equation*}
$$

where the experimental error $\Delta$ is relatively large, leading to a wide band. The hydrogen shift is closely related to $a_{\pi^{-} p \rightarrow \pi^{-} p}$, which leads to the constraint

$$
\begin{equation*}
b_{1}=b_{0}-a_{\pi^{-} p \rightarrow \pi^{-} p} \pm \Delta a_{\pi^{-} p \rightarrow \pi^{-} p} \tag{2.44}
\end{equation*}
$$

in the isospin limit.
Taking exclusively the data from pionic hydrogen, values of $b_{0}$ from $-70 \cdot 10^{-4} m_{\pi}^{-1}$ up to positive numbers are allowed from Fig. 2.7. Then, the band from pionic deuterium is added which appears with a steep slope and narrows significantly the region of allowed values of $b_{0}$. The range of $b_{0}$ is now determined by the position and width of the deuterium band, namely by Eqs. (2.41) and (2.42). This shows the necessity of having revised and extended the corrections of the $\pi^{-} d$ scattering length in the former sections. Indeed, if we would not have applied the corrections from Eq. (2.42), the deuterium band would show up in Fig. 2.7 with the same slope, but shifted by around $+55 \cdot 10^{-4} m_{\pi}^{-1}$ along the $b_{0}$ axis. This would lead to a value of $b_{0}$ being perfectly compatible with 0 . However, the situation after applying the corrections leaves us with a $b_{0} \in[-70,-40] \cdot 10^{-4} m_{\pi}^{-1}$.

## Pion Nucleon Scattering at finite energies

The unitarized coupled channel approach is applied in order to describe $\pi N$ scattering at finite energies. We fix the free parameters of the theory by fitting the model to the data above threshold. Then, the threshold prediction of the model is calculated. This is called 'Extrapolation' in the following. Comparing the predictions at threshold with the experimental data from pionic atoms, Eq. (2.41), the low energy behavior and the consistency of the model is tested.

Additionally, it is desirable to have an accurate parametrization of the $\pi N$ amplitude over some energy range, from threshold up to moderate energies. This is achieved by including the threshold data themselves in the fit, and this is referred to as 'Global fit' in the following.

Selection of experimental data: From the analyses of the CNS data base [113] for $\pi N$ scattering we choose the 'single-energy solutions' values, which are obtained by fitting narrow regions in the CM energy $\sqrt{s}$ separately. In contrast to the global fit given in Ref. [113], the single energy bins carry individual errors each. This helps to determine the statistical influence of $\pi N$ scattering on the parameters of the model. We add a small constant theoretical error of 0.002 to the amplitudes in the normalization of [113]. The channels to be included are the $s$-wave isospin $I=1 / 2$ and $I=3 / 2$ amplitudes, with real and imaginary part. This does not mean four independent data points for each point in energy $\sqrt{s}$ : Since the inelasticity is zero at the low energies included in all fits, real and imaginary part are totally determined by the phase shift $\delta$, and therefore, for the purpose of calculating the reduced $\chi_{r}^{2}$, there are only two independent values, from the $I=1 / 2$ and from the $I=3 / 2$ channel. In Fig. 2.8, the global fit and the extrapolation of $\pi N$ data to threshold are plotted. Table 2.6 displays the parameters of the fits, including the range of energies of the fitted data.

The global fit favors values for $b_{0}$ and $b_{1}$ still negative but with smaller strength than the threshold data. As this fit contains threshold and finite energy data, it is situated between the extrapolation and the intersection of the three bands in the $\left(b_{0}, b_{1}\right)$ plane. These differences between threshold and extrapolation from scattering data might be related to possible additional uncertainties in the phenomenological extraction of the partial wave amplitudes ${ }^{3}$.

In Fig. 2.8, the extrapolation and the global fit are indicated by shaded regions. These regions can be understood in the way that the reduced $\chi_{r}^{2}$ from Table 2.6 does not raise by more than 1 from the optimum $\left(\Delta \chi_{r}^{2} \leq 1\right)$ for all points $\left(b_{0}, b_{1}\right)$ inside these regions. For both fits, the optima are indicated with the black dots in Fig. 2.8. Furthermore, we plot for the global fit the region that fulfills $\Delta \chi_{r}^{2} \leq 2$. It appears as the light gray area just around

[^2]

Figure 2.8: Threshold data from D and H , extrapolations from $\pi N$ scattering, and global fits.

Table 2.6: Global fits and threshold extrapolations from Fig. 2.8. Also, the fit without the damping factor from section 2.4.1 is displayed.

|  | Global Fit | Extrapolation | No damping |
| :--- | :--- | :--- | :--- |
| fitted data $(\sqrt{s})$ | $1104-1253 \mathrm{MeV}$ <br> +threshold | $1104-1253 \mathrm{MeV}$ | $1104-1180 \mathrm{MeV}$ |
|  | $51 /(2 \cdot 10+3) \simeq 2.2$ | $24 /(2 \cdot 10)=1.2$ | $33 /(2 \cdot 6+3)=2.2$ |
| $\chi_{r}^{2}$ | $-1.143 \pm 0.109$ | $-0.990 \pm 0.083$ | $-1.528 \pm 0.28$ |
| $\alpha_{\pi N}$ | $-1.539 \pm 0.20$ | $-1.000 \pm 0.463$ | $-0.788 \pm 0.14$ |
| $2 c_{1}-c_{3}\left[\mathrm{GeV}^{-1}\right]$ | $-2.245 \pm 0.45$ | $-1.670 \pm 0.07$ |  |
| $c_{2}\left[\mathrm{GeV}^{-1}\right]$ | $-2.657 \pm 0.22$ | $0.002513 \pm 3.3 \cdot 10^{-4}$ | No $\beta$ |
| $\beta\left[\mathrm{MeV}^{-2}\right]$ | $0.002741 \pm 1.5 \cdot 10^{-4}$ | $-10 \pm 6.1$ | $10 \pm 10$ |
| $\gamma\left[10^{-5} \cdot m_{\pi}^{5}\right]$ | $5.53 \pm 7.7$ | $(91)$ | 4 |
| $\chi^{2}\left(a_{\pi^{-}-p \rightarrow \pi^{-} p}\right)$ | 3 | $(2)$ | $<1$ |
| $\chi^{2}\left(a_{\pi^{-} p \rightarrow \pi^{0} n}\right)$ | $<1$ | $(6)$ | 4 |
| $\chi^{2}\left(a_{\pi^{-}-d}\right)$ | 8 |  |  |

the $\Delta \chi_{r}^{2} \leq 1$ region.
The above explanation for the shaded regions of Fig. 2.8 has to be taken with caution: The values $\left(b_{0}, b_{1}\right)$ inside the shapes have been calculated by the use of Eq. (2.1) from the elementary scattering lengths in the particle base. They are not the free parameters of the theory, which are the $\alpha, \beta, \gamma$, $c_{2}$, and $\left(2 c_{1}-c_{3}\right)$.

This implies that the $\Delta \chi_{r}^{2} \leq 1,2$ criterion is applied to a $\chi^{2}$ that is a function of $a_{\pi^{-} p}, a_{\pi^{-} n}, a_{\pi^{0} n}$, and $a_{\pi^{-} p \rightarrow \pi^{0} n}$. Once a set of elementary scattering lengths $a_{\pi N}$ fulfills the criterion, $\left(b_{0}, b_{1}\right)$ are calculated from these values via Eq. (2.1), and give a point in the shaded regions of Fig. 2.8.

The elementary scattering lengths $a_{i}$ themselves have been calculated with the help of the CC approach by generating the fitting parameters randomly in a wide range. Every set of $a_{\pi N}$ corresponds to a set $\left(\alpha, \beta, \gamma, c_{i}\right)$ and, via the criterion, the parameter errors on $\left(\alpha, \beta, \gamma, c_{i}\right)$ in Tab. 2.6 are determined. 'Parameter error' means here: The range of a parameter of a model, that leads to a raise of the reduced $\chi_{r}^{2}$ of less than 1 from the best $\chi^{2}$, minimizing $\chi^{2}$ at the same time with respect to all other parameters.

As the final results for $\left(b_{0}, b_{1}\right)$ we take the values from the global fit:

$$
\begin{equation*}
\left(b_{0}, b_{1}\right)=(-28 \pm 40,-881 \pm 48) \cdot\left[10^{-4} m_{\pi}^{-1}\right] \tag{2.45}
\end{equation*}
$$

The errors have been taken from the maximal extension of the region in $\left(b_{0}, b_{1}\right)$, calculated from the $\Delta \chi_{r}^{2} \leq 1$ criterion described above. They can be read off Fig. 2.8. The errors on $\left(b_{0}, b_{1}\right)$ take also into account the uncertainties from $\pi N$ scattering data up to 1253 MeV .

With the caveat expressed in section 2.4 about using simultaneously data obtained through the EFT or phenomenological multiple scattering methods, we would also like to give, only as indicative, the results that we would obtain if we replaced the scattering length of $\pi^{-} p$ from Eq. (2.41) by the one given in Ref. [115]. We find then the values

$$
\begin{equation*}
\left(b_{0}, b_{1}\right)=(-39 \pm 52,-862 \pm 68) \cdot\left[10^{-4} m_{\pi}^{-1}\right] . \tag{2.46}
\end{equation*}
$$

which are compatible with the results in (2.45) within the error bars. The changes from Eqs. (2.45) to (2.46) go in the same direction as in Ref. [86] for $b_{0}$ but not for $b_{1}$. This is due to the fact that our fit puts more weight in the scattering data than that of Ref. [86].


Figure 2.9: Global fits and extrapolations in the real and imaginary part of the $I=1 / 2$ and $I=3 / 2$ channels. In the plot for $\operatorname{Im}$ S11, the range of fitted data for extrapolations and global fits is indicated.

## Finite energy behavior of the fits

In Fig. 2.9 the energy behavior of the global fit and the extrapolation from Fig. 2.8 and Tab. 2.6 is displayed. The two upper pictures show the real and imaginary parts of the isospin $I=1 / 2$ channel, the lower the same for $I=3 / 2$. The data from the CNS in the single-energy solutions [118] is displayed with errors. The global fit is displayed with the solid line, the extrapolation with the dashed line. As expected, the extrapolation provides better high energy behavior, as it is not restricted by threshold data. It is remarkable, how well the extrapolation matches the data above 1253 MeV , which is the upper limit of the fitted data.

In Ref. [45] the $\pi N$ scattering data has been fitted up to high energies, including the region of the $N^{\star}$ (1535) resonance. A fit was obtained that

Table 2.7: Isoscalar quantities: $\beta$, isoscalar generated by rescattering in $\pi N$, and final result of the fits.

|  | Global fit | Extrapolation |
| :--- | :--- | :--- |
| $b_{c}\left[10^{-4} m_{\pi}^{-1}\right]$ | -336 | -434 |
| $b_{0}\left[10^{-4} m_{\pi}^{-1}\right]$, generated | 442 | 396 |
| $b_{0}\left[10^{-4} m_{\pi}^{-1}\right]$, final | -28 | -46 |

explained well the resonance but overestimated the $I=1 / 2$ amplitude at low energies, even when including the $\rho$-meson in the $t$-channel (see sec. 2.2). The $\pi \pi N$ channel does not substantially improve the situation in Ref. [45]. It seems to be impossible to have at the same time a precise low energy fit, and a reproduction of the $N^{\star}(1535)$ resonance with the input of Ref. [45]. In the present approach we have introduced the extra isoscalar term which improves the fit of the low energy data. On the other hand, the present approach cannot reproduce the $N^{\star}$ (1535) resonance since the heavier members of the baryon and meson octet become important at these energies and one has to use the full $S U(3)$ approach as in Ref. [45].

## The size of the isoscalar piece and the $(\pi, 2 \pi)$ term

In table 2.7 we compare the two sources of isoscalar strength. In the first line we show the value of the contribution to $b_{0}$ from the isoscalar term of Eq. (2.9),

$$
\begin{equation*}
b_{c}=-\frac{1}{4 \pi} \frac{m_{N}}{m_{\pi}+m_{N}} \frac{4 c_{1}-2 c_{2}-2 c_{3}}{f_{\pi}^{2}} m_{\pi}^{2} . \tag{2.47}
\end{equation*}
$$

The second line of Tab. 2.7 shows the $b_{0}$ that is generated by the rescattering of the $\pi N$ system. It has been calculated for the fits by setting all parameters except the subtraction constant $\alpha$ to zero. In this way, one can extract the size of the isoscalar part that is generated by the multiple loop sum from the Bethe-Salpeter equation (2.3). Although the lowest order chiral Lagrangian from Ref. [45] provides pure isovector interaction, the rescattering generates an isoscalar part, where usually $90 \%$ is generated by one loop, and
most of the rest by the 2-loop rescattering (depending on the actual values of the subtraction constants). The last line of Tab. 2.7 provides the final value of $b_{0}$. The interplay of the isoscalar piece from Eq. (2.9), the isovector interaction, and the subtraction constant leads to a resulting $b_{0}$ (last line of Tab. 2.7) that cannot be explained any more as the sum of $b_{c}$ plus the generated $b_{0}$.

It is instructive to compare the results that we obtain for the isoscalar coefficients $c_{i}$ in Tab. 2.6 and those obtained in Ref. [93]. The results of the fit $2^{\dagger}$ of the Tab. 4 in Ref. [93] are:

$$
\begin{align*}
2 c_{1}-c_{3} & =-1.63 \pm 0.9 \mathrm{GeV}^{-1} \\
c_{2} & =-1.49 \pm 0.67 \mathrm{GeV}^{-1} \tag{2.48}
\end{align*}
$$

The agreement with the global fit from Tab. 2.6 is fair within errors.
A more direct comparison with the results of Ref. [93] can be achieved by removing the damping factor $\beta$ in the isoscalar term. We can only get good agreement with data up to about $\sqrt{s}=1180 \mathrm{MeV}$. Restricting ourselves to energies below $\sqrt{s}=1180 \mathrm{MeV}$, the fit to the data provides the parameters given in Tab. 2.6 as 'No damping'. One can see that the $c_{2}$ coefficient is in good agreement with Eq. (2.48) and also the combination of $2 c_{1}-c_{3}$ within errors. With this fit to the restricted data we find

$$
\begin{equation*}
\left(b_{0}, b_{1}\right)=(-37 \pm 37,-887 \pm 43) \cdot\left[10^{-4} m_{\pi}^{-1}\right] . \tag{2.49}
\end{equation*}
$$

It is worth stressing this agreement since the "standard" values of the $c_{i}$ coefficients used in chiral perturbative calculations, where the $\Delta$ is not taken into account explicitly, are much larger in size than the coefficients found here or in Ref. [93] and lead to the combinations of Eq. (2.48) with opposite sign. Given the large amount of problems originated by the use of the "standard" (large) $c_{i}$ coefficients in chiral perturbative calculations, the models to fit $\pi N$ cross sections including explicitly the $\Delta$, as in Ref. [93], and the "smaller" $c_{i}$ coefficients found here and in Ref. [93], are highly recommendable.

As for the $\gamma$ parameter corresponding to the $\pi N \rightarrow \pi \pi N$ mechanism, we should expect on physical grounds quite a small contribution. Indeed, this is the case: The size of the $\pi N \rightarrow \pi N$ term including two $\pi N \rightarrow \pi \pi N$ vertices
and the $\pi \pi N$ loop for the global fit 2 corresponds to about $5 \%$ of the tree level $\pi N$ amplitude.

## Isospin breaking

The isospin breaking in pion nucleon scattering has received much attention both phenomenologically and theoretically [87-92]. Our theoretical model uses isospin symmetry up to breaking effects from the use of different masses. We also rely for the fit upon some scattering data that has been analyzed assuming isospin symmetry [113]. A possible measure of isospin breaking can be given by the quantity $D$ which describes the deviation from the triangle identity,

$$
\begin{equation*}
D=f_{\mathrm{CEX}}-\frac{1}{\sqrt{2}}\left(f_{+}-f_{-}\right) . \tag{2.50}
\end{equation*}
$$

In Ref. [89], $D$ is calculated by fitting $s$ and $p$-wave of $f_{+}, f_{-}, f_{\text {CEX }}$ which are the amplitudes of $\pi^{+} p, \pi^{-} p$, and $\pi^{-} p \rightarrow \pi^{0} n$. The authors fit a variety of potential models to the elastic channels, and predict $f_{\text {CEX }}$ from that. Fitting in a second step $f_{\text {CEX }}$ alone, they state a discrepancy between the prediction and the fit of $f_{\text {CEX }}$ that results in a value of $D=-0.012 \pm 0.003 \mathrm{fm}$. Physical masses are included in the coupled channel approach of Ref. [89], and their value for $D$ contains effects of other sources of isospin violation than mass splitting.

In the present work we have derived a microscopical model only for the $s$ wave interaction, also using physical masses in a coupled channels approach. Unfortunately, there is no partial wave analysis available that is free of isospin assumptions, so that we cannot evaluate $D$ from Eq. (2.50) using only experimental $s$-wave amplitudes. For this reason we shall take advantage of the work done in [89].

The global fit from Tab. 2.6, which only contains an isospin violation from mass splitting, produces $D=-0.0066 \mathrm{fm}$, around half the value of Ref. [89]. In order to give more freedom to the model to violate isospin symmetry from different sources than mass splitting, we now allow different subtraction constants $\alpha_{i}$ in each channel, $\pi^{-} p, \pi^{0} n$, and $\pi^{+} p$. Then, we add extra data points taking the value of $D$ from Ref. [89] at three energies

Table 2.8: Additional fits with $D$ and different $\alpha_{\pi N}$

|  | $\mathrm{D} 3 \alpha$ |
| :--- | :--- |
| $\chi_{r}^{2}$ | 2.3 |
| $D[\mathrm{fm}]$ at 1110 MeV | -0.011 |
| $\alpha_{\pi^{-} p}$ | $-0.717 \pm 0.59$ |
| $\alpha_{\pi^{0} n}$ | $-0.823 \pm 0.26$ |
| $\alpha_{\pi^{+} p}$ | $-1.130 \pm 0.10$ |
| $2 c_{1}-c_{3}\left[\mathrm{GeV}^{-1}\right]$ | $-1.752 \pm 0.22$ |
| $c_{2}\left[\mathrm{GeV}^{-1}\right]$ | $-2.676 \pm 0.27$ |
| $\beta\left[\mathrm{MeV}^{-2}\right]$ | $0.002731 \pm 1.8 \cdot 10^{-4}$ |
| $\gamma\left[10^{-5} \cdot m_{\pi}^{5}\right]$ | $10.0 \pm 9.0$ |

covering the range of $T_{\text {lab }}=30-50 \mathrm{MeV}$ (as in Fig. 1 of Ref. [89]). The $\chi^{2}$ stays practically the same compared to the global fit in Tab. 2.6, but the fit gives $D=-0.011 \mathrm{fm}$ at $\sqrt{s}=1110 \mathrm{Mev}$, see 'D $3 \alpha$ ' in Tab. 2.8. The scattering amplitudes from this fit are practically the same as those shown in the Figs. 2.8 and 2.9 for the global fit and we do not plot them again.

As we can see in Tab. 2.8 the subtraction constant for the $\pi^{+} p$ channel barely changes with respect to the global fit in Tab. 2.6 while those for the $\pi^{-} p$ and $\pi^{0} n$ channels are reduced in size by about $30 \%$. The values that we obtain in this fit for $2 c_{1}-c_{3}$ and $c_{2}$ change little with respect to those quoted before and the values for $\left(b_{0}, b_{1}\right)$ reveal a shift compared to the ones of the global fit in Eq. (2.45) of

$$
\begin{equation*}
\delta b_{0}=6 \cdot 10^{-4} m_{\pi}^{-1}, \quad \delta b_{1}=9 \cdot 10^{-4} m_{\pi}^{-1} \tag{2.51}
\end{equation*}
$$

The changes induced by the extra isospin breaking are rather small compared with the errors that we already have.

The last value obtained for $D$ is in agreement with Ref. [89]. A warning should be given about the solution found, since the isospin $1 / 2$ amplitudes used in the fit imply isospin symmetry. However, the fact that half the amount of $D$ that we obtain comes from the use of different masses without
invoking isospin breaking from other sources, and that the threshold data, which have small error bars, and thus a large weight in our fit, do not imply isospin symmetry, makes us confident that the solution obtained accounts reasonably for isospin violation in the problem.

### 2.5 Conclusions

For a determination of the isoscalar and isovector scattering lengths of the $\pi N$ system, new calculations on the complex pion deuteron scattering length have been performed. The dispersive part from absorption has been found to be compatible with zero. This, together with corrections from crossed diagrams and the $\Delta(1232)$ resonance, and with other corrections taken from the literature, leads to a substantial shift of the real part of $a_{\pi^{-} d}$ towards positive values.

The unitary coupled channel approach of Ref. [45] has been tested for consistency at low energies. However, we have added an isoscalar term that can be matched to terms of higher orders of the chiral Lagrangians. These terms are known to play an important role at threshold. With this additional ingredient to the model, together with the $\pi N \rightarrow 2 \pi N$ channel, an acceptable global fit for the $\pi N$ amplitude up to intermediate energies has been obtained.

One of the results of the present work concerns the values of the $c_{i}$ parameters used in chiral perturbation theory at low energies. We find them compatible with values obtained from fits to data when the $\Delta$ is explicitly taken into consideration. On the other hand, we have addressed the isospin violation issue and found that our fit to data accounts for about half the isospin breaking only from mass splittings. The model has been extended to account for other sources of isospin breaking and then can match results of isospin breaking found in other works. We find that the effect of this breaking in the $b_{0}, b_{1}$ parameters is well within uncertainties from other sources.

Attention to the sources of errors has been paid and we find larger values than in former studies. Altogether, we have here a new determination of the $\pi N$ scattering lengths and a new parametrization of the $\pi N s$-wave amplitudes at low energies that can be used as input in studies of other elementary processes or as input to construct optical potentials from pio-
nic atoms, where problems tied to the strength of the isoscalar part of the potential still remain

## Chapter 3

## The $s$-wave pion nucleus optical potential


#### Abstract

We calculate the $s$-wave part of the pion-nucleus optical potential using a unitarized chiral approach that has been previously used to simultaneously describe pionic hydrogen and deuterium data as well as low energy $\pi N$ scattering in the vacuum. This energy dependent model allows for additional isoscalar parts in the potential from multiple rescattering. We consider Pauli blocking and pion polarization in an asymmetric nuclear matter environment. Also, higher order corrections of the $\pi N$ amplitude are included. The model can accommodate the repulsion required by phenomenological fits, though the theoretical uncertainties are bigger than previously thought.


### 3.1 Introduction

The problem of the missing repulsion in pionic atoms has attracted much attention in the past $[80,99,108,120-122]$ and recently [123-129] and was further motivated by the discovery of deeply bound pionic atoms at GSI [130-133].

Due to the repulsion of the $s$-state pion in nuclear matter, the $\pi^{-}$wave function is strongly repelled and overlaps only little with the nucleus. The wave function tests mainly the peripheral zone of the nucleus and, thus, nuclear matter at less than nuclear density. However, even at half the nuclear matter density difficulties in the theoretical description persist. From phe-
nomenological fits to pionic atoms reaching from C to Pb , a strong repulsion is needed for a consistent description of the combined data. However, theoretical calculations consistently failed to deliver this "missing repulsion" (see, e.g., [99]) although there has been recent progress [126].

The $s$-wave pion-nucleus optical potential is the basic input for a calculation of the $s$-levels of pionic atoms. Usually, the optical potential is calculated for infinite nuclear matter as a function of the Fermi momentum. Explicit calculations for finite nuclei done in Ref. [134] provide a prescription to pass from nuclear matter to finite nuclei: the $s$-wave part of the potential is provided by the corresponding nuclear matter results changing $\rho$ to $\rho(r)$ (local density approximation), while for the $p$-wave the prescription is slightly more complicated.

The $s$-wave pion optical potential $2 \omega V_{\text {opt }}(r)=\Pi_{S}(r)$ is closely connected to the $s$-wave pion selfenergy which is usually [120] parametrized as

$$
\begin{align*}
\Pi_{S}(r) & =-4 \pi\left[\left(1+\frac{m_{\pi}}{m_{N}}\right) b_{0}\left(\rho_{p}+\rho_{n}\right)+\left(1+\frac{m_{\pi}}{m_{N}}\right) b_{1}\left(\rho_{n}-\rho_{p}\right)\right. \\
& \left.+\left(1+\frac{m_{\pi}}{2 m_{N}}\right) B_{0}(\rho)\left(\rho_{p}+\rho_{n}\right)^{2}\right] \tag{3.1}
\end{align*}
$$

where the density $\rho$ is a function of the radial distance, $\rho \equiv \rho(r)$, given by the density profile of the nucleus. From this expression the sensitivity of the selfenergy to the isoscalar $b_{0}$ becomes visible, as in symmetric nuclear matter the isovector term $b_{1}$ vanishes. However, heavy nuclei such as ${ }_{82}^{208} \mathrm{~Pb}$ recently used in experiments [131-133] contain more neutrons than protons; it is therefore interesting to study asymmetric matter, in particular with respect to a possible renormalization of the isovector $b_{1}$ [127, 135-137]. The last term in Eq. (3.1) takes into account corrections from higher order in density. This quantity has also an imaginary part due to pion absorption, which is mainly a two-body process, and the imaginary part of the optical potential determines the width of the pionic atom.

Traditional fits to pionic atoms $[134,138,139]$ provide the set of parameters displayed in Tab. 3.1. Although the sets of parameters are quite different from each other they result in similar pion self energies at $\rho=\rho_{0} / 2$, half the nuclear density. Therefore these sets are not contradictory but tell us that the pionic atom data require this magnitude of selfenergy at $\rho_{0} / 2$.

Table 3.1: Typical fits of pionic atom data.

| Ref. | $b_{0}\left[m_{\pi}^{-1}\right]$ | $b_{1}\left[m_{\pi}^{-1}\right]$ | $B_{0}\left[m_{\pi}^{-4}\right]$ |
| :--- | :--- | :--- | :--- |
| $[138]$ | -0.0045 | -0.0873 | $-0.049+i 0.046$ |
| $[139]$ | -0.0325 | -0.0947 | $0.002+i 0.047$ |
| $[134]$ | -0.0183 | -0.105 | $i 0.0434$ |

This equivalence of pion optical potentials using the concept of $\rho_{\text {eff }}=\rho_{0} / 2$ was early established in $[140,141]$. Furthermore, Tab. 3.1 suggests that the smaller value of $\left|b_{0}\right|$ in Ref. [138] needs to be compensated by a large negative real part of the $\rho^{2}$-term $B_{0}$; thus, corrections of higher order in the density are important.

The model of Ref. [1] is of interest in this context as a good part of the $\pi N$ vacuum isoscalar is generated by the multiple rescattering of the dominant Weinberg-Tomozawa term of the $\pi N$ interaction. This realization is important because rescattering terms are appreciably modified in the nuclear medium. Indeed, the Pauli blocking in the intermediate nucleon states is well known to generate a repulsion, the Ericson-Ericson Pauli corrected rescattering term [120]. On the other hand, the pion polarization due to particle-hole $(p h)$ and $\Delta$-hole $(\Delta h)$ excitation of the intermediate pions also produces corrections and accounts for the imaginary part of the potential from pion absorption [99, 145].

Another point is the energy dependence of the $\pi N$ interaction [126]. Ref. [1] focuses on the precise determination of the scattering lengths but also provides the energy dependence close to threshold. For pionic atoms where the pion is practically at rest with respect to the nucleus this is still relevant due to the Fermi motion of the nucleons (see also [142-144] for low energy $\pi$-nucleus scattering. Note that in this context, the vacuum model [1] already contains certain information about the nucleon-nucleon correlations as one of the fitted data points has been the $\pi^{-}$-deuteron scattering length. The deuteron wave function that enters the theoretical description provides the $N N$ momentum distribution and allows for an inclusion of the Fermi motion in the deuteron. The issue of the energy dependence is a relevant
one and in the medium it induces corrections which, accidentally (because of the smallness of the $b_{0}$ parameter), has an effect similar to the one of the renormalization of $b_{1}$ [126].

On the other hand there are some medium corrections coming from vertex corrections, off-shell effects, and wave function renormalization which, if desired, can also be recast as renormalization of $b_{1}$ and $b_{0}$. We shall also introduce novel terms in the pion selfenergy related to the $N^{*}$ (1440) decay into $N \pi \pi$, with the two pions in a scalar isoscalar state. This mechanism has already been used in [146] to estimate some uncertainties in the study of the $\pi$-deuteron interaction.

Another novelty in the present work is that we shall start from a free model for $\pi N$ scattering which is constructed using a chiral unitary approach, incorporating the lowest order ( LO ) and the needed next to lowest order (NLO) chiral Lagrangians together with multiple scattering of the pions [1].

The vacuum model from Ref. [1] will be modified in various steps: In Sec. 3.2.1 Pauli blocking of the intermediate nucleonic states together with the appropriate spectral function for the intermediate pions will lead to nonlinear corrections in the density with preliminary numerical results given in Sec. 6.4. Also, a self consistent calculation is presented in Sec. 3.3.1 where the overall pion $s$-wave selfenergy serves as an input for the intermediate pions in the $\pi N$ loops. In Secs. 3.4, 3.5, the diagrammatic model will be extended to the above mentioned higher order vertex corrections. Final numerical results are provided in Sec. 8.8.

### 3.2 Low energy pion nucleon interaction in vacuum and matter

The vacuum $\pi N$ isoscalar term $b_{0}$ is around ten times smaller than the vacuum isovector $b_{1}$-term and its precise determination is a complex task due to large cancellations in the amplitude. With the advent of new experimental data [70-74] for the $\pi^{-} p \rightarrow \pi^{-} p, \pi^{-} p \rightarrow \pi^{0} p$, and $\pi^{-} d \rightarrow \pi^{-} d$ scattering lengths from pionic hydrogen and deuterium theoretical efforts in several directions have been made to precisely determine the parameters of low energy $\pi N$ scattering. In this context, $\pi^{-}$-deuteron scattering at threshold plays an


Figure 3.1: Rescattering of the $\pi^{-} N$ system generated by the Bethe-Salpeter equation.
important role as the complex scattering length $a_{\pi^{-}{ }_{d}}$ puts tight constraints on the size of $b_{0}$.

Pion deuteron scattering has been recently treated in chiral perturbative approaches [110, 146] including also corrections from isospin breaking [147] and effects like Fermi motion [86]. These and other higher order corrections have been taken into account in another theoretical framework in Ref. [75]. In the extraction of the strong scattering lengths from experiment, special attention has to be paid to the Coulomb corrections in the extraction of the scattering lengths from pionic hydrogen [114, 115, 148].

In the present study we rely upon the results from chapter 2 [1] on low energy $\pi N$ scattering in $s$-wave which is summarized below. This model simultaneously describes the available data at threshold from pionic hydrogen and deuterium and also low energy $\pi N$ scattering. In a restriction to the coupled channels $\pi^{-} p, \pi^{0} n$, and $\pi^{-} n$ the $\pi N s$-wave amplitude $T(\sqrt{s})$ is unitarized by the use of the Bethe-Salpeter equation (BSE)

$$
\begin{equation*}
T(\sqrt{s})=[1-V(\sqrt{s}) G(\sqrt{s}))]^{-1} V(\sqrt{s}) . \tag{3.2}
\end{equation*}
$$

Here, the kernel $V$ is given by the elementary isovector interaction from the Weinberg-Tomozawa term of the LO chiral Lagrangian [11-13] as discussed in Sec. 2.2. The $\pi N$ loop function $G$ in Eq. (3.2) provides the unitarity cut and is regularized in dimensional regularization with one free parameter, the subtraction constant $\alpha_{\pi N}$ (see Eq. (2.8)). In Fig. 3.1 we show a diagrammatic representation of the BSE equation (3.2), including also the $\pi \pi N$ channel which is also incorporated in [1].

In the framework of the heavy baryon approach the vertices are factorized on-shell (see Eq. (2.7)) because the off-shell part of the vertices in the loops can be absorbed renormalizing the lowest order tree level amplitude [22]. However, we will see in Sec. 3.4 that in a nuclear matter environment these
renormalizations are modified leading to finite, density dependent, corrections of the amplitude.

The multiple rescattering which is provided by Eq. (3.2) generates isoscalar pieces from the isovector interaction providing a large $b_{0}$ term. However, it is known $[19,87,93]$ that the NLO chiral Lagrangian is a necessary ingredient in $\pi N$ scattering at low energies. In order to provide the necessary degrees of freedom in the model, the isoscalar $s$-wave piece with the chiral coefficients $c_{i}$ in the notation of Ref. [93], given in Eq. (2.9), is added to the kernel of Eq. (3.2). The term $c_{3} q^{2}$ in ref. [93] has been taken as $c_{3} m_{\pi}^{2}$, consistently with the approach of refs. [24, 45] which uses the on-shell values for the vertices in the scattering equations. The free fit parameters up to this point are the subtraction constant $\alpha_{\pi N}$ and the two combinations of $c_{i}$ from Eq. (2.9) as well as a damping factor parametrized with $\beta$ from Eq. (2.10) which is of no relevance here because we stay close to threshold. There are further refinements of the model, described in detail in chapter 2 [1], such as the inclusion of the $\pi \pi N$ two-loop diagram which introduces one additional fit parameter, $\gamma$, from the real part of this loop.

In order to include the complex pion-deuteron scattering length $a_{\pi^{-} d}$ in the data fit, one has to employ the elementary $\pi N$ scattering model from above in the framework of a three body process. In chapter $2[1]$ this has been carried out by using the $\pi N$ amplitudes in a Faddeev multiple scattering approach. The interesting point is that the impulse approximation vanishes making the double rescattering off the two nucleons the dominant term. This term is sensitive to the isoscalar amplitude so that the experimental scattering length $a_{\pi^{-} d}$ provides valuable information on the vacuum $b_{0}$ term and sets tight constraints on it.

Additional corrections of higher order Ref. [75] in $\pi d$ scattering such as absorption, dispersion, the influence of the $\Delta$ (1232), and Fermi motion have been treated in a separate Feynman diagrammatic approach, together with other corrections from the literature, see Ref. [75] and references therein. In this context it is interesting to note that many of the diagrams from the recent study [110] are effectively included in chapter 2 [1] in the Fermi motion in double scattering. Once having included these various corrections in $a_{\pi d}$, the model parameters are fixed from data, namely the scattering


Figure 3.2: Diagrammatic representation of the $\pi^{-}$selfenergy from $s$-wave interaction with nucleons.
lengths $a_{\pi^{-} p \rightarrow \pi^{-} p}, a_{\pi^{-} p \rightarrow \pi^{0} n}, a_{\pi^{-} d}$, and low energy $\pi N$ data from [113]. The parameter values are quoted in the left column of Tab. 3.2. The values of the $c_{i}$ from Eq. (2.9) are in agreement with other works [93]; furthermore, the isospin violations found in the study qualitatively agree with Ref. [89]. In the following, we concentrate on the in-medium modifications of the approach.

### 3.2.1 The model in nuclear matter

The $s$-wave $\pi N \rightarrow \pi N$ vacuum model from chapter 2 [1], summarized in Sec. 3.2, provides the driving interaction of the $\pi^{-}$with the nucleus. In order to obtain the pion selfenergy $\Pi_{S}$ from Eq. (3.1) of the $\pi^{-}$in asymmetric nuclear matter with proton and neutron densities $\rho_{p}$ and $\rho_{n}\left(k_{F}^{p}, k_{F}^{n}\right.$ the respective Fermi momenta), the $\pi^{-} N \rightarrow \pi^{-} N$ amplitude $T$ is summed over the nucleons in the Fermi sea as schematically indicated in Fig. 3.2. The $s$-wave selfenergy for a $\pi^{-}$at momentum $\left(k^{0}, \mathbf{k}\right)$ with respect to the nuclear matter rest frame reads

$$
\begin{align*}
\Pi_{S}\left(k^{0}, \mathbf{k} ; \rho_{p}, \rho_{n}\right) & =2 \int^{k_{F}^{p}} \frac{d^{3} \mathbf{p}_{p}}{(2 \pi)^{3}} T_{\pi^{-} p}\left(P^{0}, \mathbf{P} ; \rho_{p}, \rho_{n}\right) \\
& +2 \int^{k_{F}^{n}} \frac{d^{3} \mathbf{p}_{n}}{(2 \pi)^{3}} T_{\pi^{-} n}\left(P^{0}, \mathbf{P} ; \rho_{p}, \rho_{n}\right) \tag{3.3}
\end{align*}
$$

where $\mathbf{p}_{p, n}$ are the nucleon momenta. Due to the breaking of Lorentz invariance, the amplitudes $T_{\pi^{-} p, n}$ depend independently on the components of $\left(P^{0}, \mathbf{P}\right)$, the total 4 -momentum of the $\pi N$ system in the nuclear matter frame, namely $P^{0}=k^{0}+E_{p, n}\left(\mathbf{p}_{p, n}\right)$ and $\mathbf{P}=\mathbf{k}+\mathbf{p}_{p, n}$. The factor of 2 in

Eq. (3.3) accounts for the sum over the nucleon spins. Note that Eq. (3.3) allows for isospin breaking by using different masses for particles of the same isospin multiplet.

In analogy to the vacuum case, $T_{\pi^{-} p}$ and $T_{\pi^{-} n}$ are given by the solutions of Bethe-Salpeter equations (BSE)

$$
\begin{equation*}
\left.T\left(P^{0}, \mathbf{P} ; \rho\right)=\left[1-V(\sqrt{s}) G\left(P^{0}, \mathbf{P} ; \rho\right)\right)\right]^{-1} V(\sqrt{s}) \tag{3.4}
\end{equation*}
$$

where $s=\left(P^{0}\right)^{2}-\mathbf{P}^{2}$ and the loop function $G$ which is modified as described below. In Sec. 3.4 we will apply in-medium changes also to the kernel $V$ from off-shell parts of the vertices and other sources. For the charge $C=0$ sector, the BSE is represented by $(2 \times 2)$ matrices accounting for the coupled channels $\pi^{-} p$ and $\pi^{0} n$. For the $\pi^{-} n$ interaction there is only one channel.

The diagonal matrix $G$ from Eq. (3.4) contains the loop functions $G_{\pi N}$ which have been formulated in dimensional regularization in Eq. (2.8) for the vacuum case. Alternatively, one can use a cut-off scheme [45] with $\Lambda$ the three momentum cut-off. The vacuum $G_{\pi N}$ is then given by

$$
\begin{align*}
G_{\pi N}\left(P^{0}, \mathbf{P}\right) & =a_{\pi N}+i \int \frac{d^{4} q}{\left.(2 \pi)^{4}\right)} \frac{M_{N}}{E(\mathbf{P}-\mathbf{q})} \\
& \times \frac{1}{P^{0}-q^{0}-E(\mathbf{P}-\mathbf{q})+i \epsilon} \frac{1}{\left(q^{0}\right)^{2}-\mathbf{q}^{2}-m+i \epsilon} \tag{3.5}
\end{align*}
$$

with a cut-off for the three-momentum integration $\Lambda=1 \mathrm{GeV}$ and $m\left(M_{N}\right)$ being the $\pi^{-}, \pi^{0}(p, n)$ masses. Over wide energy ranges, a change in $\Lambda$ can be written as an additive constant to the real part of $G_{\pi N}$. Therefore, we have denoted a separate piece $a_{\pi N}$ in Eq. (3.5) in the same way as in Ref. [45]. For the free case the propagator in the cut-off scheme agrees with the propagator from dimensional regularization over a wide energy range by choosing the appropriate subtraction constant. In the nuclear medium with Lorentz covariance explicitly broken, a cut-off scheme is more convenient in order to implement the in-medium dressing. Thus, we will employ the propagator from Eq. (3.5) in this work. This requires a refit of the vacuum data. The values of the model parameters with the cut-off propagator from Eq. (3.5) instead of dimensional regularization are displayed in Tab. 3.2 on the right hand side. The new fit shows that the model is insensitive to the used regularization scheme. Parameters, $\chi^{2}$, and predictions for isoscalar and

Table 3.2: Global fits to pionic hydrogen, deuteron, and low energy $\pi N$ scattering data, using dimensional regularization from Ref. Eq. (2.8) and cut-off scheme. Also, the resulting $b_{0}, b_{1}$ are shown.

|  | DimReg | Cut-off |
| :--- | :--- | :--- |
| fitted data $(\sqrt{s})$ | $1104-1253 \mathrm{MeV}+$ threshold | $1104-1253 \mathrm{MeV}+$ threshold |
| $\chi_{r}^{2}$ | $51 /(2 \cdot 10+3) \simeq 2.2$ | $48 /(2 \cdot 10+3) \simeq 2.1$ |
| $\alpha_{\pi N}[-]$ | $-1.143 \pm 0.109$ | - |
| $a_{\pi N}[\mathrm{MeV}]$ | - | $-2.025 \pm 1.28$ |
| $2 c_{1}-c_{3}\left[\mathrm{GeV}^{-1}\right]$ | $-1.539 \pm 0.20$ | $-1.487 \pm 0.20$ |
| $c_{2}\left[\mathrm{GeV}^{-1}\right]$ | $-2.657 \pm 0.22$ | $-2.656 \pm 0.22$ |
| $\beta\left[\mathrm{MeV}^{-2}\right]$ | $0.002741 \pm 1.5 \cdot 10^{-4}$ | $0.002752 \pm 1.5 \cdot 10^{-4}$ |
| $\frac{\gamma\left[10^{-5} \cdot m_{\pi}^{5}\right]}{\chi^{2}\left(a_{\pi^{-} p \rightarrow \pi^{-} p}\right)}$ | $5.53 \pm 7.7$ | $6.27-7.8$ |
| $\chi^{2}\left(a_{\pi^{-} p \rightarrow \pi^{0} n}\right)$ | $<1$ | 3 |
| $\chi^{2}\left(a_{\pi^{-} d}\right)$ | 8 | $<1$ |
| $b_{0}\left[10^{-4} m_{\pi^{-}}^{-1}\right]$ | $-28 \pm 40$ | 7 |
| $b_{1}\left[10^{-4} m_{\pi^{-}}^{-1}\right]$ | $-881 \pm 48$ | -29 |

isovector terms $b_{0}$ and $b_{1}$ are stable. For notation of the parameters, see Sec. 3.2. In Tab. 3.2, $\alpha_{\pi N}$ is the subtraction constant of the loop in dimensional regularization and $a_{\pi N}$ the subtraction constant from Eq. (3.5).

The more important parameters are the $c_{i}$ and $\alpha_{\pi N}\left(a_{\pi N}\right)$. The real part of the $\pi \pi N$ loop $(\gamma)$ is tiny at threshold. For pionic atoms, the damping factor $\beta$ from chapter 2 [1] which is more important for the higher energy $\pi N$ data is of no relevance because the c.m. energy of $\pi N$ due to Fermi motion in the nucleus is small.

The two major medium modifications of $G_{\pi N}$ are the Pauli blocking of the nucleon propagator and the polarization of the pion. The corresponding diagram is displayed in Fig. 3.3. For the amplitude of the in-medium $\pi N$ loop function a similar expression as in Ref. [150] is obtained. Here, we give the generalization to asymmetric nuclear matter for the $\pi^{-} p, \pi^{0} n$, and $\pi^{-} n$


Figure 3.3: In-medium correction of $s$-wave $\pi N$ scattering: Renormalization of the pion and Pauli blocking of the nucleon, symbolized by a crossed propagator. The pion $p$-wave selfenergy stands for resummed $p h, \Delta h$ insertions and includes $N N, N \Delta, \Delta \Delta$ short-range correlations.
loops. With $N=p, n$ and $\pi_{i}=\pi^{-}, \pi^{0}$,

$$
\begin{align*}
& G_{\pi_{i} N}\left(P^{0}, \mathbf{P} ; \rho_{p}, \rho_{n}\right)=a_{\pi N}+i \int \frac{d^{4} q}{(2 \pi)^{4}} \theta\left(q_{\mathrm{cm}}^{\max }-\left|\mathbf{q}_{\mathrm{cm}}\right|\right) \frac{M_{N}}{E_{N}(\mathbf{P}-\mathbf{q})} \\
\times & \left(\frac{\theta\left(|\mathbf{P}-\mathbf{q}|-k_{F}^{N}\right)}{P^{0}-q^{0}-E_{N}(\mathbf{P}-\mathbf{q})+i \epsilon}+\frac{\theta\left(k_{F}^{N}-|\mathbf{P}-\mathbf{q}|\right)}{P^{0}-q^{0}-E_{N}(\mathbf{P}-\mathbf{q})-i \epsilon}\right) \\
\times & \int_{0}^{\infty} d \omega \frac{2 \omega}{\left(q^{0}\right)^{2}-\omega^{2}+i \epsilon} S_{\pi_{i}}\left(\omega, \mathbf{q} ; \rho_{p}, \rho_{n}\right) . \tag{3.6}
\end{align*}
$$

The cut-off in the vacuum model is applied in the $\pi N$ c.m. frame, whereas Eq. (3.6) is defined in the nuclear matter rest frame. Since in the free case $q_{\text {max }}$ is given in the c.m. frame we boost $\mathbf{q}_{\mathrm{cm}} \equiv \Lambda=1 \mathrm{GeV}$ to this frame and demand it to be smaller in modulus than $q_{\mathrm{cm}}^{\max }$,

$$
\begin{equation*}
\mathbf{q}_{\mathrm{cm}}=\left[\left(\frac{P^{0}}{\sqrt{s}}-1\right) \frac{\mathbf{P} \cdot \mathbf{q}}{|\mathbf{P}|^{2}}-\frac{q^{0}}{\sqrt{s}}\right] \mathbf{P}+\mathbf{q} \tag{3.7}
\end{equation*}
$$

where $s=\left(P^{0}\right)^{2}-\mathbf{P}^{2}$. In Eq. (3.6) we have also taken into account the hole part of the $N$-propagator as in Ref. [149] which can play a role at the low pion energies we are studying. This term has been neglected in Ref. [150] which works at higher energies. The pion spectral function $S_{\pi_{i}}$ is different for $\pi^{-}$and $\pi^{0}$ for asymmetric nuclear matter. For $S$ we include the particle hole ( $p h$ ) excitation and $N N$ short-range correlations as described in the next section.

In the model from chapter 2 [1], the $\Delta(1232)$ has been explicitly taken into account in pion-deuteron scattering, leading to corrections in the $\pi d$


Figure 3.4: Integration over the Fermi sea of the medium $\pi N$ amplitude. The crosses represent Pauli blocking of the nucleon propagators and the large dots, the $p$-wave pion selfenergy.
scattering lengths which finally have an influence in the value of the vacuum isoscalar amplitude. In the present situation we can take the corresponding effect into account by including also the $\Delta$-hole $(\Delta h)$ excitation in the pion selfenergy; in fact, closing the nucleon lines of the deuteron in the $\Delta$-box and $\Delta$-crossed box diagrams of Figs. 2.2 and 2.6 one obtains a pion selfenergy corresponding to Fig. 3.3 substituting the $p h$ by a $\Delta h$ excitation of the pion. The $N^{*}(1440)$ Roper-hole excitation can be in principle also included in the pion selfenergy but has been found small in Ref. [150] for low energy pions. However, in Sec. 3.7.3 the Roper resonance will be included at tree level in a different configuration with an additional coupling to a scalar-isoscalar pion pair which will result in large contributions to the isoscalar $b_{0}$ term.

In Ref. [150] Pauli blocking for the intermediate $\pi \pi N$ loop - see Fig. 3.1 - has been included for the imaginary part. In the present case the pion has very little momentum in the $\pi N$ c.m. frame and the system is below the $\pi \pi N$ threshold so that no change is required.

Combining all the ingredients of the in-medium model, the $s$-wave pion selfenergy can be symbolized by the diagram in Fig. 3.4: The in-medium propagator from Eq. (3.6) is resummed in the BSE (3.4), and the remaining integral over the Fermi seas from Eq. (3.3) corresponds to closing the nucleon line.

### 3.2.2 Pion polarization in asymmetric nuclear matter

The spectral function of the pion $\pi_{i}$ at momentum $\left(q^{0}, \mathbf{q}\right)$ from Eq. (3.6) is given by the imaginary part of the propagator,

$$
\begin{equation*}
S_{\pi_{i}}\left(q^{0}, \mathbf{q} ; \rho_{p}, \rho_{n}\right)=-\frac{1}{\pi} \operatorname{Im} D_{\pi}, \quad D_{\pi}=\frac{1}{\left(q^{0}\right)^{2}-\mathbf{q}^{2}-m_{\pi_{i}}^{2}-\Pi_{\pi_{i}}\left(q^{0}, \mathbf{q} ; \rho_{p}, \rho_{n}\right)} \tag{3.8}
\end{equation*}
$$

For the pion selfenergy inside loops the $p$-wave part is dominant because $\mathbf{q}$ is a running variable and $\Pi_{\pi_{i}} \propto \mathbf{q}^{2}$. The s-wave part will be included in the self consistent treatment in Sec. 3.3.1. For the selfenergy we take into consideration the $(p h)-(p h)$ short-range repulsion parametrized in terms of the Migdal parameter, which is chosen $g^{\prime}=0.7$,

$$
\begin{align*}
\Pi_{\pi_{i}}\left(q^{0}, \mathbf{q} ; \rho_{p}, \rho_{n}\right) & =\left(\frac{D+F}{2 f_{\pi}}\right)^{2} F^{2}(q) \mathbf{q}^{2} \\
& \times \frac{\bar{U}_{\pi_{i}}\left(q^{0}, \mathbf{q} ; \rho_{p}, \rho_{n}\right)}{1-\left(\frac{D+F}{2 f_{\pi}}\right)^{2} F^{2}(q) g^{\prime} \bar{U}_{\pi_{i}}\left(q^{0}, \mathbf{q} ; \rho_{p}, \rho_{n}\right)} \tag{3.9}
\end{align*}
$$

For a diagrammatic representation of the pion selfenergy, see, e.g., Ref. [108]. The Lindhard functions for asymmetric matter for $(p h)$ and $(\Delta h)$ excitations, evaluated below, are added in Eq. (3.9), $\bar{U}=\bar{U}^{(p h)}+U^{(\Delta h)}$. Note that we have here for simplicity assigned the same $g^{\prime}$ to $(p h)$ and $(\Delta h)$ excitations. For the form factor that takes into account the off-shell pions coupling to ph or $\Delta h$ we have chosen the same function $F(q)=\Lambda^{2} /\left(\Lambda^{2}+\mathbf{q}^{2}\right)$ with $\Lambda=0.9$ GeV .

The Lindhard function for symmetric nuclear matter, $\bar{U}\left(q, k_{F}\right)$, can be found in the literature, e.g. in Ref. [151], and here, we concentrate on an extension to asymmetric matter. In the non-relativistic reduction, the Lindhard function for pions turns out to be

$$
\begin{align*}
\bar{U}_{\pi_{i}}\left(q, k_{F}^{1}, k_{F}^{2}\right) & =4 \int \frac{d^{3} \mathbf{k}}{(2 \pi)^{3}}\left[\frac{\Theta\left(k_{F}^{1}-|\mathbf{k}|\right) \Theta\left(|\mathbf{k}+\mathbf{q}|-k_{F}^{2}\right)}{q^{0}+\epsilon(\mathbf{k})-\epsilon(\mathbf{k}+\mathbf{q})+i \eta}\right. \\
& \left.+\frac{\Theta\left(k_{F}^{2}-|\mathbf{k}|\right) \Theta\left(|\mathbf{k}-\mathbf{q}|-k_{F}^{1}\right)}{-q^{0}+\epsilon(\mathbf{k})-\epsilon(\mathbf{k}-\mathbf{q})+i \eta}\right] \tag{3.10}
\end{align*}
$$

The first term is the contribution of the forward going $p h$ excitation (direct term) and the second term the pion crossed-term selfenergy. The index 1 (2)
labels the Fermi sea corresponding to the hole (particle) part of the direct and the particle (hole) part of the crossed contribution. E.g., for a $\pi^{-}, k_{F}^{1}=k_{F}^{p}$ and $k_{F}^{2}=k_{F}^{n}$. For a $\pi^{+}, k_{F}^{1}=k_{F}^{n}, k_{F}^{2}=k_{F}^{p}$. The integral (3.10) can be solved analytically. For this, we split the ordinary Lindhard function from Ref. [151] in direct and crossed part by $\bar{U}\left(q^{0}, \mathbf{q}, k_{F}\right)=\bar{U}_{d}\left(q^{0}, \mathbf{q}, k_{F}\right)+\bar{U}_{c}\left(q^{0}, \mathbf{q}, k_{F}\right)$ with $\bar{U}_{c}\left(q^{0}, \mathbf{q}, k_{F}\right)=\bar{U}_{d}\left(-q^{0}, \mathbf{q}, k_{F}\right)$ and

$$
\begin{align*}
\bar{U}_{d}\left(q^{0}, \mathbf{q}, k_{F}\right) & =\frac{3}{2} \frac{\rho M_{N}}{|\mathbf{q}| k_{F}}\left(z+\frac{1}{2}\left(1-z^{2}\right) \log \left(\frac{z+1}{z-1}\right)\right), \\
z & =\frac{M_{N}}{|\mathbf{q}| k_{F}}\left(q^{0}-\frac{\mathbf{q}^{2}}{2 M_{N}}\right) \tag{3.11}
\end{align*}
$$

where $\rho=2 /\left(3 \pi^{2}\right) k_{F}^{3}$ and $M_{N}$ the proton or neutron mass. Evaluating the integral in Eq. (3.10) one obtains for the $p h$ Lindhard function in asymmetric matter

$$
\begin{align*}
& \bar{U}_{\pi^{+}}^{(p h)}\left(q^{0}, \mathbf{q} ; \rho_{p}, \rho_{n}\right)=\bar{U}_{d}\left(q^{0}, \mathbf{q}, k_{F}^{n}\right)+\bar{U}_{c}\left(q^{0}, \mathbf{q}, k_{F}^{p}\right), \\
& \bar{U}_{\pi^{-}}^{(p h)}\left(q^{0}, \mathbf{q} ; \rho_{p}, \rho_{n}\right)=\bar{U}_{d}\left(q^{0}, \mathbf{q}, k_{F}^{p}\right)+\bar{U}_{c}\left(q^{0}, \mathbf{q}, k_{F}^{n}\right), \\
& \bar{U}_{\pi^{0}}^{(p h)}\left(q^{0}, \mathbf{q} ; \rho_{p}, \rho_{n}\right)=\frac{1}{2}\left(\bar{U}\left(q^{0}, \mathbf{q}, k_{F}^{p}\right)+\bar{U}\left(q^{0}, \mathbf{q}, k_{F}^{n}\right)\right) . \tag{3.12}
\end{align*}
$$

These are the expressions to be used in Eq. (3.9). The result in Eqs. (3.10) and (3.12) is in agreement with Ref. [152], correcting a typographical error in their Eq. (A.5).

For the $\Delta h$ Lindhard function $U^{(\Delta h)}\left(q, k_{F}^{p}, k_{F}^{n}\right)$, no new calculation is required, as the $\Delta$ always plays the role of a particle and is not affected by the Fermi sea. It is therefore sufficient to split $U_{\Delta}\left(k_{F}\right)$ from Ref. [151] into its charge states and direct plus crossed parts, and use as argument the $k_{F}$ that corresponds to the hole part,

$$
\begin{align*}
U_{\pi^{-}}^{(\Delta h)}\left(q^{0}, \mathbf{q} ; \rho_{p}, \rho_{n}\right) & =\frac{1}{4} U_{d}^{(\Delta h)}\left(q^{0}, \mathbf{q} ; k_{F}^{p}\right)+\frac{3}{4} U_{c}^{(\Delta h)}\left(q^{0}, \mathbf{q} ; k_{F}^{p}\right) \\
& +\frac{1}{4} U_{c}^{(\Delta h)}\left(q^{0}, \mathbf{q} ; k_{F}^{n}\right)+\frac{3}{4} U_{d}^{(\Delta h)}\left(q^{0}, \mathbf{q} ; k_{F}^{n}\right), \\
U_{\pi^{0}}^{(\Delta h)}\left(q^{0}, \mathbf{q} ; \rho_{p}, \rho_{n}\right) & =\frac{1}{2}\left(U_{d}^{(\Delta h)}\left(q^{0}, \mathbf{q} ; k_{F}^{p}\right)+U_{c}^{(\Delta h)}\left(q^{0}, \mathbf{q} ; k_{F}^{p}\right)\right. \\
& \left.+U_{c}^{(\Delta h)}\left(q^{0}, \mathbf{q} ; k_{F}^{n}\right)+U_{d}^{(\Delta h)}\left(q^{0}, \mathbf{q} ; k_{F}^{n}\right)\right) . \tag{3.13}
\end{align*}
$$

Analytic expressions for the direct and crossed part of the $\Delta h$ Lindhard function can be found in Ref. [151].


Figure 3.5: Real and imaginary part of the pion propagator $D_{\pi}$ for symmetric and asymmetric nuclear matter at a pion momentum of 500 MeV . The position of the quasielastic pion peak is indicated with the arrows. In the vacuum the peak is at $q^{0}=518 \mathrm{MeV}$. The asymmetric matter corresponds to the ratio of $n$ to $p$ in ${ }_{82}^{208} \mathrm{~Pb}$.

In order to see the effects of asymmetric nuclear matter we plot the pion propagator for normal nuclear density $\rho_{0}=0.483 m_{\pi}^{3}$ which corresponds to $k_{F}=268 \mathrm{MeV}$ for symmetric matter. For asymmetric matter we set $k_{F}^{n}=1.154 k_{F}^{p}$ which corresponds to the ratio of neutron rich nuclei such as ${ }_{82}^{208} \mathrm{~Pb}$. Then, $\rho_{0}=\rho_{p}+\rho_{n}$ is obtained with $k_{F}^{p}=247 \mathrm{MeV}$ and $k_{F}^{n}=286$ MeV . In the plots in Fig. 3.5 the propagator from Eq. (3.8) for pions at $|\mathbf{q}|=500 \mathrm{MeV}$ is shown. The $\pi^{0}$ in asymmetric matter is very similar to the case of symmetric matter. The $\pi^{-}$shows some minor deviations.

### 3.3 Numerical results

In Fig. 3.6, the real part of the $s$-wave pion selfenergy from the full model and from several approximations is plotted. The solid lines show the results for the model from Sec. 3.2.1 in symmetric and asymmetric nuclear matter, with and without the $p$-wave renormalization from Eq. (3.9) of the pion propagator in the intermediate $\pi N$ loops. For the cases with asymmetric matter, the x-axis is given by $k_{F}^{p}$. The neutron Fermi momentum is then chosen to be $k_{F}^{n}=1.154 k_{F}^{p}$. This ratio corresponds to the ratio of neutron rich nuclei as ${ }_{82}^{208} \mathrm{~Pb}$ with $k_{F}^{p}=241 \mathrm{MeV}$ and $k_{F}^{n}=278 \mathrm{MeV}$. The selfenergy


Figure 3.6: Real part of the $s$-wave pion selfenergy for the pion at rest. Note that for asymmetric nuclear matter, $k_{F}$ of the proton is plotted on the abscissa and we always take $k_{F}^{n}=1.157 k_{F}^{p}$. Fit results to pionic atom data from Refs. [134, 138, 139] are also plotted. Theoretical calculation of Ref. [99] indicated as "Garcia et al." (dashed line).


Figure 3.7: Imaginary part of the $s$-wave pion selfenergy for the pion at rest. Phenomenological fits as in Fig. 3.6.
in asymmetric nuclear matter is larger than in symmetric matter which can be easily understood from the large and positive term $(-4 \pi) b_{1}\left(\rho_{n}-\rho_{p}\right)$ from Eq. (3.1).

The effect of Pauli blocking in the intermediate loops of the $s$-wave rescattering (see Eq. (3.6)) can be approximated [99, 120] by

$$
\begin{equation*}
\Delta b_{0}\left(k_{F}\right)=-\frac{6 k_{F}}{\pi m_{\pi}^{2}} \frac{m_{N}}{m_{\pi}+m_{N}}\left(\lambda_{1}^{2}+2 \lambda_{2}^{2}\right) . \tag{3.14}
\end{equation*}
$$

As pointed out in Ref. [99] the quantities $\lambda_{1,2}$ are closely related to the vacuum isoscalar and isovector $b_{0}, b_{1}$ terms (generated from rescattering, not the elementary ones) for which we take from Ref. [1],

$$
\begin{align*}
& b_{0, \mathrm{vac}}=-0.0028 m_{\pi}^{-1} \hat{=}-\frac{1}{1+\frac{m_{\pi}}{m_{N}}} \frac{2 \lambda_{1}}{m_{\pi}} \\
& b_{1, \mathrm{vac}}=-0.0881 m_{\pi}^{-1} \hat{=}-\frac{1}{1+\frac{m_{\pi}}{m_{N}}} \frac{2 \lambda_{2}}{m_{\pi}} . \tag{3.15}
\end{align*}
$$

With these values and $B_{0}=0$ (no pion medium modification) one obtains from Eq. (3.1) the dotted curve in Fig. 3.6 upper left panel. Adding the approximate medium change of $b_{0}$ from Eq. (3.14) according to $b_{0}=b_{0 \text {, vac }}+\Delta b_{0}$, the dashed curve is obtained. Thus, the $t \rho$ approximation is not sufficient, whereas the inclusion of $\Delta b_{0}$ leads to a good agreement with the rescattering model. This shows also that effects from more than one loop in the $\pi N$
rescattering of the $\pi N$ amplitude are small, because Eq. (3.14) corresponds to exactly one Pauli blocked loop in the rescattering series [99].

Next, we compare to asymmetric nuclear matter but still without pion modification. This is displayed in Fig. 3.6 upper right panel. Now, the isovector term contributes and we can derive a similar approximation as Eq. (3.14) for the $b_{1}$ renormalization in nuclear matter,

$$
\begin{equation*}
\Delta b_{1}\left(k_{F}\right)=-\frac{6 k_{F}}{\pi m_{\pi}^{2}} \frac{m_{N}}{m_{\pi}+m_{N}}\left(2 \lambda_{1} \lambda_{2}-\lambda_{2}^{2}\right) \tag{3.16}
\end{equation*}
$$

by simply comparing the isospin structure of $\pi N$ scattering at one loop. The result from Eq. (3.1) using $b_{0}=b_{0, \text { vac }}+\Delta b_{0}$ and $b_{1}=b_{1 \text {, vac is indicated as }}$ the dotted line. As the dashed line we plot the result from Eq. (3.1) using $b_{0}=b_{0, \mathrm{vac}}+\Delta b_{0}$ and $b_{1}=b_{1, \mathrm{vac}}+\Delta b_{1}$. Obviously, the correction from Eq. (3.16) is small. However, In Sec. 3.4 we will find additional vertex corrections that will modify considerably the isovector strength of $\pi N$ scattering.

When including the pion renormalization in the model according to Eqs. $(3.6,3.8)$ the real part of the $s$-wave pion selfenergy for symmetric and asymmetric nuclear matter decreases as shown in the two lower plots of Fig. 3.6. We can compare to Ref. [99]. For this, we take the final values for $B_{0}$ from there, $B_{0}=0.032+i 0.040 m_{\pi}^{-4}$. Note that this is only qualitative because we do not take the density dependence of $B_{0}$ from Ref. [99] into account but use a mean value. The values from [99] for $b_{0}$ and $b_{1}$ are $-0.013 m_{\pi}^{-1}$ and $-0.092 m_{\pi}^{-1}$, respectively. With these values and adding $\Delta b_{0}$ from Eq. (3.14) to $b_{0}$, the selfenergy is calculated according to Eq. (3.1) and plotted in Fig. 3.6, lower left panel for symmetric nuclear matter. In the same plot the $s$-wave selfenergy from fits to the bulk of pionic atom data from Refs. $[134,138,139]$ with the values given in Tab. 3.1 is shown. Both the present model and results from Ref. [99] are systematically below the phenomenological values. Neither the present model nor Ref. [99] reach the required size for the real part of $\Pi_{S}$ and thus the problem of missing repulsion persists.

The imaginary part of the pion $s$-wave selfenergy is displayed in Fig. 3.7. The result from Ref. [99] (dashed line) agrees well with the phenomenological values from Refs. [138] and [139] (gray band) whereas the present model shows a $30 \%$ discrepancy.

The differences between the results from Ref. [99] and the present calculation (dashed vs. solid line for the symmetric matter case including the pion renormalization) should be attributed to a different input used in [99], such as form factors plus the fact that extra crossed terms of $\rho^{2}$ character (smaller than those incorporated here) were also evaluated in [99]. The larger repulsion from [99] can be partly explained by the large vacuum $\left|b_{0}\right|,\left|b_{1}\right|$ used there, whereas nowadays values for $b_{0}$ compatible with zero as in Eq. (3.15) are regarded as more realistic.

### 3.3.1 Self consistent treatment of the amplitude

For the pion polarization in intermediate $\pi N$ loops, so far only the $p$-wave pion selfenergy has been taken into account. For the $s$-wave part we can include the selfenergy determined in the last section in a self consistent approach. For this, the $\pi^{-}$selfenergy $\Pi_{S}$ from Eq. (3.3) is included in the pion propagator from Eq. (3.8). Additionally, the selfenergy is resummed so that it can be included in the same way as the $p$-wave selfenergy in the pion propagator,

$$
\begin{equation*}
D_{\pi^{-}}=\frac{1}{\left(q^{0}\right)^{2}-\mathbf{q}^{2}-m_{\pi_{i}}^{2}-\Pi_{S, \pi^{-}}\left(q^{0}, \mathbf{q} ; \rho_{p}, \rho_{n}\right)-\Pi_{S}\left(q^{0}=m_{\pi}, \mathbf{q}=0 ; \rho_{p}, \rho_{n}\right)} \tag{3.17}
\end{equation*}
$$

We have approximated here the energy and momentum dependence of $\Pi_{S}$ by the static case $\left(q^{0}=m_{\pi}, \mathbf{q}=0\right)$. Solving for $\Pi_{S}$ by iteration one obtains the results in Tab. 3.3 for asymmetric matter. As in Sec. 3.3 we set $k_{F}^{n}=$ $1.157 k_{F}^{p}$ and show the results for $k_{F}^{p}=213 \mathrm{MeV}$ and $k_{F}^{p}=241 \mathrm{MeV}$ which corresponds to densities of around $\rho_{0} / 2$ and $\rho_{0}$. Three iteration steps are shown with step 0 being the selfenergy without iteration. Comparing the size of $\Pi_{S}$ from Figs. 3.6 and 3.7 with $m_{\pi}^{2}$ from the propagator, the result is expected to change only little. Indeed, the iteration converges rapidly and changes are small. At densities higher than $\rho_{0}$, self consistency would play a more important role because $\Pi_{S}$ rises rapidly as a function of $\rho$. One could think of including also the higher order corrections from the next sections in the self consistent treatment. However, given the smallness of the changes found here this would not induce any new effects. At this point one can

Table 3.3: Self consistent treatment of the $s$-wave selfenergy $\Pi_{S}\left(q^{0}=m_{\pi}, \mathbf{q}=\right.$ $0)$ for asymmetric matter. To the left, the case with $k_{F}^{p}=213 \mathrm{MeV}$, to the right $k_{F}^{p}=241 \mathrm{MeV}$. Three iteration steps are shown.

|  | $\operatorname{Re}\left(\Pi_{S}\right)_{[213 \mathrm{MeV}]}$ | $\operatorname{Im}\left(\Pi_{S}\right)_{[213 \mathrm{MeV}]}$ | $\operatorname{Re}\left(\Pi_{S}\right)_{[241 \mathrm{MeV}]}$ | $\operatorname{Im}\left(\Pi_{S}\right)_{[241 \mathrm{MeV}]}$ |
| :--- | :--- | :--- | :--- | :--- |
| Step 0 | 2470.4 | -570.8 | 3423.6 | -1233.8 |
| Step 1 | 2503.9 | -562.4 | 3491.3 | -1207.4 |
| Step 2 | 2504.3 | -562.1 | 3492.2 | -1205.8 |

improve the calculation by including the $s$-wave pion self energy not in the approximation $\left(q^{0}=m_{\pi}, \mathbf{q}=0\right)$ but with the full $q^{0}, \mathbf{q}$ dependence since it is known at least for the vacuum case that the isoscalar $\pi N$ amplitude is small at threshold but then grows rapidly at finite scattering energies. Taking only the $q^{0}$-dependence - the $\mathbf{q}$ dependence is small - the self consistent calculation delivers indeed a larger change than before, of about $10 \%$ of additional repulsion at $\rho=\rho_{0} / 2$.

### 3.4 Higher order corrections of the isovector interaction

In this section additional corrections are introduced that go beyond the medium modifications from Sec. 3.2.1, namely medium corrections affecting the kernel of the Bethe-Salpeter equation itself. In our model the kernel is given by the Weinberg-Tomozawa isovector $\pi N \rightarrow \pi N$ transition and the NLO isoscalar $\pi N \rightarrow \pi N$ transition. Considering vertex corrections of the rescattering is advantageous because it allows to include higher order corrections to the Ericson-Ericson rescattering piece, that is a large source of isoscalar strength. The corresponding $s$-wave pion selfenergy diagrams appear at high orders in density that are difficult to access through a systematic expansion of the selfenergy (see, e.g., Ref. [124]). In this section we will consider the renormalization of the Weinberg-Tomozawa interaction through vertex corrections. In Sec. 3.5 similar changes to the NLO isoscalar piece will be applied. From now on only symmetric nuclear matter will be considered.


Figure 3.8: The vertex tadpole at $1 / f_{\pi}^{4}$ (diagram 1) and the corresponding medium diagram at $1 / f_{\pi}^{6}$ (diagram $1^{\prime}$ ).

### 3.4.1 Tadpoles and off-shell contributions

In the vacuum the vertex renormalizations can be partly absorbed in the coupling constant $f_{\pi}$. In the nuclear medium, these diagrams should be explicitly taken into account. Fig. 3.8 (1) shows a tadpole diagram that involves a four-pion nucleon vertex. This term is accounted for implicitly in the free case in a renormalization of the lowest order Weinberg-Tomozawa term. However, in the medium the virtual pion can be polarized by exciting $p h$ or $\Delta h$ excitations and this leads to diagram (1') of Fig. 3.8. The difference between these two terms should be considered a genuine many body correction.

A diagram with the same geometry but within a linear $\sigma$ model has been also proposed in Ref. [127]. The $4 \pi 2 N$ vertex in diagram (1) of Fig. 3.8 is obtained from the LO chiral Lagrangian with two baryons,

$$
\begin{equation*}
\mathcal{L}_{\pi N}^{(2)}=i \operatorname{Tr}\left[\bar{B} \gamma^{\mu}\left[\Gamma_{\mu}, B\right]\right] \tag{3.18}
\end{equation*}
$$

with $\Gamma_{\mu}$ expanded up to four meson fields,

$$
\begin{equation*}
\Gamma_{\mu}=\frac{1}{32 f_{\pi}^{4}}\left[\frac{1}{3} \partial_{\mu} \Phi \Phi^{3}-\Phi \partial_{\mu} \Phi \Phi^{2}+\Phi^{2} \partial_{\mu} \Phi \Phi-\frac{1}{3} \Phi^{3} \partial_{\mu} \Phi\right] \tag{3.19}
\end{equation*}
$$

where $\Phi$ is the standard $S U(2)$ representation of the pion field, $\Phi_{11}=$ $1 / \sqrt{2} \pi^{0}, \Phi_{12}=\pi^{+}, \Phi_{21}=\pi^{-}, \Phi_{22}=-1 / \sqrt{2} \pi^{0}$. For processes with nucleon $N$ in the initial and $N^{\prime}$ in final state in diagram ( $1^{\prime}$ ), the resulting

Lagrangians $\mathcal{L}_{N N^{\prime}}$ are

$$
\begin{align*}
\mathcal{L}_{p p}=-\mathcal{L}_{n n} & =\frac{i}{48 f_{\pi}^{4}} \bar{p} \gamma^{\mu} p\left(2 \pi^{-} \pi^{+}+\left(\pi^{0}\right)^{2}\right)\left(\pi^{-} \partial_{\mu} \pi^{+}-\pi^{+} \partial_{\mu} \pi^{-}\right) \\
\mathcal{L}_{p n} & =-\frac{\sqrt{2} i}{48 f_{\pi}^{4}} \bar{n} \gamma^{\mu} p\left(2 \pi^{-} \pi^{+}+\left(\pi^{0}\right)^{2}\right)\left(\pi^{-} \partial_{\mu} \pi^{0}-\pi^{0} \partial_{\mu} \pi^{-}\right) \\
\mathcal{L}_{n p} & =\frac{\sqrt{2} i}{48 f_{\pi}^{4}} \bar{p} \gamma^{\mu} n\left(2 \pi^{-} \pi^{+}+\left(\pi^{0}\right)^{2}\right)\left(\pi^{+} \partial_{\mu} \pi^{0}-\pi^{0} \partial_{\mu} \pi^{+}\right) \tag{3.20}
\end{align*}
$$

For the process $\pi^{-} n \rightarrow \pi^{-} n$ where the external pions have on-shell momenta $k, k^{\prime}$ the diagrams (1) and (1') are given by
which can be approximated by $k^{0}+k^{0}=2 \sqrt{s}-M_{i}-M_{j}$ with $M_{i}, M_{j}, E_{i}, E_{j}$ the masses and energies of the incoming and outgoing nucleons $i$ and $j$. In Eq. (3.21) we have made the same $s$-wave projection as for the ordinary $\pi N \rightarrow \pi N$ amplitude [1, 45]. The meson propagators for diagram (1) and $\left(1^{\prime}\right)$ are given by

$$
\begin{equation*}
D_{(1)}=\frac{1}{p^{2}-m_{\pi}^{2}+i \epsilon}, \quad D_{\left(1^{\prime}\right)}=\int_{0}^{\infty} d \omega \frac{2 \omega S_{\pi}(\omega, \mathbf{p}, \rho)}{\left(p^{0}\right)^{2}-\omega^{2}+i \epsilon} \tag{3.22}
\end{equation*}
$$

where $S_{\pi}$ is the pion in-medium spectral function from Eq. (3.8). The contribution of the vertex correction can then be written as a correction to the kernel $V \rightarrow V+\delta V$ of the Bethe-Salpeter Eq. (3.4) where $\delta V=V^{\left(1^{\prime}\right)}-V^{(1)}$. This is because $D_{\left(1^{\prime}\right)}$ contains also $D_{(1)}$ and the vacuum diagram has to be subtracted explicitly.

One can see from Eq. (3.21) that $\delta V$ has explicitly order $1 / f_{\pi}^{4}$. However, in Eq. (3.22) $D_{\left(1^{\prime}\right)}-D_{(1)}$ is of order $1 / f_{\pi}^{2}$ (and higher from $p h, \Delta h$ iterations in the spectral function $S_{\pi}$ ) since the $p h$ excitation $p$-wave pion selfenergy is of order $1 / f_{\pi}^{2}$. Thus, the correction $\delta V$ is of order $1 / f_{\pi}^{6}$ and higher.

For other transitions such as $\pi^{-} p \rightarrow \pi^{-} p$ or $\pi^{-} p \rightarrow \pi^{0} n$ we observe that the contributions from Eq. (3.20) scale in the same way as the $C_{i j}$ coefficients
of the coupled channels [1,45], i.e. the overall correction is of isovector nature. This means that one can absorb the vertex correction as a common factor in the definition of $f_{\pi}$, as it appears in the Weinberg-Tomozawa term, resulting in an in-medium renormalized $f_{\pi, \text { med }}^{2}$,

$$
\begin{equation*}
\frac{b_{1}^{*}(\rho)}{b_{1 \text { free }}} \equiv \frac{f_{\pi}^{2}}{f_{\pi, \text { med }}^{2}(\rho)}=1+\frac{r}{f_{\pi}^{2}} \int_{0}^{\infty} \frac{d p p^{2}}{2 \pi^{2}}\left(-\frac{1}{2 \eta}+\int_{0}^{\infty} d \omega S_{\pi}(\omega, p, \rho)\right) \tag{3.23}
\end{equation*}
$$

with $r=-5 / 12$ and $\eta^{2}=p^{2}+m_{\pi}^{2}$. In Eq. (3.23), $b_{1}^{*}(\rho)$ and $b_{1 \text { free }}$ are the density dependent isovector term and vacuum isovector term, respectively. Although the diagrams (1) and (1') are linearly divergent, their difference which gives the medium correction is not and the $p$-integration in Eq. (3.23) is well defined. Note that casting the vertex correction as a correction to the coupling $f_{\pi}$ in Eq. (3.23) is just for convenience. E.g. for the $\pi N N p$-wave coupling, where $f_{\pi}$ also appears, such a procedure is not possible. Hence, the warning here that one should be careful not to talk about a universal renormalization of $f_{\pi}$.

In the on-shell reduction scheme of the $\pi N$ amplitude from chapter 2 [1] the on-shell and off-shell part of the $\pi N$ loop is separated and it can be shown that the off-shell part can be absorbed in the coupling of the $\pi N$ interaction [22]. However, in the nuclear medium, this is no longer the case and one has to take the off-shell part explicitly into account. In the free case, the off-shell part in the vertices of the rescattering diagram (2) and the crossed diagram (3) in Fig. 3.9 cancel the intermediate nucleon propagator, leading to a diagram with the same structure as (1) in Fig. 3.8. As an example we consider $\pi^{-} n \rightarrow \pi^{-} n$ scattering via a $\pi^{-} n$ loop as shown in diagram (2) in Fig. 3.9. The amplitude is then, with $k=k^{\prime}(p)$ the momentum of the external $\pi^{-}$(external neutron) and $q$ the momentum of the $\pi^{-}$in the loop,

$$
\begin{align*}
V_{\pi^{-} n \rightarrow \pi^{-} n}^{(2)} & =i \int \frac{d^{4} q}{(2 \pi)^{4}} \frac{M_{N}}{E(\mathbf{q})}\left(\frac{k^{0}+q^{0}}{4 f_{\pi}^{2}}\right)^{2} \\
& \times \frac{1}{k^{0}+p^{0}-q^{0}-E(\mathbf{q})+i \epsilon} \frac{1}{q^{2}-m_{\pi}^{2}+i \epsilon} . \tag{3.24}
\end{align*}
$$

Using in the heavy baryon approach $p^{0}-E(\mathbf{q}) \sim 0$ and expanding the numerator as $\left(2 k^{0}+q^{0}-k^{0}\right)^{2}=4\left(k^{0}\right)^{2}+4 k^{0}\left(q^{0}-k^{0}\right)+\left(q^{0}-k^{0}\right)^{2}$, the on-shell


Figure 3.9: Additional medium renormalizations at $1 / f_{\pi}^{6}$ and higher. From off-shell parts of direct and crossed term, diagram (2) and (3). Renormalization of the pion propagator in (4) and additional vertex correction with a loop in the $t$-channel, diagram (5). The shaded circles indicate resummed insertions of $p h, \Delta h$ pion $p$-wave selfenergies into the medium, including also $N N, N \Delta, \Delta \Delta$ short range correlations (SRC).
part is given by the $4\left(k^{0}\right)^{2}$-term. For the other terms, the baryon propagator is canceled and Eq. (3.24) reads

$$
\begin{equation*}
V_{\pi^{-} n \rightarrow \pi^{-} n}^{(2)} \approx V_{\text {on }} G_{\pi N} V_{\text {on }}+\left(2 k^{0}\right)\left(\frac{-3 i}{32 f_{\pi}^{4}}\right) \int \frac{d^{4} q}{(2 \pi)^{4}} \frac{1}{q^{2}-m_{\pi}^{2}+i \epsilon} \tag{3.25}
\end{equation*}
$$

with $V_{\text {on }}$ the usual on-shell transition $\pi^{-} n \rightarrow \pi^{-} n$ and $G_{\pi N}$ the $\pi^{-} n$ loop function. In Sec. 3.2.1 medium corrections have been applied to the first term in Eq. (3.25), the on-shell one-loop rescattering. The second term is the product of the usual $\pi N$ on-shell amplitude times a pion tadpole and has, thus, the same structure as diagram (1) of Fig. 3.8. Dressing the remaining pion tadpole in the way it is done for diagram ( $1^{\prime}$ ) of Fig. 3.8 and subtracting the vacuum tadpole can again be expressed in a renormalization of $f_{\pi}$ in Eq. (3.23), this time with $r=-3 / 8$. The crossed term in $\pi^{-} n \rightarrow \pi^{-} n$ scattering via one loop is displayed in Fig. 3.9, (3). Note that the intermediate states can in this case be $\pi^{+} n$ and $\pi^{0} p$. Evaluating the off-shell parts as before, again the structure of tadpole and on-shell scattering of diagram ( $1^{\prime}$ ) is obtained. Summing both off-shell parts from diagrams (2) and (3) the result can be cast in a modification of $f_{\pi}$ as in Eq. (3.23) with $r=+3 / 4$. The calculation is repeated for the other coupled channels $\pi^{-} p \rightarrow \pi^{-} p, \pi^{-} p \rightarrow \pi^{0} n$, and $\pi^{0} n \rightarrow \pi^{0} n$ and it is interesting to note that the off-shell parts of the oneloop amplitude have pure isovector character. This is in contrast to the on-shell one-loop amplitude with two pure isovector scatterings that results in a mixture of isovector and isoscalar contributions.

In addition we have to consider structures as in Fig. 3.9 (4), (5) at the same order in $f_{\pi}$ and density. For the tadpole pion selfenergy in diagram (4) of Fig. 3.9 we consider the process $\pi^{-} n \rightarrow \pi^{-} n$ with the external pions at momentum $k$. The $\pi \pi$ vertex is obtained from the LO chiral Lagrangian which reads up to four fields

$$
\begin{align*}
\mathcal{L}_{\pi \pi}^{(2)} & =\frac{1}{6 f_{\pi}^{2}}\left[\pi^{+} \pi^{+} \partial \pi^{-} \partial \pi^{-}-2 \pi^{+} \pi^{-} \partial \pi^{+} \partial \pi^{-}+\pi^{-} \pi^{-} \partial \pi^{+} \partial \pi^{+}\right] \\
& -\frac{1}{3 f_{\pi}^{2}}\left[\pi^{+} \pi^{-} \partial \pi^{0} \partial \pi^{0}+\pi^{0} \pi^{0} \partial \pi^{+} \partial \pi^{-}-\pi^{0} \partial \pi^{0}\left(\pi^{-} \partial \pi^{+}+\pi^{+} \partial \pi^{-}\right)\right] \\
& +\frac{m_{\pi}^{2}}{6 f_{\pi}^{2}}\left(\pi^{+} \pi^{+} \pi^{-} \pi^{-}+\pi^{+} \pi^{-} \pi^{0} \pi^{0}+\frac{\pi^{0} \pi^{0} \pi^{0} \pi^{0}}{4}\right) . \tag{3.26}
\end{align*}
$$

The $\pi^{-}$selfenergy of this external pion line consists then in charged and neutral pion loops and can be written as

$$
\begin{equation*}
(-i \Pi)=\frac{1}{6 f_{\pi}^{2}} \int \frac{d^{4} p}{(2 \pi)^{4}}\left(4\left(p^{2}-m_{\pi}^{2}\right)+4\left(k^{2}-m_{\pi}^{2}\right)+5 m_{\pi}^{2}\right) \frac{1}{p^{2}-m_{\pi}^{2}} \tag{3.27}
\end{equation*}
$$

We have written the momentum structure from the $\pi \pi$ vertex in a form where it becomes visible that the first and third term contribute to the pion wave function renormalization. They renormalize the free pion mass. Dressing the pion of the tadpole-loop as in the other diagrams gives rise to a medium correction of the pion selfenergy.

The pion tadpole (4) can appear attached to an intermediate pion in the rescattering scheme (see Fig. 3.1). In this case, the intermediate pion propagator at momentum $k$ cancels with the second term in Eq. (3.27). As a result, the tadpole is attached to the $\pi N$-vertex with the structure of diagram (1) in Fig. 3.8. The medium corrections arises then from the dressing of the pion as displayed in diagram ( $1^{\prime}$ ).

However, the pion tadpole (4) from Fig. 3.9 from can also appear in an external pion line of the rescattering displayed in Fig. 3.1. In this case, the diagram contributes to the external wave function renormalization in the medium. In other words, this is a reducible diagram, because two pieces are separated by a pion propagator. In the search for pion selfenergy terms we must only look for irreducible diagrams. However, the second term in the bracket is special because it exactly cancels the pion propagator $\left(k^{2}-m_{\pi}^{2}\right)^{-1}$ leading to a genuine irreducible diagram, that must be taken into account and is of the tadpole type of Fig. 3.8 ( $1^{\prime}$ ).

Inserting the pion tadpole in this way in internal as well as external pion lines, the corresponding $\delta V^{(4)}$ from Fig. 3.9 (4) is given by

$$
\begin{equation*}
\delta V^{(4)}=\frac{1}{k^{2}-m_{\pi}^{2}}\left(\frac{2 k^{0}}{4 f_{\pi}^{2}}\right)\left(k^{2}-m_{\pi}^{2}\right) \frac{2}{3 f_{\pi}^{2}} i \int \frac{d^{4} p}{(2 \pi)^{4}}\left(D_{\left(1^{\prime}\right)}-D_{(1)}\right) \tag{3.28}
\end{equation*}
$$

which, by analogy to the terms calculated before, can be recast into a renormalization of $f_{\pi}$ (for the purpose of the isovector term) given in Eq. (3.23) with $r=2 / 3$. In this case the isovector character is obvious as the pion selfenergy is the same for all charge states of the pion.

Note that there should be a symmetry factor of 2 as one can insert the selfenergy also at the other external pion line in diagram (4). However, if
the pion selfenergy is inserted in an intermediate $\pi N$ loop of the rescattering series, this symmetry factor is not present - each intermediate pion has only one pion selfenergy insertion. Note that for the contribution from inserting the pion tadpole in an external pion line of the rescattering scheme of Fig. 3.1 there is a factor $1 / 2$ to be taken into account in the wave function renormalization when considering the adiabatic introduction of the interaction [153]. Considering this, it is easy to see that Eq. (3.28) takes already correctly into account all multiplicity factors.

### 3.4.2 Loop corrections in the $t$-channel

For the vertex correction (5) in Fig. 3.9 we consider the process $\pi^{-} p \rightarrow \pi^{-} p$. The loop of the vertex correction is charged, because a neutral pion in the loop can not couple to the Weinberg-Tomozawa term. The diagram will be evaluated for forward scattering $k=q$ which simplifies the calculation this kind of approximation will be made several times in the following and is discussed in Sec. 3.5.4. Then, the amplitude for $\pi^{-} p \rightarrow \pi^{-} p$ is given by

$$
\begin{equation*}
(-i t)^{(5)}=\left(2 q^{0}\right)\left(-\frac{1}{2 f_{\pi}^{4}}\right) \int \frac{d^{4} p}{(2 \pi)^{4}} D^{2}(p)\left(p^{0}\right)^{2} . \tag{3.29}
\end{equation*}
$$

The pion propagators $D(p)$ are given by Eq. (3.22) for the vacuum and the medium case. As it is easy to see the one-loop correction is again of isovector type although the analytical result,

$$
\begin{equation*}
\frac{\delta V^{(5)}}{V_{\mathrm{WT}}}=\frac{2}{f_{\pi}^{2}} \int_{0}^{\infty} \frac{d p p^{2}}{2 \pi^{2}}\left(-\frac{1}{2 \eta}+\int_{0}^{\infty} d \omega \frac{2 \omega}{\eta+\omega} S_{\pi}(\omega, p, \rho)\right) \tag{3.30}
\end{equation*}
$$

is slightly different from the previously discussed diagrams and one can not include this diagram in the scheme provided by Eq. (3.23). In Eq. (3.30), $\delta V^{(5)}=V_{\text {mat }}^{(5)}-V_{\text {vac }}^{(5)}$ is the difference between in-medium and vacuum diagram; the result is normalized to the leading order Weinberg-Tomozawa vertex $V_{\mathrm{WT}}$. Vacuum and medium part for this and the other diagrams have been calculated independently; it is however a good check to take the vacuum limit of the spectral function

$$
\begin{equation*}
S_{\pi}(\omega, p, \rho) \rightarrow \frac{1}{2 \eta} \delta(\omega-\eta) \tag{3.31}
\end{equation*}
$$



Figure 3.10: Numerical results for different medium dressings of diagram (5) in Fig. 3.9.
and observe that the right hand sides of Eqs. (3.23) and (3.30) indeed vanish.
In the evaluation of Eq. (3.30) only one of the two intermediate pion propagators has been dressed. A factor of two has been supplied in order to account for the two possibilities to insert the medium dressing in either of the intermediate propagators. We can check this approximation by performing the full calculation with medium dressings in both propagators. The calculation is numerically involved; however, using a similar trick as in Eq. (2.30) the $\omega$-integrations factorize and the full result reads

$$
\begin{equation*}
\frac{\delta V_{\mathrm{full}}^{(5)}}{V_{\mathrm{WT}}}=\frac{1}{f_{\pi}^{2}} \int_{0}^{\infty} \frac{d p p^{2}}{2 \pi^{2}}\left[-\frac{1}{2 \eta}+\int_{0}^{\infty} d x\left(\int_{0}^{\infty} d \omega 2 \omega e^{-\omega x} S_{\pi}(\omega, p, \rho)\right)^{2}\right] \tag{3.32}
\end{equation*}
$$

For practical purposes, an integration up to $\mathrm{x}=0.2$ is sufficient. In Fig. 3.10 the results from Eq. (3.30) and (3.32) are compared (dashed vs. dotted line). Indeed, the dressing of one propagator is sufficient.

It is wort while to inspect the off-shell behavior of the Weinberg-Tomozawa term in diagram (5). In a recent work on $N N \rightarrow N N \pi$ [154] an interesting cancellation pattern of the NLO-diagrams has been found. As a result, the


Figure 3.11: Additional diagram which cancels the off-shell part of the $\pi \pi$ vertex of diagram (5) of Fig. 3.9. There is a complementing diagram with the $p h$ insertion in the other pion line of the loop (not drawn here).
authors show that it is enough to take the LO Weinberg-Tomozawa vertex, but on-shell because the off-shell contribution is canceled by NLO diagrams. In the medium dressing of diagram (5) the nucleon from the $p h$-insertion, together with the external nucleon line of (5) and the intermediate pion propagator, have the same structure as the LO pion production diagram in $N N \rightarrow N N \pi$. Thus, there are higher order diagrams which are not evaluated here but which will cancel the off-shell part of the Weinberg-Tomozawa (WT) vertex in diagram (5). Substituting $2 p^{0} \rightarrow 2 m_{\pi}$ for the WT-term, we obtain quite a different contribution than before,

$$
\begin{equation*}
\delta V_{\text {on-shell }}^{(5)}=\frac{m_{\pi}^{3}}{3 f_{\pi}^{4}} \int_{0}^{\infty} \frac{d p p^{2}}{2 \pi^{2}} \frac{1}{\eta}\left(-\frac{1}{4 \eta^{2}}+\int_{0}^{\infty} d \omega \frac{S_{\pi}(\omega, p, \rho)}{\omega+\eta}\right) . \tag{3.33}
\end{equation*}
$$

which is plotted in Fig. 3.10 with the solid line (after division by $V_{\mathrm{WT}}$ ). The large deviation from the previous results indicate that there is a strong off-shell dependence; in other words, high loop momenta contribute much if the full off-shell vertex is used, and this dependence implies large theoretical uncertainties. A rigorous proof of the off-shell cancellation in the present calculation is still pending.

In Fig. 3.11 another in-medium diagram is shown with a $p h$ or $\Delta h$ directly coupling to three pions. The $3 \pi N N$-interaction is obtained from an expansion of the part with $D, F$ of the LO chiral $\pi N$ Lagrangian up to three pion fields $[155,156]$. It is interesting to note that this diagram cancels precisely the off-shell part of the $\pi \pi$-vertex of diagram (5) in Fig. 3.9. For a similar


Figure 3.12: Additional vertex corrections. The two diagrams come with a relative minus sign due to the isovector character of the $\pi N s$-wave interaction.
diagram, this cancellation has been found in Ref. [155, 156]; in Sec. 3.4.4 we show explicitly the off-shell cancellation for a related family of diagrams. In any case, the on-shell condition for diagram (5), required by the new diagram from Fig. 3.11, further reduces the contribution so that the overall contribution from diagram (5) and the diagram in Fig. 3.11 is negligible.

In the following we would like to discuss another type of loop corrections in the $t$-channel, the diagrams of Fig. 3.12. The sum of the two diagrams involves the contribution $\bar{U}\left(q^{0}+k^{0}\right)-\bar{U}\left(q^{0}-k^{0}\right)$, with $\bar{U}$ the Lindhard function for only forward going bubbles. Terms involving this combination are found very small in Appendix B of [157] and we do not consider them.

We do not consider selfenergy insertions in the nucleon lines. The reason is that summing over occupied states in Eq. (3.3) corresponds to a $p h$ excitation; a local selfenergy in the particle and the hole lines cancels in the $p h$ propagator.

### 3.4.3 Vertex corrections from $\pi N N$ and $\pi N \Delta$ related terms

Next we want to take into account the renormalization of the isovector amplitude from the diagram shown in Fig. 3.13. For the process $\pi^{-} n \rightarrow \pi^{-} n$ we can have $\pi^{0}$ or $\pi^{-}$for the pion of the loop. In the first case the two $n n \pi^{0}$ vertices provide an isospin coefficient of of 1 while in the second case we have $(\sqrt{2})^{2}$ and a change of sign in the $\pi^{-} p \rightarrow \pi^{-} p$ isovector amplitude. Thus, the two pion combinations in the loop provide an isospin coefficient (1-2).


Figure 3.13: Additional vertex correction. Dressing the pion and introducing Pauli blocking for the intermediate nucleons gives a density dependent correction of the isovector amplitude.

Once again we calculate the term at zero momentum transfer (see discussion in Sec. 3.5.4) and find for the vacuum case:

$$
\begin{align*}
\frac{\delta V^{(3.13)}}{V_{\mathrm{WT}}} & =-i \int \frac{d^{4} p}{(2 \pi)^{4}}\left(\frac{D+F}{2 f_{\pi}}\right)^{2} \mathbf{p}^{2} F^{2}(p)\left(\frac{M_{N}}{E(\mathbf{q}-\mathbf{p})}\right)^{2} \\
& \times\left(\frac{1}{q^{0}-p^{0}-E(\mathbf{q}-\mathbf{p})+i \epsilon}\right)^{2} \frac{1}{p^{2}-m_{\pi}^{2}+i \epsilon} \tag{3.34}
\end{align*}
$$

where $F(q)$ is a monopole form factor accompanying the $p$-wave $\pi N N$ vertex as defined following Eq. (3.9). An explicit calculation shows that the correction of Fig. 3.13 is of isovector type. Recall that what we want are the medium corrections, hence we now substitute the free pion propagator by the one in the medium of Eq. (3.22) and include a factor of $1-n(\mathbf{p}-\mathbf{q})$ that takes into account Pauli blocking of the intermediate nucleon states. The correction is then given by the difference of the in-medium diagram minus the vacuum diagram from Eq. (3.34). Integrating $p^{0}$ one obtains

$$
\begin{align*}
\frac{\delta V^{(3.13)}}{V_{\mathrm{WT}}} & =-\int \frac{d^{3} \mathbf{p}}{(2 \pi)^{3}}\left(\frac{D+F}{2 f_{\pi}}\right)^{2} \mathbf{p}^{2} F^{2}(p)\left(\frac{M_{N}}{E(\mathbf{q}-\mathbf{p})}\right)^{2} \\
& \times\left[-\frac{1}{2 \eta} \frac{1}{\left[q^{0}-\eta-E(\mathbf{q}-\mathbf{p})+i \epsilon\right]^{2}}\right. \\
& \left.+[1-n(\mathbf{q}-\mathbf{p})] \int_{0}^{\infty} d \omega \frac{S_{\pi}(\omega,|\mathbf{p}|, \rho)}{\left[q^{0}-\omega-E(\mathbf{q}-\mathbf{p})+i \epsilon\right]^{2}}\right] \tag{3.35}
\end{align*}
$$

where $\eta^{2}=p^{2}+m_{\pi}^{2}$. The diagram exhibits two $p$-wave $\pi N N$ vertices; the same, important, short-range correlations between $p h$ and $\Delta h$, that are included in the dressed pion propagator (see Eq. (3.9)), should also be taken into account between a $p$-wave vertex of the diagram and the adjoint $p h$ or $\Delta h$ insertion in the pion propagator. The inclusion of these short range correlations (SRC) is most easily achieved by decomposing the pion selfenergy in a longitudinal and a transversal part $V_{l}$ and $V_{t}$. The matter part of the diagram after subtracting the vacuum loop, divided by the tree level isovector $V_{\mathrm{WT}}$, is given by

$$
\begin{align*}
\frac{\delta V_{\mathrm{SRC}}^{(3.13)}}{V_{\mathrm{WT}}}= & -i \int \frac{d^{4} p}{(2 \pi)^{4}}\left(\frac{M_{N}}{E_{N}(\mathbf{p})} \frac{1}{M_{N}-p^{0}-E_{N}(\mathbf{p})+i \epsilon}\right)^{2} \\
\times & \left(\theta ( k _ { F } - p ) \left[\left(V_{l}\left(p^{0}, \mathbf{p}\right)-V_{t}\left(p^{0}, \mathbf{p}\right)\right]\right.\right. \\
\times & \theta\left(p-k_{F}\right)\left[\frac{V_{l}\left(p^{0}, \mathbf{p}\right)}{1-U\left(p^{0}, \mathbf{p}\right) V_{l}\left(p^{0}, \mathbf{p}\right)}+\frac{2 V_{t}\left(p^{0}, \mathbf{p}\right)}{1-U\left(p^{0}, \mathbf{p}\right) V_{t}\left(p^{0}, \mathbf{p}\right)}\right. \\
& \left.\left.-V_{l}\left(p^{0}, \mathbf{p}\right)-2 V_{t}\left(p^{0}, \mathbf{p}\right)\right]\right) \tag{3.36}
\end{align*}
$$

where

$$
\begin{align*}
& V_{l}\left(p^{0}, \mathbf{p}\right)=\left(\frac{f_{\pi N N}}{m_{\pi}}\right)^{2} F^{2}(p)\left(\frac{\mathbf{p}^{2}}{\left(p^{0}\right)^{2}-\mathbf{p}^{2}-m_{\pi}^{2}+i \epsilon}+g^{\prime}\right) \\
& V_{t}\left(p^{0}, \mathbf{p}\right)=\left(\frac{f_{\pi N N}}{m_{\pi}}\right)^{2} F^{2}(p) g^{\prime} \tag{3.37}
\end{align*}
$$

and $U$ is the sum of $p h$ and $\Delta h$ Lindhard functions, $U=U_{N}+\left(f_{\pi N \Delta} / f_{\pi N N}\right)^{2} U_{\Delta}$. The term with $\theta\left(k_{F}-p\right)$ accounts for the small correction from Pauli blocking of the intermediate nucleons without any modification of the pion, whereas the term with $\theta\left(p-k_{F}\right)$ comes from diagrams with pion polarization through $p h$ and $\Delta h$ insertions. The $p^{0}$-integration is performed numerically. There is one technical complication resulting from the non-analyticity of the $\Delta$ width in the $\Delta h$ Lindhard function (step function $\Theta\left(\sqrt{s}-M_{N}-m_{\pi}\right)$; see Appendix of [151]). This leads to unphysical imaginary parts in $\delta V$ from the $p^{0}$-integration; the $\Delta$-width is, thus, set to zero for this diagram. The additional short-range correlations reduce the contribution from the diagram


Figure 3.14: The $\Delta$ as intermediate baryon in $\pi^{-} p \rightarrow \pi^{-} p$ scattering. The $\pi \Delta$-vertex is in $s$-wave and taken from [44].
strongly. This is in agreement with findings from Ref. [155] in the study of similar in-medium corrections for the isoscalar $N N$ interaction.

The intermediate nucleons in the diagram of Fig. 3.13 can also be excited. Close to threshold, even if it is off-shell, the $\Delta(1232)$ is important as we will see. The corresponding vertex correction is shown in Fig. 3.14. For the $\pi \Delta \rightarrow \pi \Delta$ interaction we take the Weinberg-Tomozawa interaction of isovector type from Ref. [66] in the $s$-wave approximation of Ref. [44]. Note that parity does not allow an $s$-wave vertex of the type $\pi \pi \Delta N$, thus both intermediate baryons have to be $\Delta$ (or, as already included, both have to be nucleons). The calculation is straightforward with the result for $\pi^{-} n \rightarrow \pi^{-} n$

$$
\begin{align*}
\delta V^{(3.14)} & =\frac{5 f_{\pi N \Delta}^{2}}{9 f_{\pi}^{2} m_{\pi}^{2}}\left(2 k^{0}\right) \int_{0}^{\infty} \frac{d p p^{2}}{2 \pi^{2}} F^{2}(p) p^{2} \\
& \times\left(-\frac{1}{2 \eta} \frac{1}{\left(M_{N}-\eta-E_{\Delta}\right)^{2}}+\int_{0}^{\infty} d \omega S_{\pi}(\omega, p, \rho) \frac{1}{\left(M_{N}-\omega-E_{\Delta}\right)^{2}}\right) \tag{3.38}
\end{align*}
$$

In Eq. (3.38), $M_{N}\left(M_{\Delta}\right)$ is the nucleon ( $\Delta$ ) mass, $\eta^{2}=p^{2}+m_{\pi}^{2}, k$ is the momentum of the external pions which is taken on-shell as in the other cases, $F$ is the monopole form factor for the off-shell pions at the $\pi N \Delta$ vertices, and $f_{\pi N \Delta}=2.13$ is the strong coupling of $\Delta$ to $\pi N$. An explicit evaluation of different charge states shows that the correction from this diagram is of pure isovector nature. The correction to the tree level $\pi N$ Weinberg-Tomozawa
term with amplitude $V_{\text {WT }}$ is then

$$
\begin{align*}
\frac{\delta V^{(3.14)}}{V_{\mathrm{WT}}}= & \frac{20 f_{\pi N \Delta}^{* 2}}{9 m_{\pi}^{2}} \int_{0}^{\infty} \frac{d p p^{2}}{2 \pi^{2}} F^{2}(p) p^{2}\left(-\frac{1}{2 \eta} \frac{1}{\left(M_{N}-\eta-E_{\Delta}\right)^{2}}\right. \\
& \left.+\int_{0}^{\infty} d \omega S_{\pi}(\omega, p, \rho) \frac{1}{\left(M_{N}-\omega-E_{\Delta}\right)^{2}}\right) \tag{3.39}
\end{align*}
$$

As in case of the corresponding diagram with nucleons discussed above, the introduction of additional SRC for the two $\pi N \Delta$ vertices is important; using the same projection technique as above, the result reads

$$
\begin{align*}
\frac{\delta V_{\mathrm{SRC}}^{(3.14)}}{V_{\mathrm{WT}}}= & i \frac{20}{9}\left(\frac{f_{\pi N \Delta}}{f_{\pi N N}}\right)^{2} \int \frac{d^{4} p}{(2 \pi)^{4}}\left(\frac{1}{M_{N}-p^{0}-E_{\Delta}+i \epsilon}\right)^{2} \\
\times & {\left[\frac{V_{l}\left(p^{0}, \mathbf{p}\right)}{1-U\left(p^{0}, \mathbf{p}\right) V_{l}\left(p^{0}, \mathbf{p}\right)}+\frac{2 V_{t}\left(p^{0}, \mathbf{p}\right)}{1-U\left(p^{0}, \mathbf{p}\right) V_{t}\left(p^{0}, \mathbf{p}\right)}\right.} \\
& \left.-V_{l}\left(p^{0}, \mathbf{p}\right)-2 V_{t}\left(p^{0}, \mathbf{p}\right)\right] \tag{3.40}
\end{align*}
$$

with $V_{t}$, $V_{l}$ from Eq. (3.37). Evaluating Eq. (3.39), the ratio of medium correction over tree level amplitude can reach a value of one at the highest densities $\rho \sim \rho_{0}$. However, once the additional short-range correlations are introduced, the ratio does not exceed some $40 \%$. Although this is a large reduction, a $40 \%$ correction at $\rho \sim \rho_{0}$ (and still $20 \%$ at $\rho \sim \rho_{0} / 2$ ) to the driving Weinberg-Tomozawa term, from a diagram, where all particles are off-shell, indicates a very poor convergence of the perturbative expansion and large theoretical uncertainties. The $\pi N \Delta$ vertex is defined in the $\Delta$ restframe; one has to boost the pion momentum to this frame. This leads to a reduction of the contribution by another factor of 0.68 ; yet the correction is large.

We have no control over the isoscalar $\pi \Delta$ interaction which accompanies the isovector one. It is, thus, inconsistent to consider only the isovector $\pi \Delta$ interaction as done here, and then regard the large correction provided by this diagram as a realistic effect. Therefore, the diagram will not be considered in the final results; it just gives an idea of the intrinsic uncertainties of the present medium calculation.


Figure 3.15: Triangle diagrams. The labels "off-shell" and "on-shell" refer to the $\pi \pi$-vertex. For diagram (b), we have explicitly drawn one $p h$-excitation that is otherwise, in its resummed version, represented by the gray shaded circle. Diagram (b) is complemented with a diagram with the $p h$-insertion in the other internal pion line. Diagram (c) is complemented with a diagram that has the loop on the other side of the $\pi N$ vertex.

### 3.4.4 Triangle diagrams

There is another family of diagrams displayed in Fig. 3.15. As indicated in the figure, it is enough to calculate the diagram on the right hand side with the $\pi \pi$ vertex taken at its on-shell value. This is equivalent to calculating the same diagram on-shell plus off-shell, plus the set of other diagrams displayed on the left hand side of Fig. 3.15 [155] as we show in Sec. 3.4.4.

The LO chiral Lagrangian for $\pi \pi$ interaction is given by Eq. (3.26). For the calculation, we consider first the reaction $\pi^{-} p \rightarrow \pi^{-} p$. For this configuration of external particles we can have charged or neutral pions for the loop lines. Summing both possibilities and inserting a factor of two from inserting the medium correction in either internal pion line, the medium amplitude takes the form

$$
\begin{align*}
T^{(3.15, d)} & =\frac{2 i(D+F)^{2}}{3 f_{\pi}^{4}} \int \frac{d^{4} p}{(2 \pi)^{4}} F^{2}(\mathbf{p}) \mathbf{p}^{2} \frac{M_{N}}{E(p)} \\
& \times \int_{0}^{\infty} d \omega \frac{2 \omega S_{\pi}(\omega, p, \rho)}{\left(p^{0}\right)^{2}-\omega^{2}+i \epsilon} \frac{1}{\left(p^{0}\right)^{2}-\eta^{2}+i \epsilon} \frac{1}{p^{0}+M_{N}-E(p)+i \epsilon} \\
& \times\left(3 p^{0} k^{0}+\frac{3}{4} m_{\pi}^{2}\right) \tag{3.41}
\end{align*}
$$

where the momentum of the external (internal) pions is $k(p)$, respectively. Again, we take the limit of forward scattering (see discussion in Sec. 3.5.4), and, moreover, that the external pions and nucleons are at rest. In Eq. (3.41), $E(p)=\sqrt{M_{N}^{2}+|\mathbf{p}|^{2}}$ is the nucleon energy. There is also a form factor $F(|\mathbf{p}|)$
for the off-shell pions in the $\pi N N$ vertex and the factor $M_{N} / E(p)$ from the non-relativistic reduction of the nucleon propagator.

A straightforward calculation reveals that the term $3 p^{0} k^{0}$ is of isovector nature, whereas the contribution with $3 / 4 m_{\pi}^{4}$ is isoscalar. In the heavy baryon approximation we can neglect $M_{N}-E(p)$ in the baryon propagator and in this limit the isoscalar term cancels due to symmetric integration. We are left with a purely isovector contribution in which the $p^{0}$ from the numerator cancels the baryon propagator in the heavy baryon limit; the correction is given by

$$
\begin{align*}
\delta V^{(3.15, d)} & =-\left(2 k^{0}\right) \frac{(D+F)^{2}}{f_{\pi}^{4}} \int_{0}^{\infty} \frac{d p p^{2}}{2 \pi^{2}} p^{2} F^{2}(p) \frac{M_{N}}{E(p)} \\
& \times \frac{1}{\eta}\left(-\frac{\theta\left(k_{F}-p\right)}{8 \eta^{2}}+\theta\left(p-k_{F}\right)\left[-\frac{1}{4 \eta^{2}}+\int_{0}^{\infty} d \omega \frac{S_{\pi}(\omega, p, \rho)}{\eta+\omega}\right]\right) . \tag{3.42}
\end{align*}
$$

The term with $\theta\left(k_{F}-p\right)$ accounts for the small medium correction from Pauli blocking of the intermediate nucleon but without any modification of the pion, whereas the term with $\theta\left(p-k_{F}\right)$ contains all diagrams with pion polarization. This is the amplitude for $\pi^{-} p \rightarrow \pi^{-} p$ including a factor of two according to the two possibilities to insert the medium correction in either pion line of the loop.

## Off-shell cancellation in triangle diagrams

Next, we would like to discuss the treatment of the momenta from the $\pi \pi$ vertex in the diagrams from Fig. 3.15. The off-shell cancellation concerns only the isoscalar amplitude which has been found to be small and has been neglected in the last section. However, the triangle diagrams with $\Delta$, discussed in Sec. 3.4.4, exhibit the same off-shell cancellations and hence the present considerations are of relevance.

For the external pion momentum $k$ we replace $k^{2} \rightarrow m_{\pi}^{2}$. This is in the line of our on-shell treatment of the amplitude and an approximation which we have already made at several points of the analysis. In Fig. 3.15 an off-shell cancellation for the internal pion momentum is indicated which we
discuss in the following. The off-shell cancellation concerns only the isoscalar part given by

$$
\begin{align*}
-i t^{(3.15, a)} & =\frac{2(D+F)^{2}}{3 f_{\pi}^{4}} \int \frac{d^{4} p}{(2 \pi)^{4}} F^{2}(\mathbf{p}) \mathbf{p}^{2}\left[\left(p^{2}-m_{\pi}^{2}\right)+\frac{3}{4} m_{\pi}^{2}\right] \\
& \times \frac{M_{N}}{E(p)} \frac{1}{\left(\left(p^{0}\right)^{2}-\eta^{2}+i \epsilon\right)^{2}} \frac{1}{p^{0}+M_{N}-E(p)+i \epsilon} \tag{3.43}
\end{align*}
$$

where we have have separated on-shell and off-shell part in the square brackets (compare to Eq. (3.41)). The evaluation of the diagrams (c) and the complementing diagram where the loop is before the $\pi N$ interaction is straightforward using the Feynman rules from Appendix A of [155]. The result for all transitions $\pi^{-} n \rightarrow \pi^{-} n, \pi^{-} p \rightarrow \pi^{-} p$, and $\pi^{0} n \rightarrow \pi^{0} n$ is given by (external nucleon at rest and the pion scatters forward)

$$
\begin{align*}
-i t^{(3.15, c)} & =-\frac{1}{3 f_{\pi}^{4}}(D+F)^{2} \int \frac{d^{4} p}{(2 \pi)^{2}} \vec{p}^{2} F^{2}(p) \\
& \times \frac{M_{N}}{E(p)} \frac{1}{p^{2}-m_{\pi}^{2}+i \epsilon} \frac{1}{p^{0}+M_{N}-E(p)+i \epsilon} \tag{3.44}
\end{align*}
$$

and for $\pi^{-} p \rightarrow \pi^{0} n$ the contribution vanishes. The contribution is thus of pure isoscalar nature. One sees immediately that the contribution from Eq. (3.44) cancels half of the off-shell term with $\left(p^{2}-m_{\pi}^{2}\right)$ from Eq. (3.43).

As for the diagram (b) in Fig. 3.15 the $3 \pi N N[155,156]$ vertex in the $p h$ loop is either proportional to $\vec{\sigma} \mathbf{k}$ or $\vec{\sigma} \mathbf{p}$ where $k(p)$ is momentum of the external (internal) pion. The terms proportional to $\vec{\sigma} \mathbf{k}$ do not contribute to the $s$-wave scattering. The terms proportional to $\vec{\sigma} \mathbf{k}$ give zero anyways because in this case there is an odd power of loop momenta $\mathbf{p}$ and the three momentum integral vanishes. However, the terms proportional to $\vec{\sigma} \mathbf{p}$ for diagram (b) do not vanish. Let us first consider a $\pi^{+}$running in the triangle loop and the overall process $\pi^{-} n \rightarrow \pi^{-} n$. The $p h$-insertion can be at the left or at the right internal pion line. Then,

$$
\begin{equation*}
V_{(b)}^{ \pm}=-2 \frac{1}{6 f_{\pi}^{2}} V_{\pi^{+} p n} \tag{3.45}
\end{equation*}
$$

with $V_{(b)}^{ \pm}$the vertex of $\pi^{-} p \rightarrow \pi^{+} \pi^{-} n$ given by (a) of Appendix A of [155] and $V_{\pi^{+} p n}$ is the normal $\pi^{+} p n$ vertex as it would appear in a $p h$ in the $\pi^{+}$
propagator. Eq. (3.45) contains already the two possibilities of having the $p h$-insertion at the left or right internal pion line of diagram (b).

Consider now the off-shell part of the $\pi \pi$ vertex in diagram (a), again with a $\pi^{+}$running, and insert one $p h$ in the pion propagator. Then, we have the structure

$$
\begin{equation*}
-2 \frac{i}{3 f_{\pi}^{2}}\left(p^{2}-m_{\pi}^{2}\right) \frac{i}{p^{2}-m_{\pi}^{2}+i \epsilon} V_{\pi^{+} p n} \equiv \frac{2}{3 f_{\pi}^{2}} V_{\pi^{+} p n} \tag{3.46}
\end{equation*}
$$

from the off-shell part of the $\pi \pi$-vertex together with one of the $\pi N N$ vertices of the $p h$ insertion and the intermediate propagator. Again, there is a factor of 2 from inserting the $p h$ in either of the internal pion lines. Diagrammatically, Eq. (3.46) means that the off-shell part of the $\pi \pi$ vertex cancels the intermediate pion propagator and leads to a new diagram of the structure of diagram (b). Comparing the last expression in Eq. (3.46) with Eq. (3.45) we see that Eq. (3.46) is twice as large and with opposite sign than diagram (b). Thus, diagram (b) cancels half of the off-shell part of diagram (a).

For the process $\pi^{-} n \rightarrow \pi^{-} n$ we can also have a $\pi^{0}$ running in the loop of diagram (b). The expression corresponding to Eq. (3.45) is then

$$
\begin{equation*}
V_{(b)}^{0}=-\frac{1}{2} \frac{2}{3 f_{\pi}^{2}}\left(V_{\pi^{0} p p}+V_{\pi^{0} n n}\right) \tag{3.47}
\end{equation*}
$$

There is an additional symmetry factor of $1 / 2$ that appears when contracting the neutral pion fields in the loop with the geometry of diagram (b) as it is easy to see. The expression corresponding to Eq. (3.46), i.e. a $\pi^{0}$ running in the triangle loop (a), considering only the off-shell part of the $\pi \pi$ vertex, is given by

$$
\begin{equation*}
-2 \frac{i}{3 f_{\pi}^{2}}\left(p^{2}-m_{\pi}^{2}\right) \frac{i}{p^{2}-m_{\pi}^{2}+i \epsilon}\left(V_{\pi^{0} p p}+V_{\pi^{0} n n}\right) \equiv \frac{2}{3 f_{\pi}^{2}}\left(V_{\pi^{0} p p}+V_{\pi^{0} n n}\right) . \tag{3.48}
\end{equation*}
$$

Comparing to Eq. (3.47) it becomes clear that also for a $\pi^{0}$ running in the loop, diagram (b) cancels half of the off-shell part of diagram (a).

Concluding, in any case diagram (b) cancels half of the off-shell part of diagram (a). We have already seen above that diagram (c) cancels the other half. This means that the off-shell cancellation depicted in Fig. 3.15 is indeed


Figure 3.16: Additional set of triangle diagrams with $\Delta$ (1232). The figure caption from Fig. 3.15 also applies here.
correct, and it is enough to calculate diagram (d), i.e. the triangle loop with an on-shell $\pi \pi$ vertex, where "on-shell" refers to the loop momentum. Note that we also the external pions on-shell as discussed above. Note also that the off-shell cancellation shown here is only correct for zero momentum transfer, i.e., forward scattering.

## Triangle diagrams with intermediate $\Delta$

In Fig. 3.16 another family of diagrams is displayed. The same type of of shell cancellation found for the diagrams in Fig. 3.15 holds also here as has been shown in $[155,156]$ for a similar configuration. This means with $p$ $(k)$ being the loop momentum (external momentum): $p^{2} \rightarrow m_{\pi}^{2}, k^{2} \rightarrow m_{\pi}^{2}$, $p k \rightarrow p^{0} k^{0}$ (the mixed term $p k$ is not affected by the off-shell cancellation). In the last substitution, the integration over the spatial part $\mathbf{p k}$ vanishes by symmetry.

The correction for $\pi^{-} p \rightarrow \pi^{-} p$ is, including a factor of two from inserting the medium correction in either pion line,

$$
\begin{align*}
\delta V^{(3.16, d)} & =-\frac{4 f_{\pi N \Delta}^{* 2}}{9 f_{\pi}^{2} m_{\pi}^{2}} \int_{0}^{\infty} \frac{d p p^{2}}{2 \pi^{2}} F^{2}(p) \frac{p^{2}}{\eta}\left(-\frac{m_{\pi}^{2}\left(M_{N}-E_{\Delta}-2 \eta\right)-2 \eta^{2} k^{0}}{2 \eta^{2}\left(M_{N}-E_{\Delta}-\eta\right)^{2}}\right. \\
& \left.+\int_{0}^{\infty} d \omega \frac{S_{\pi}(\omega, p, \rho)}{\omega+\eta} \frac{2 m_{\pi}^{2}\left(M_{N}-E_{\Delta}-\omega-\eta\right)-4 \omega \eta k^{0}}{\left(M_{N}-E_{\Delta}-\eta\right)\left(M_{N}-E_{\Delta}-\omega\right)}\right) . \tag{3.49}
\end{align*}
$$

Here, $E_{\Delta}$ is the $\Delta(1232)$ energy. If one uses the $\Delta$ propagator $\left(l^{0}-M_{\Delta}+\right.$ $i \Gamma / 2)^{-1}$ instead of $\left(l^{0}-E_{\Delta}+i \Gamma / 2\right)^{-1}$, then one would have to replace $E_{\Delta} \rightarrow$ $M_{\Delta}$ in Eq. (3.49). Results are similar anyways.

As an explicit calculation shows, the term with $2 m_{\pi}^{2}$ inside the $\omega$-integral is of isoscalar nature whereas the term with $k^{0}$ is of isovector nature. This
means that the term with $2 m_{\pi}^{2}$ is the same for all channels of our coupled channel approach whereas one has to multiply $\delta V^{(3.16, d)}$ by $-1,-\sqrt{2}$, and 0 for $\pi^{-} n \rightarrow \pi^{-} n, \pi^{-} p \rightarrow \pi^{0} n$, and $\pi^{0} n \rightarrow \pi^{0} n$, respectively. As nucleon and $\Delta$ mass are non-degenerate, the isoscalar part does not cancel as it had been the case for the diagrams of Fig. 3.15.

## SRC and results for the triangle diagrams

In Eq. (3.36) we have already seen an example for additional SRC: Between the nucleon emitting a pion and the $p h$ or $\Delta h$ medium insertions in the pion propagator, there are also SRC. If this is the case for both ends of the pion propagator, the substitution of the pion propagator $D_{\pi}$ is given by the projection technique employed in Eq. (3.36). In the triangle diagrams, there is only one side of the pion line affected and the additional SRC can be cast in a substitution of the in-medium pion propagator $D_{\left(1^{\prime}\right)}$ from Eq. (3.22) according to

$$
\begin{equation*}
D_{\left(1^{\prime}\right)} \rightarrow D_{\left(1^{\prime}\right)} \times \frac{1}{1-g^{\prime}\left(\frac{D+F}{2 f_{\pi}}\right)^{2} F^{2}(p) U} \tag{3.50}
\end{equation*}
$$

as an explicit calculation shows. This does not affect the off-shell cancellation behavior discussed in Sec. 3.4.4. Remember that we always use the same Migdal parameter for $N N$ and $N \Delta$ SRC. Although the contributions from Eqs. $(3.42,3.49)$ are sizable, they are suppressed by a factor of 5 by the additional SRC from Eq. (3.50). In the final numerical results they introduce a small correction.

### 3.4.5 Isovector correction from the NLO $\pi N$ interaction

The Weinberg-Tomozawa term is also renormalized by higher orders in the isoscalar $\pi N$ interaction. A correction of this type comes from the nucleon tadpole in the pion propagator shown in Fig. 3.17. The $\pi N$ interaction of the tadpole is from the $s$-wave isoscalar interaction from the NLO chiral Lagrangian whereas the other $\pi N$ interaction is given by the WeinbergTomozawa term. The nucleon tadpole with isovector interaction vanishes in symmetric nuclear matter.


Figure 3.17: Nucleon tadpole in the pion propagator with NLO chiral $\pi N$ interaction.

The isoscalar $\pi N$ interaction is given by

$$
\begin{align*}
t_{\pi N} & =\frac{4 c_{1}}{f_{\pi}^{2}} m_{\pi}^{2}-\frac{2 c_{2}}{f_{\pi}^{2}}\left(k^{0}\right)^{2}-\frac{2 c_{3}}{f_{\pi}^{2}} k^{2} \\
& =\left(\frac{4 c_{1}}{f_{\pi}^{2}} m_{\pi}^{2}-\frac{2 c_{2}}{f_{\pi}^{2}} \omega(k)^{2}-\frac{2 c_{3}}{f_{\pi}^{2}} m_{\pi}^{2}\right)-\frac{2 c_{2}+2 c_{3}}{f_{\pi}^{2}}\left(k^{2}-m_{\pi}^{2}\right) \\
& \equiv t_{\pi N}^{\text {on }}+t_{\pi N}^{\mathrm{off}} \tag{3.51}
\end{align*}
$$

Here the interaction has been separated into on-shell and an off-shell part [158] (the latter term with $\left(k^{2}-m_{\pi}^{2}\right)$ ). For the nucleon tadpole in Fig. 3.17 and considering first the off-shell part, the selfenergy is given by $\Pi=$ $-\left(2 c_{2}+2 c_{3}\right)\left(k^{2}-m_{\pi}^{2}\right) \rho / f_{\pi}^{2}$. The entire diagram is then given by

$$
\begin{equation*}
V^{(3.17)}=-t_{\pi N \rightarrow \pi N} \frac{2 c_{2}+2 c_{3}}{f_{\pi}^{2}}\left(k^{2}-m_{\pi}^{2}\right) D(k) \rho \tag{3.52}
\end{equation*}
$$

where $t_{\pi N \rightarrow \pi N}$ is the isovector interaction from the Weinberg-Tomozawa term and $D(k)$ the intermediate pion propagator which cancels the term $\left(k^{2}-m_{\pi}^{2}\right)$ from the isoscalar vertex. This means a vertex renormalization by a similar mechanism as we have already seen for diagram (4) in Fig. 3.9 and

$$
\begin{equation*}
\frac{\delta V^{(3.17)}}{V_{\mathrm{WT}}}=-\frac{2 c_{2}+2 c_{3}}{f_{\pi}^{2}} \rho \tag{3.53}
\end{equation*}
$$

For the numerical evaluation we use the values of the $c$-coefficients of the fit $2^{\dagger}$ from Ref. [93],

$$
\begin{align*}
& c_{1}=-0.35 \pm 0.1 \mathrm{GeV}^{-1}, \\
& c_{2}=-1.49 \pm 0.67 \mathrm{GeV}^{-1}, \\
& c_{3}=0.93 \pm 0.87 \mathrm{GeV}^{-1} \tag{3.54}
\end{align*}
$$

It would be more consistent to use the values of the present fit in Tab. 3.2 instead. However, in the present model we have only access to $c_{2}$ and the combination $2 c_{1}-c_{3}$. In Eq. (3.53) the $c$-coefficients are combined in a different way, and we have to resort to the values of [93]. In any case the values from Eq. (3.54) are compatible within errors with ours from Tab. 3.2 (see also Sec. 2.4.1). For an estimate of the theoretical error, we can also use the $c$-values from fit $2^{*}$ instead of $2^{\dagger}$ [93]. This induces a theoretical error of the order of $20 \%$ for the contribution which by itself is smaller than other diagrams.

As for the on-shell part of the interaction in Eq. (3.51) we notice that the intermediate pion propagator, between the nucleon tadpole and the Weinberg-Tomozawa vertex, does not cancel. This means that the on-shell nucleon tadpole contributes to the pion selfenergy and not to the vertex renormalization. However, compared to the $p$-wave pion selfenergy in the rescattering loops, the $s$-wave selfenergy is small and can be neglected as we have also seen in the self consistent calculation in Sec. 3.3.1.

### 3.4.6 Results for the isovector renormalization

For all corrections evaluated in this section 3.4, the vertex corrections can be recast as a correction to the isovector interaction strength $b_{1}^{*}(\rho)$ or, in other words, an in-medium change of $f_{\pi}$. Note that we refer to the $f_{\pi}$ that appears in the Weinberg-Tomozawa term of Eq. (2.7); we do not claim a universal change of $f_{\pi}$ in the nuclear medium (see also the caveat following Eq. (3.23)). For example, Eq. (3.23) gives the renormalization of $b_{1 \text { vac }} / b_{1}^{*}(\rho)$ from the diagrams (1) to (4) of Figs. 3.8,3.9 with an overall value of $r=1$. Including these diagram as well as all other isovector corrections found, the in-medium change of $b_{1}$ is plotted in Fig. 3.18. The result in Fig. 3.18 is given at $\Lambda=0.9 \mathrm{GeV}$ for the monopole form factor that appears in the $p h$ and $\Delta h$ pion selfenergies of the vertex corrections (see Eqs. (3.9,3.37,3.50)). The dependence on $\Lambda$ is considerable and at $\Lambda=1 \mathrm{GeV} f_{\pi, \text { med }}$ is reduced by another $10 \%$. With the decrease of $f_{\pi}$ in the medium the isovector $\pi N$ interaction effectively increases, in quantitative agreement with a recent analysis of deeply bound pionic atoms [137] and the phenomenological fit


Figure 3.18: In-medium isovector $b_{1}^{*}(\rho)$ compared to the vacuum isovector term $b_{1}$ free. The gray band from Suzuki et al. [137] is from a phenomenological fit, as well as the point from Nieves et al. [134]. Also shown are chiral calculations from Meißner et al. [136] and Weise et al.(Friedman et al.) [125, 129].
from [134], whereas there is some discrepancy with Refs. [125, 129, 136]. In the Ericson-Ericson rescattering piece, the isovector interaction generates isoscalar contributions. Thus, from the change of the isovector in Fig. 3.18 we expect also a significant change of the overall pion $s$-wave selfenergy in the final results in Sec. 8.8.

### 3.5 Renormalization of the NLO isoscalar term in $\pi N$ scattering

The model from chapter $2[1]$ for the $\pi N$ interaction in the vacuum has two sources for isoscalar contributions: one is the NLO, point-like, interaction from Eq. (2.9) and the other one comes from the rescattering of the pion generated in the Bethe-Salpeter equation. In fact, the latter is quite large, $b_{g}=442 \cdot 10^{-4} m_{\pi}^{-1}$ (see Tab. 2.7). This large contribution is partly canceled by the NLO contact term from Eq. (2.9) that is $b_{c}=-336 \cdot 10^{-4} m_{\pi}^{-1}$, leading to a final value of $b_{0}=-28 \cdot 10^{-4} m_{\pi}^{-1}$.

For the application of the model in nuclear matter this partial cancellation has consequences. Renormalizing the isovector strength changes the in-medium isoscalar term through the Ericson-Ericson rescattering piece, as we have seen in the last section. The sum of this term and the point-like NLO interaction will then not show the partial cancellation of the vacuum case any more. It is therefore important to treat the NLO isoscalar term on the same footing as the isovector renormalization.

The diagrams from the last section will be the guideline for the renormalization of the NLO isoscalar. We do not redraw these diagrams, but the $\pi N \rightarrow \pi N$ contact interactions is now given by the NLO isoscalar term instead of the LO Weinberg-Tomozawa isovector term.

### 3.5.1 Tadpole and off-shell contributions

We start with the pion tadpole (1) from Fig. 3.8. The NLO Lagrangian has to be expanded to four pion pion lines in order to provide the $4 \pi 2 N$ vertex needed for this diagram. As shown in the following, to this end we can utilize the in-medium Lagrangian derived in Ref. [159] (see also [158,160]) by taking the mean-field approximation for the nucleon field. The terms with a medium correction $\rho$ of the nuclear density read

$$
\begin{equation*}
\langle\mathcal{L}\rangle=\frac{1}{2} \rho\left(c_{3} \operatorname{Tr}\left[\partial U \partial U^{\dagger}\right]+c_{2} \operatorname{Tr}\left[\partial_{0} U \partial_{0} U^{\dagger}\right]+c_{1} \operatorname{Tr}\left[U^{\dagger} \chi+\chi^{\dagger} U\right]\right) \tag{3.55}
\end{equation*}
$$

by keeping only the isoscalar terms which are parametrized in terms of $c_{1}, c_{2}, c_{3}$. Expanding this term up to four external pion lines leads to a $\pi \pi$ vertex with a nucleon tadpole as displayed in Fig. 3.19 to the left. Contracting two of the pion fields leads to a diagram that appears as a pion selfenergy with a pion tadpole and a nucleon tadpole as displayed in the center of Fig. 3.19. For a $\pi^{-}$this selfenergy is given by
$\Pi^{(3.19)}=\frac{2 \rho}{3 f_{\pi}^{4}} i \int \frac{d^{4} p}{(2 \pi)^{4}} D(p)\left(2 c_{3}\left(k^{2}+p^{2}\right)+2 c_{2}\left(\left(k^{0}\right)^{2}+\left(p^{0}\right)^{2}\right)-\frac{5}{2} c_{1} m_{\pi}^{2}\right)$
with the pion propagator $D(p)$ and the momenta as assigned in Fig. 3.19, center. Note the factor of $\rho$ of the nuclear density in Eq. (3.56): the selfenergy is of the type $t \rho$ with a matrix element $t$ that can be extracted by opening


Figure 3.19: Pion-pion interaction with a nucleon tadpole from the NLO $\pi N$ interaction (left). Closing one pion line produces a pion selfenergy of the $t \rho$ type (center). This $t$ is a $\pi N$ vertex correction (right), with the same geometry as diagram (1) in Fig. 3.8.
the nucleon line of the nucleon tadpole, meaning the division of Eq. (3.56) by $\rho$. This is displayed to the right in Fig. 3.19. The resulting diagram is a $\pi N$ vertex correction with the same geometry as diagram (1) in Fig. 3.8 but using the NLO isoscalar interaction for the $4 \pi 2 N$ vertex. As a result, the isoscalar vertex correction for the coupled channels $i, j$ reads

$$
\begin{align*}
\delta V_{i j}^{(3.19)} & =\delta_{i j} \frac{m_{\pi}^{2}}{3 f_{\pi} 4} \int_{0}^{\infty} \frac{d p p^{2}}{2 \pi^{2}} \int_{0}^{\infty} \frac{d \omega}{\omega}\left(S_{\pi}(\omega, p, \rho)-\frac{\delta(\eta-\omega)}{2 \eta}\right) \\
& \times\left[4 c_{3}\left(m_{\pi}^{2}+\omega^{2}-p^{2}\right)+4 c_{2}\left(\left(k^{0}\right)^{2}+\omega^{2}\right)-5 c_{1} m_{\pi}^{2}\right] \tag{3.57}
\end{align*}
$$

The in-medium correction from Eq. (3.57) has to be added to the kernel of the Bethe-Salpeter equation (3.4). The term with $S_{\pi}$ corresponds to the dressed pion propagator as in diagram ( $1^{\prime}$ ) of Fig. 3.8 and the term with $\delta(\eta-\omega)$ to the vacuum diagram (1) which is subtracted. Note that the $4 \pi 2 N$ vertex in Eq. (3.56) is quadratic in the loop momentum $p$. Thus, there will be a large contribution to the integrand from high values of $p$ beyond the range of applicability of the chiral expansion. One possible simplification is to take the on-shell, at rest, value for the vertex $p^{2}=\left(p^{0}\right)^{2}=m_{\pi}^{2}$ in the bracket in Eq. (3.56). With this replacement the square bracket in Eq. (3.57) is substituted by

$$
\begin{equation*}
[\cdots] \rightarrow\left[m_{\pi}^{2}\left(8 c_{3}+4 c_{2}-5 c_{1}\right)+c_{2}\left(2 k^{0}\right)^{2}\right] . \tag{3.58}
\end{equation*}
$$

From the theoretical point of view, this is the more realistic choice; however, potential off-shell contributions introduce unknown theoretical uncertainties
at this point.
For the diagrams (2) and (3) from Fig. 3.9 one or two of the $\pi N$ vertices can be given by the NLO isoscalar interaction: the Bethe-Salpeter equation (3.4) iterates the kernel and allows for any combination of isoscalar (see Eq. (2.9)) and isovector vertices (see Eq. (2.7)) in the rescattering series. For two iterated Weinberg-Tomozawa vertices we have already determined the off-shell contributions from the direct and crossed term; for iterated isoscalar interactions we should do in principle the same. However, the contributions are smaller and we neglect them. This can be seen as following: the strength of the isoscalar interaction is $b_{c}=-336 \cdot 10^{-4} m_{\pi}^{-1}$ which is around one third of the isovector strength. A combination of two isoscalar vertices in $\pi N$ rescattering would, thus, approximately lead to an off-shell contribution nine times smaller than the off-shell effect from the combination of two isovector vertices studied before, and we can safely neglect it. A combination of one isoscalar vertex and one isovector vertex results in an overall isovector interaction and is of no interest in the present case where only symmetric nuclear matter is considered.

The diagram (4) from Fig. 3.9 renormalizes the NLO isoscalar interaction in the same way as affecting the isovector interaction studied before. The renormalization can be cast in a change of $f_{\pi}$ as in Eq. (3.23), with $r=2 / 3$ as before.

### 3.5.2 Loop corrections in the $t$-channel

In diagram (5) of Fig. 3.9 the Weinberg-Tomozawa term can be replaced by the NLO isoscalar interaction. We will consider on-shell and off-shell part of this interaction, given in Eq. (3.51), separately.

## On-shell part

We consider the process $\pi^{-} p \rightarrow \pi^{-} p$ and a charged pion running in the loop. Then, the contribution from the on-shell part of the NLO Lagrangian reads

$$
\begin{align*}
-i T_{\mathrm{NLO}}^{(5), \text { on }} & =\int \frac{d^{4} p}{(2 \pi)^{4}}\left(-i t_{\pi N}\right) i D(p) i D(p)\left(-i t_{\pi \pi}\right) \\
& =\frac{1}{3 f_{\pi}^{2}} \int \frac{d^{4} p}{(2 \pi)^{4}} \frac{4 c_{1} m_{\pi}^{2}-2 c_{3} m_{\pi}^{2}-2 c_{2} \eta^{2}}{f_{\pi}^{2}} \\
& \times \frac{1}{\left(p^{2}-m_{\pi}^{2}+i \epsilon\right)^{2}}\left(p^{2}+6 p q+q^{2}-2 m_{\pi}^{2}\right) \tag{3.59}
\end{align*}
$$

with $\eta^{2}=m_{\pi}^{2}+\vec{p}^{2}$ and $D$ being the pion propagator as before. We have taken here already the limit of forward scattering. The last term comes from the $\pi \pi$ vertex with $q$ the momentum of the external pions. As always, we take the on-shell value for the external pions $q^{2}=m_{\pi}^{2}$ and can substitute the last term in Eq. (3.59) by $\left(p^{2}-m_{\pi}^{2}\right)$, taking into account that the mixed term $6 p q$ vanishes due to symmetric integration. As in Eq. (3.30) we dress only one of the propagators in order to stay in line with the other corrections evaluated (then, a multiplicity factor of 2 appears according to the two possibilities of inserting the in-medium correction in either pion propagator of the loop). The propagators in Eq. (3.59) are then given by $D^{2} \rightarrow D_{(1)} D_{\left(1^{\prime}\right)}$ in Eq. (3.59) for the medium part and $D^{2} \rightarrow D_{(1)}^{2}$ for the vacuum part with $D_{(1)}, D_{\left(1^{\prime}\right)}$ from Eq. (3.22). For the process $\pi^{-} p \rightarrow \pi^{-} p$ the pion in the loop can also be a $\pi^{0}$. Taking this into account, integrating the $p^{0}$-component, and subtracting the vacuum part from the medium part results in

$$
\begin{align*}
\delta V_{\mathrm{NLO}}^{(5), \text { on }} & =\frac{2}{3 f_{\pi}^{4}} \int_{0}^{\infty} \frac{d p p^{2}}{2 \pi^{2}}\left(4 c_{1} m_{\pi}^{2}-2 c_{3} m_{\pi}^{2}-2 c_{2} \eta^{2}\right) \\
& \times\left[-\frac{5 m_{\pi}^{2}+8 p^{2}}{8 \eta^{3}}+\int_{0}^{\infty} d \omega \frac{2 S_{\pi}(\omega, p, \rho)}{\eta+\omega}\left(\omega+\frac{4 p^{2}+m_{\pi}^{2}}{4 \eta}\right)\right] \tag{3.60}
\end{align*}
$$

It is easy to see that this correction is of of isoscalar type. The amplitude in Eq. (3.60) has a term $2 c_{2} \eta^{2}$ which introduces additional powers of $p$ in the integration. Although the integral is still convergent, the value of the
$c_{2}$ coefficient is only valid for small momenta, where it has been determined in fits to low energy $\pi N$ scattering data. The term introduces unrealistic contributions as it gives a large weight to high momenta. Therefore, we replace $2 c_{2} \eta^{2} \rightarrow 2 c_{2} m_{\pi}^{2}$ in Eq. (3.60), taking thus the threshold value of the NLO scalar interaction.

However, there is another off-shell cancellation, the one of the $\pi \pi$ vertex. This is due to the additional diagram shown in Fig. 3.11. In Sec. 3.4.2 we have seen that this diagram cancels the off-shell part of the $\pi \pi$ vertex and in Sec. 3.4.4 the off-shell cancellation is shown explicitly for a related family of diagrams. Also here, with the $\pi N$ interaction given by the NLO chiral Lagrangian, the same cancellation occurs. The vertex correction is then given by

$$
\begin{align*}
\delta V_{\mathrm{NLO}}^{(5), \text { on }} & =-\frac{m_{\pi}^{4}}{f_{\pi}^{4}}\left(4 c_{1} m-2 c_{3}-2 c_{2}\right) \\
& \times \int_{0}^{\infty} \frac{d p p^{2}}{2 \pi^{2}} \frac{1}{\eta}\left[-\frac{1}{4 \eta^{2}}+\int_{0}^{\infty} d \omega \frac{S_{\pi}(\omega, p, \rho)}{\eta+\omega}\right] . \tag{3.61}
\end{align*}
$$

This is a tiny correction to the isoscalar renormalization and neglected in the final numerical results.

## Off-shell part

The off-shell part of the NLO isoscalar interaction is renormalized in a similar way as before. The vacuum amplitude for $\pi^{-} p \rightarrow \pi^{-} p$ is in this case given by

$$
\begin{align*}
-i T_{\mathrm{NLO}}^{(5), \text { off }} & =-2 \frac{2 c_{2}+2 c_{3}}{f_{\pi}^{2}} \frac{1}{3 f_{\pi}^{2}} \int \frac{d^{4} p}{(2 \pi)^{4}} \frac{1}{\left(p^{2}-m_{\pi}^{2}+i \epsilon\right)^{2}}\left(p^{2}-m_{\pi}^{2}\right) \\
& \times\left[\left(p^{2}+6 p q+q^{2}-2 m_{\pi}^{2}\right)+\left(p^{2}+q^{2}-\frac{1}{2} m_{\pi}^{2}\right)\right] \tag{3.62}
\end{align*}
$$

where the external pions are again at momentum $q$ and $\left(p^{2}-m_{\pi}^{2}\right)$ is the off-shell part of the NLO isoscalar vertex which cancels one of the propagators. In the square brackets, the contributions from having a charged pion or a neutral one are denoted separately. It is easy to see that the overall contribution is again of isoscalar nature.

In the term $\left(p^{2}+6 p q+q^{2}-2 m_{\pi}^{2}\right)$ from the $\pi \pi$ vertex, the mixed product cancels due to symmetric integration. The $p^{0}$-integration is straightforward for both vacuum and medium loop. The resulting medium correction reads

$$
\begin{align*}
\delta V_{\mathrm{NLO}}^{(5), \text { off }} & =-4 \frac{2 c_{2}+2 c_{3}}{3 f_{\pi}^{4}} \int_{0}^{\infty} \frac{d p p^{2}}{2 \pi^{2}} \\
& \times\left[-\frac{3 m_{\pi}^{2}}{8 \eta}+\int_{0}^{\infty} d \omega S_{\pi}(\omega, p, \rho)\left(\omega^{2}-p^{2}-\frac{1}{4} m_{\pi}^{2}\right)\right] . \tag{3.63}
\end{align*}
$$

The factor ( $\omega^{2}-p^{2}-m_{\pi}^{2} / 4$ ) comes from on- and off-shell part of the $\pi \pi$ vertex. As before, the off-shell part cancels with the diagram from Fig. 3.11 (using, of course the same interaction for the $\pi N$-vertex). Keeping only the on-shell part of the $\pi \pi$ interaction, the final result can be easily obtained and is given in Eq. (3.63) with the replacement

$$
\begin{equation*}
\left(\omega^{2}-p^{2}-\frac{1}{4} m_{\pi}^{2}\right) \rightarrow \frac{3}{4} m_{\pi}^{2} \tag{3.64}
\end{equation*}
$$

### 3.5.3 Further renormalizations of the isoscalar $\pi N$ interaction

Next, the diagram from Fig. 3.13 is considered which is now given by two $\pi N N$ vertices and one $\pi^{-} N \rightarrow \pi^{-} N$ transition from the NLO isoscalar interaction. The only change with respect to the previous result from Eq. (3.34) is a change in isospin factors: Instead of a combination of the form $(1-2)$ one obtains $(1+2)$ so that the new vertex correction is given by $\delta V_{\mathrm{NLO}}^{(3.13)} / V_{\mathrm{NLO}}=-3 \delta V_{\mathrm{SRC}}^{(3.13)} / V_{\mathrm{WT}}$ with $\delta V_{\mathrm{SRC}}^{(3.13)} / V_{\mathrm{WT}}$ given in Eq. (3.36).

Considering the nucleon tadpole in Fig. 3.17 it is clear that this correction renormalizes in the same way the isoscalar as the isovector interaction because the tadpole factorizes with the $\pi N$ amplitude. Thus, the renormalization of the isoscalar amplitude $\delta V_{\mathrm{NLO}}^{(3.17)} / V_{\mathrm{NLO}}$ is again given by the right side of Eq. (3.53).

When renormalizing the Weinberg-Tomozawa interaction, we have also considered the diagrams in Fig. 3.12. They have been found small as argued at the end of Sec. 3.4.1 due to the occurrence of the difference of Lindhard
functions $\bar{U}\left(q^{0}+k^{0}\right)-\bar{U}\left(q^{0}-k^{0}\right)$. The minus sign was a consequence of the isovector nature of the Weinberg-Tomozawa vertex. However, if one replaces the Weinberg-Tomozawa vertices at the bottom of the diagrams in Fig. 3.12 with isoscalar ones, one obtains the combination $\bar{U}\left(q^{0}+k^{0}\right)+\bar{U}\left(q^{0}-k^{0}\right)$. This might result in a significant contribution. However, as the following argument shows, we should not consider this contribution as it would mean double counting: We consider the diagrams in Fig. 3.12 with the nucleon line closed. This corresponds to a contribution to the $s$-wave pion selfenergy. However, the resulting selfenergy, let it be $\Pi_{(3.12)}$, is already generated in a different piece: Imagine the Ericson-Ericson one-loop rescattering of two isovector interactions. Insert now a nucleon tadpole in the intermediate pion line as displayed in Fig. 3.17. Joining the external nucleon lines in this rescattering diagram generates again the pion selfenergy $\Pi_{(3.12)}$. For similar reasons of double counting we also discard vertex corrections that occur when one replaces the vertices at the bottom of Fig. 3.12 with the NLO isoscalar interaction and additionally also the other $\pi N$ vertices.

Finally, there is a small vertex correction to the isoscalar interaction from the triangle diagrams with intermediate $\Delta$ which we have already discussed and evaluated in Sec. 3.4.4, Eqs. (3.49,3.50).

The sum of the isoscalar corrections calculated in this section results in an increase of the NLO interaction similar to that of the isovector shown in Fig. 3.18 and also similar in size.

### 3.5.4 Finite momentum transfer in vertex corrections

In the calculation of several vertex corrections we have assumed that the external pions and nucleons are at rest. This approximation is good when rescattering is ignored; the pion in nuclear matter is practically at rest and only Fermi motion and binding energy introduce small corrections to the assumption. However, in rescattering loops (see, e.g. Fig. 3.1) the off-shell pion can have any $k^{\prime 0}, \mathbf{k}^{\prime}$. For an estimate, we consider the diagram in Fig. 3.14 which gives a large contribution (which, however, finally was not included in the numerical results due to the reasons given in Sec. 3.4.3). In
the threshold approximation, one $\Delta$ propagator is given by

$$
\begin{equation*}
\frac{1}{q^{0}-p^{0}-E_{\Delta}(\mathbf{q}-\mathbf{p})} \approx \frac{1}{M-\omega(\mathbf{p})-E_{\Delta}(\mathbf{p})} \tag{3.65}
\end{equation*}
$$

Consider now rescattering, i.e. the incoming pion at momentum $k^{0}=m_{\pi}$, $\mathbf{k}=\mathbf{0}$ and the incoming nucleon are approximately at threshold, whereas the outgoing pion at $k^{\prime}$ and the outgoing nucleon are inside a rescattering loop. Then, there is a momentum mismatch and the above $\Delta$ propagator is given by

$$
\approx \frac{1}{q^{0}-p^{0}+k^{0}-k^{\prime 0}-E_{\Delta}\left(\mathbf{q}-\mathbf{p}+\mathbf{k}-\mathbf{k}^{\prime}\right)} \frac{1}{M-\omega(\mathbf{p})+m_{\pi}-\omega\left(\mathbf{k}^{\prime}\right)-E_{\Delta}\left(\mathbf{p}+\mathbf{k}^{\prime}\right)} .
$$

Take a high momentum of $\left|\mathbf{k}^{\prime}\right|=1 \mathrm{GeV}$ for the rescattering loop and an average momentum for the loop of the diagram, $|\mathbf{p}|=400 \mathrm{MeV}$. Then, Eq. (3.65) takes the value $-1 / 780 \mathrm{MeV}^{-1}$, whereas Eq. (3.66) takes $1 / 1942 \mathrm{MeV}^{-1}$ (averaged over the angle) which is 2.5 times smaller. Under the same conditions but for the diagram in Fig. 3.13, the factor is even 3.5. Making similar estimates we find for the loop correction in the $t$-channel from Fig. 3.9 (5) a reduction of a factor of around 2 when the momenta are taken as above. Although a loop momentum of 1 GeV is certainly high, we have shown that the momentum mismatch between external particles and the intermediate states in the loop in any case softens the vertex correction. For loop momenta close to threshold, this reduction will be weaker, of course. A more elaborate calculation can clarify this point but note that this involves the evaluation of non-factorizing multiple loops.

### 3.6 Numerical results

In the last sections 3.4 and 3.5, vertex corrections for both the isovector and the isoscalar interaction have been found. Together with the in-medium $\pi N$ loops $G_{\pi N}$, shown in Eq. (3.6), we obtain a new $\pi N \rightarrow \pi N$ transition $T$ in Eq. (3.4). Integrating over the nucleons of the Fermi seas according to Eq. (3.3), the $s$-wave pion selfenergy is evaluated. The results are for symmetric
nuclear matter as the calculations from Secs. 3.4 and 3.5 are performed in this limit.

The contributions to the $s$-wave pion self energy can be ordered in powers of the density. We restrict the calculation up to order $\rho^{2} \rho^{1 / 3}$ which corresponds to the order of the previous results from Sec. 3.3: the pion $p$-wave polarization counts with $\rho$ whereas the closing of the external nucleon line, as indicated in Fig. 3.4, corresponds to another power of $\rho$. The Pauli blocking of the intermediate nucleon in the $\pi N$ rescattering is a power $\rho^{1 / 3}$ effect as can be seen from Eq. (3.14). Of course, higher powers are also contained: from the multiple rescattering generated by the BSE equation (3.4) on one hand, and the resummation of $p h, \Delta h$ pion selfenergies from Eq. (3.9) on the other hand; however, these higher order corrections are small.

The vertex diagrams from Secs. 3.4, 3.5, which are at order $\rho$ through the $p$-wave pion polarization, introduce additional corrections to $\Pi_{S}$. Closing the nucleon line (order $\rho$ ) for these diagrams, corrections of order $\rho^{2}$ are obtained which are added to the pion $s$-wave self energy $\Pi_{S}$. Besides the corrections discussed so far, there are other diagrams at $\rho^{2} \rho^{1 / 3}$ : they consist of the Ericson-Ericson rescattering piece from Fig. 3.3 without any pion polarization but with the order $\rho$ vertex corrections from Secs. 3.4 and 3.5 for exactly one of the Weinberg-Tomozawa $\pi N$ interactions. These contributions are most easily included by multiplying the rescattering term with $\left(b_{1}^{*}(\rho) / b_{1, \text { free }}-1\right)>$ 0 , with $b_{1}^{*}(\rho) / b_{1, \text { free }}$ from Fig. 3.18.

The corrections evaluated in this way include the set of diagrams from Ref. [124] but furthermore provide many additional corrections as can be seen in Secs. 3.4, 3.5. Summing all contributions, the pion $s$-wave self energy up to order $\rho^{2} \rho^{1 / 3}$ is plotted in Fig. 3.20 with the black solid lines. For the pion three-momentum, we have taken a typical value of $\mathbf{p}=50 \mathrm{MeV}$ although $\Pi_{S}$ depends only weakly on $\mathbf{p}$. The gray band shows the area between the experimental fits to pionic atoms from Refs. [138, 139] (see also Fig. 3.6) whereas the dark band represents the phenomenological fit from [134]. The present result for the external pion energy $k^{0}=m_{\pi}$ stays some $30 \%$ below the phenomenological fit. Note that at the order $\rho^{2} \rho^{1 / 3}$ considered here, the imaginary part $\operatorname{Im} \Pi_{S}$ is the same as in Fig. 3.7.

In Ref. [126] it has been claimed, that a possible way of understanding


Figure 3.20: The $s$-wave pion selfenergy in nuclear matter up to order $\rho^{2} \rho^{1 / 3}$, for three pion energies ( $p_{0}=m_{\pi}, p_{0}=m_{\pi}+10 \mathrm{MeV}, p_{0}=m_{\pi}+20 \mathrm{MeV}$ ). The gray band shows the area between the phenomenological fits from Refs. [138, 139] and the dark band the fit from [134]. For $\operatorname{Im} \Pi_{S}$ all phenomenological fits lie in the gray shaded area.
the repulsion in pionic atoms comes from the energy dependence of the pion self energy together with a consistent incorporation of gauge invariance; a consistent treatment of the Coulomb potential requires that the argument of the pion self energy is $\Pi_{S}\left(\omega-V_{c}\right)$ [126] rather than $\Pi_{S}\left(\omega=m_{\pi}\right)$. Furthermore, the small isoscalar $\pi N$ potential at threshold rises rapidly with increasing energy and, thus, large effects from the energy dependence of $\Pi_{S}$ can be expected. Note that for $\Pi_{S}\left(p^{0}=m_{\pi}\right)$ the chiral calculation from Ref. [124] which is the basis for [126] delivers only around half of the repulsion needed, and the good agreement with experiment in [126] comes from the energy dependence.

In order to see this effect in the present calculation we have plotted $\Pi_{S}$ also for $p^{0}=m_{\pi}+10 \mathrm{MeV}$ and $p^{0}=m_{\pi}+20 \mathrm{MeV}$ (dashed and dotted line, respectively). The Coulomb potential for a nucleus with $A=100, Z=50$ can reach $V_{c} \sim 16 \mathrm{MeV}$ at an effective density of $\rho=\rho_{0} / 2$, and even more for heavier nuclei. As Fig. 3.20 shows, the energy dependence leads indeed to an extra repulsion which agrees well with the phenomenological fits. However, theoretical uncertainties are larger than thought as will be discussed in Sec. 3.6.1 and put into question a reliable determination of the optical potential.


Figure 3.21: $s$-wave pion selfenergy to all orders in $\rho$. Different pion energies $\left(p_{0}=m_{\pi}, p_{0}=m_{\pi}+10 \mathrm{MeV}, p_{0}=m_{\pi}+20 \mathrm{MeV}\right)$. Phenomenological fits as in Fig. 3.20.

### 3.6.1 Theoretical uncertainties

Higher order corrections in density play an important role. E.g., at order $\rho^{3} \rho^{1 / 3}$, diagrams appear where both isovector vertices of the large EricsonEricson rescattering piece contain the corrections from Sec. 3.4. These corrections reduce Re $\Pi_{S}$ from Fig. 3.20 almost down to the values of the first calculation from Fig. 3.6: Dressing all vertices with the corrections found and using the $\pi N$ loop function with Pauli blocking and pion polarization, i.e. including all corrections found, to all orders, the result is indicated in Fig. 3.21. Changes with respect to Fig. 3.20 are mainly due to the fact, that the vertex corrections can occur quadratically and higher while before at order $\rho^{2} \rho^{1 / 3}$ only one vertex correction enters the rescattering series.

The imaginary part on the right hand side is more negative and closer to the values from phenomenological fits: the imaginary part comes from the $p h$ insertions in the $\pi N$ loop function. This rescattering loop is enhanced by the larger strength in both $\pi N$ vertices due to the vertex corrections.

We have also treated the pion self energy selfconsistent as in Sec. 3.3.1, including the $k^{0}$ energy dependence of the $s$-wave potential. This leads to another $10 \%$ increase at $\rho=\rho_{0} / 2$.

The decrease of the real part can be understood as following: in the vacuum model the rescattering piece introduces an attraction which is compensated by a repulsion $b_{c}$ from the NLO isoscalar interaction at tree level [1]. The isovector interaction in the medium is increased from vertex corrections
as we have seen in Fig. 3.18; this leads to an increase of the attraction from the rescattering piece. The Pauli blocking of the intermediate nucleon, namely the Ericson-Ericson effect from Eq. (3.14), which is repulsive, can not fully compensate this effect; as a result, the net repulsion is smaller than without vertex corrections. However, for external pion energies of $k^{0}=m_{\pi}+10$ $\mathrm{MeV}, k^{0}=m_{\pi}+20 \mathrm{MeV}$, the results are in the region of the required repulsion as Fig. 3.21 shows; as outlined above, these energies are indeed required by the Coulomb potential.

Apart from the corrections at $\rho^{3} \rho^{1 / 3}$ discussed before, there are many more corrections at that order which are hard to access in a systematic expansion in density and, thus, at this point unknown systematical uncertainties appear.

Obviously, these higher order effects, and also the corrections at $\rho^{2} \rho^{1 / 3}$, depend on the parameters of the vacuum model from Ref. [1]. In particular, the results should be tested for stability with respect to the size of $b_{c}$ and the rescattering term. For this, we have performed a refit of the vacuum amplitude. Results and details will be given in the next Sec. 3.7.

Further theoretical uncertainties come from the regularization scale $\Lambda$ that appears in the monopole form factors of the pion $p$-wave polarization. The result depends on $\Lambda$; a smaller value than the one used of $\Lambda=0.9 \mathrm{GeV}$ would provide more repulsion. Nevertheless, the good agreement with the phenomenological analysis on the $b_{1}$ renormalization from [137], which has been noted in Sec. 3.4.6, provides support for this choice of $\Lambda$.

### 3.7 Dependence on the vacuum renormalization

As we have seen in the last section, the renormalization of the isovector and isoscalar $\pi N$ interaction induces a large negative contribution for the real part of the $s$-wave pion selfenergy $\Pi_{S}$. This is a consequence of a fine-tuned balance in the vacuum model, broken in the medium, between the isoscalar piece from rescattering and the isoscalar from the NLO chiral Lagrangian. In the following we gain further insight into this issue and study the effect of additional constraints on the NLO isoscalar piece in the vacuum. To this
end we improve the Ericson-Ericson approximation for $\pi N$ rescattering.

### 3.7.1 Improvement of the Ericson-Ericson approximation

In Refs. $[99,120]$ it has been shown that the pion selfenergy from one-loop rescattering,

$$
\begin{align*}
& \Pi_{S}^{\mathrm{res}}(p) \\
= & 4 \int \frac{d^{3} \mathbf{k}}{(2 \pi)^{3}} \int \frac{d^{3} \mathbf{q}}{(2 \pi)^{3}} n(\mathbf{k}) n(\mathbf{q}) \frac{1}{-p^{0}+E(\mathbf{q})+\omega(\mathbf{p}+\mathbf{k}-\mathbf{q})-E(\mathbf{k})+i \epsilon} \\
\times & \frac{1}{2 \omega(\mathbf{p}+\mathbf{k}-\mathbf{q})}(4 \pi)^{2}\left[\left(\frac{2 \lambda_{1}}{\mu}\right)^{2}+2\left(\frac{\lambda_{2}}{\mu^{2}}\right)^{2}\left[p^{0}+\omega(\mathbf{p}+\mathbf{k}-\mathbf{q})\right]^{2}\right] \tag{3.67}
\end{align*}
$$

can be approximated by

$$
\begin{equation*}
\Pi_{S}\left(p^{0}=\mu, \mathbf{p}=\mathbf{0}\right)=24 \rho \frac{k_{F}}{\mu^{2}}\left(\lambda_{1}^{2}+2 \lambda_{2}^{2}\right) \tag{3.68}
\end{equation*}
$$

in the limit of low density and for the pion mass $\mu \equiv m_{\pi} \ll M_{N}$. We have adopted here the notation of [99] where $E,(\omega)$ is the nucleon, (pion) energy, $n$ is the nucleon distribution function, and $\lambda_{1}, \lambda_{2}$ are the isoscalar and isovector coupling strengths (see Eq. (3.15)), respectively.

The derivation of Eq. (3.68) from Eq. (3.67) is straightforward; however, one of the approximations made in the evaluation of the integrals, the replacement $\omega(\mathbf{p}+\mathbf{k}-\mathbf{q}) \rightarrow \mu$ in the first term of the third line in Eq. (3.67), can be easily improved. We replace

$$
\begin{equation*}
\frac{1}{2 \omega(\mathbf{p}+\mathbf{k}-\mathbf{q})} \rightarrow \frac{\mu}{2 \mu^{2}+\mathbf{k}^{2}+\mathbf{q}^{2}-2 \mathbf{k} \cdot \mathbf{q}} \tag{3.69}
\end{equation*}
$$

and use for the rest of the evaluation the same approximations as in the derivation of Eq. (3.68), such as $\mu \ll M_{N}$. The final result reads

$$
\begin{align*}
\Pi_{S}\left(p^{0}=\mu, \mathbf{p}=\mathbf{0}\right) & =\frac{2 \mu}{3 \pi^{2}}\left(1+\frac{\mu}{M_{N}}\right)^{2}\left(b_{0}^{2}+2 b_{1}^{2}\right) \\
& \times\left[2 k_{F}^{2} \mu+8 \sqrt{2} k_{F}^{3} \arctan \left(\frac{\sqrt{2} k_{F}}{\mu}\right)\right. \\
& \left.+\mu\left(6 k_{F}^{2}+\mu^{2}\right) \log \left(\frac{\mu^{2}}{2 k_{F}^{2}+\mu^{2}}\right)\right] \tag{3.70}
\end{align*}
$$

The first non-vanishing coefficient in a Taylor expansion of this expression around $k_{F}=0$ is at $k_{F}^{4}$, and at that order Eq. (3.68) is indeed recovered. It is interesting to demonstrate the improvement of the EE-approximation in Eq. (3.70) over Eq. (3.68) in a numerical example.

In the results from the last section we have seen that an increased isovector interaction decreases the real part of the pion selfenergy. In a "toy model" we can simulate the effect of an increased isovector due to vertex corrections. To this end, the Weinberg-Tomozawa term is multiplied with a factor of 1.5 and inserted into the full coupled channel vacuum model, including all other ingredients as before, and with the parameters of the best fit from Tab. 3.2.

In the vacuum model, there is a large, negative isoscalar contribution from the NLO chiral Lagrangian, $b_{c}$, and a large attractive piece from rescattering with two isovector vertices that compensates the NLO piece. Increasing the isovector interaction should, thus, increase the attraction in the vacuum model. Indeed, the "toy model" results in a large change of the isoscalar and isovector $\left(b_{0}, b_{1}\right)$ from the original vacuum values $\left(-0.0029 \mu^{-1},-0.0883 \mu^{-1}\right)$ towards a large attractive isoscalar and also increased isovector, $\left(+0.0634 \mu^{-1},-0.1752 \mu^{-1}\right)$.

In a next step, we include Pauli blocking in the "toy model" and evaluate the pion selfenergy in the medium. The result is plotted in Fig. 3.22 with the solid line. The dotted line indicates the $t \rho$-approximation using the vacuum $b_{0}=+0.0634 \mu^{-1}$. Adding to that the usual EE-rescattering piece from Eq. (3.68), the result is indicated with the dashed line. Adding instead the improved expression from Eq. (3.70), the result is indicated with the dashed-dotted line and agrees well with the full numerical solution.

There are several observations worth noting: First, the $t \rho$-approximation gives a huge negative contribution to $\operatorname{Re} \Pi_{S}$ because the increase of the elementary isovector interaction has generated a large and positive vacuum isoscalar value from rescattering. Second, the ordinary EE-rescattering piece from Eq. (3.68) is also huge because the vacuum isovector has been increased from $b_{1}=-0.0883 \mu^{-1}$ to $b_{1}=-0.1752$ in the "toy model". However, the repulsive piece from rescattering, using the better approximation from Eq. (3.70), is not strong enough to overcome the negative increase from the $t \rho$ contribution. Thus, the overall result from increasing the Weinberg-


Figure 3.22: Real part of the $s$-wave pion selfenergy $\Pi_{S}$ in rescattering with a 1.5 times increased isovector. Solid line: Full model with Pauli blocking for the intermediate nucleons. Dotted line: t $\rho$-approximation. Dashed line: Including the EE-approximation from Eq. (3.68) for the rescattering piece. Dashed-dotted line: Including the improved EE-approximation from Eq. (3.70).

Tomozawa term is more attractive instead of more repulsive. One would expect an even more negative result once introducing the in-medium polarization of the pion as we have already seen in Fig. 3.6 by comparing the two upper panels.

Summarizing the arguments up to this point, the increase of the isovector interaction leads to a large attraction in rescattering in the nuclear medium. Then, the question arises why the final result for $\operatorname{Re} \Pi_{S}$ in Fig. 3.20 at $\Lambda=0.9$ GeV is almost of the same size as the previous result in Fig. 3.6 without any vertex corrections. This is due to the other class of vertex corrections, the renormalization of the isoscalar interaction from Sec. 3.5. These corrections induce an increase of the isoscalar term from the NLO Lagrangian which gives a repulsive contribution in the vacuum model. In the nuclear medium the leading order contribution, $t \rho$, thus also induces more repulsion; the result after isovector and isoscalar vertex corrections is similar to the result without any vertex corrections. Note that for larger cut-offs than the used value of $\Lambda=0.9 \mathrm{GeV}$, the above effects do not cancel and $\operatorname{Re} \Pi_{S}$ becomes
even smaller.
Summarizing, the size of $\operatorname{Re} \Pi_{S}$ is difficult to determine. This is due to the in-medium breaking of a fine-tuned balance of the vacuum model, a balance in which large contributions from the NLO isoscalar term and the isoscalar term from rescattering almost cancel. The present model should be tested for stability with respect to this balance which will be done in the next section.

### 3.7.2 The size of the isoscalar contribution from the NLO Lagrangian

In the last section we have seen that for $\Lambda=0.9 \mathrm{GeV}$ both the vertex corrections of the isoscalar and the isovector interaction induce large effects in nuclear matter but the final results remain similar as these two renormalizations almost cancel. It is interesting to study the stability of the results when varying the vacuum fit itself, i.e. the size of $b_{c}$ that has been obtained from the fit to the vacuum data.

For a faster convergence of the chiral expansion it is desirable to have a smaller contribution from the next-to-leading order chiral Lagrangian, i.e. a smaller $b_{c}$ with $b_{c}$ given in Eq. (2.47). This will also help make more reliable predictions for the nuclear medium, because the cancellation of a smaller $b_{c}$ with a smaller isoscalar from rescattering will lead to more predictable nuclear effects. Thus, it is straightforward to perform a refit of the vacuum data and at the same time imposing a smaller $b_{c}$. To this end, we include an additional term in the expression for $\chi^{2}, \chi^{2} \rightarrow \chi^{2}+b_{c}^{2} / r^{2}$ with $b_{c}$ in $\mathrm{MeV}^{-1}$. Setting $r=\infty, 50,20 \mathrm{MeV}^{-1}$ we can impose smaller and smaller $b_{c}$. The resulting fits are labeled $1,2,3$, respectively.

The original vacuum fit with the values from Tab. 3.2 includes data from experimental phase shift analyses up to $\sqrt{s}=1253 \mathrm{MeV}$. In the nuclear medium, these energies are never reached: The maximal $k_{F}=270 \mathrm{MeV}$ corresponds to a c.m. energy $\sqrt{s}$ of

$$
\begin{equation*}
s=m_{\pi}^{2}+M_{N}^{2}+2 m_{\pi} \sqrt{M_{N}^{2}+k_{F}^{2}} \tag{3.71}
\end{equation*}
$$

or $\sqrt{s}=1081 \mathrm{MeV}$ which is even below the first single energy bin of the SAID [113] analysis. Thus, we have restricted the fitted energy regions in

Table 3.4: Fit parameters of the refits 1,2,3. Data from threshold to $\sqrt{s}=$ 1.154 MeV is fitted.

|  | Fit 1 | Fit 2 | Fit 3 |
| :--- | :--- | :--- | :--- |
| $\chi^{2}$ | 7.5 | 28 | 43 |
| $a_{\pi N}[\mathrm{MeV}]$ | -2.794 | 8.829 | 14.96 |
| $2 c_{1}-c_{3}\left[\mathrm{GeV}^{-1}\right]$ | -0.933 | -1.316 | -1.43 |
| $c_{2}\left[\mathrm{GeV}^{-1}\right]$ | -1.719 | -1.673 | -1.537 |
| $\gamma\left[10^{-5} m_{\pi}^{5}\right]$ | 10 | 10 | 10 |
| $b_{0}\left[10^{-4} m_{\pi}^{-1}\right]$ | -33 | -25 | -10 |
| $b_{1}\left[10^{-4} m_{\pi}^{-1}\right]$ | -892 | -882 | -861 |
| $b_{c}\left[10^{-4} m_{\pi}^{-1}\right]$ | -341 | -155 | -46 |

the new fits 1,2 , and 3 from threshold up to $\sqrt{s}=1154 \mathrm{MeV}$ and additionally given a larger theoretical error of 0.01 to the data at finite energy because we prefer a good description of the threshold over a precise prescription at higher energies. In order to reduce the correlations of the fit parameters, we also remove the exponential damping factor (the term from Eq. (2.9) with $\beta$ ) that is only important at much higher energies.

The results for the fits 1,2 , and 3 are shown in Tab. 3.4. Fit 1 does not contain any restriction on $b_{c}$ and indeed the resulting $b_{c}$ is very similar to that of the original fit from Tab. 3.2. Note that the absence of the fit parameter $\beta$ and the different choice and weight of fitted data produces a different parameter set for the fit 1 compared to the original fit in Tab. 3.2; however, the subtraction constant $a_{\pi N}$ is similar in both fits.

Imposing a smaller $b_{c}$ (fit 2,3 ) leads to a change in $a_{\pi N}$; in other words, the loop regularization changes the isoscalar piece from rescattering in order to compensate the changed $b_{c}$; remember that the final $b_{0}$ from pionic hydrogen and deuterium data is determined to be very small.

It is interesting to compare the $c$-values from Tab. 3.4 to the fit $2^{\dagger}$ of Tab. IV of [93] which provides the combinations $2 c_{1}-c_{3}=-1.63 \pm 0.9 \mathrm{GeV}^{-1}$ and $c_{2}=-1.49 \pm 0.67 \mathrm{GeV}^{-1}$. Although all $c$-values from Tab. 3.4 are inside the error bars from [93], the fit 2 shows a better agreement and fit 3 , which has the smallest $b_{c}$ of all fits, agrees well with the values of fit $2^{\dagger}$ from [93]. We can not expect a perfect agreement as in the present vacuum model from
chapter 2 [1] some diagrams that vary slowly with energy are absorbed in the $c$-values while in [93] they have been explicitly taken into account. However, we have seen that the present model does allow for a small $b_{c}$ and then also agrees well with the systematic chiral expansion [93].

The data description is still sufficient for the fits 2 and 3 as the plots in the two upper rows of Fig. 3.23 show. We have here divided the partial wave amplitudes from the SAID solution by $r(s)=-M_{N} Q(\sqrt{s}) /(4 \pi \sqrt{s})$ with $Q$ the c.m. momentum. This allows for a closer inspection of the very low energy data and one can also plot the experimental information from threshold (see error bars at $\sqrt{s}=1077 \mathrm{MeV}$ ). This data point has been calculated from the intersection region of the three bands from pionic hydrogen and deuterium from Fig. 2.7. Note that for all fits there is a conflict between threshold and finite energy scattering data and the model, even allowing for a large $b_{c}$, can not fit all points simultaneously. Apart from potential shortcomings of the model, there might be a conflict in the data for the lowest energy data points ${ }^{1}$.

The lower row in Fig. 3.23 shows the pion selfenergy in the nuclear medium, using the new fit results. Although $b_{c}$ in fit 3 is almost 10 times smaller than in fit 1 , the real part of the $s$-wave self energy hardly changes. The present model is stable under changes of the isoscalar term from the NLO chiral Lagrangian.

### 3.7.3 Uncertainties from the Roper resonance in $\pi N$ scattering

There is another type of medium effect which has not been considered so far and will introduce additional uncertainties. This is related to the Roper excitation and its decay into nucleon and two pions in $I=0$ and $s$-wave. The Roper is the lightest resonance with the same quantum numbers as the nucleon and allows for a decay into a nucleon and two pions which are in isospin zero and $s$-wave relative to each other and also relative to the nucleon. In Ref. [161] the mechanism of Roper excitation from an isoscalar source and subsequent decay into two pions has been found dominant at low energies in

[^3]

Figure 3.23: Dependence on the vacuum renormalization. For the fit conditions see text. Upper two rows: The $S_{11}$ and $S_{31}$ channel in $\pi N$ scattering. The amplitudes are divided by $r(s)$ as described in the text. The phase shift analysis is from [113]. Lower row: pion $s$-wave selfenergy in the nuclear medium for the various fits. The phenomenological fit (gray bands) is from [134].


Figure 3.24: The Roper resonance in isoscalar, $s$-wave $\pi N$ scattering in the medium. On the right hand side the interaction in the heavy baryon limit is shown.
the $N N \rightarrow N N \pi \pi$ production for pions in $I=0$. The isoscalar source can be described by an effective $\sigma$ exchange $\sigma N N^{*}$ between the nucleons, whose strength has been fitted independently for the ( $\alpha, \alpha^{\prime}$ ) reaction on a proton target [162]. Based on that finding, the relevance of this mechanism in $\pi d$ scattering at low energies was also stressed in [147]. We shall also consider it here in connection with the $s$-wave pion nucleus optical potential.

For the present purposes the mechanism described above can be adapted by having the two pions one in the initial state and the other one in the final state as indicated in Fig. 3.24 on the left hand side. As the two pions are in a relative $I=0$ state, we obtain an isoscalar contribution to the $\pi N$ amplitude. The second nucleon line to which the isoscalar $\sigma$ couples is closed and gives a medium contribution to $\pi N$ scattering. In the heavy baryon limit the diagram reduces to a point-like interaction of a pion with two nucleons as indicated in Fig. 3.24 on the right hand side.

For the $N^{*} N \pi \pi$ coupling an effective Lagrangian from Ref. [163] is used which leads to the effective vertex [161]

$$
\begin{equation*}
-i \delta \tilde{H}_{N^{*} N \pi \pi}=-2 i \frac{m_{\pi}^{2}}{f_{\pi}^{2}}\left(c_{1}^{*}-c_{2}^{*} \frac{\omega_{1} \omega_{2}}{m_{\pi}^{2}}\right) \tag{3.72}
\end{equation*}
$$

for $\pi^{+} \pi^{+}, \pi^{-} \pi^{-}$and $\pi^{0} \pi^{0}$ and zero otherwise (note a minus sign in $c_{2}^{*}$ with respect to [161] because now one of the pions is incoming). Here $\omega_{1}, \omega_{2}$ are the energies of the pions. The values for the couplings are obtained in Ref. [161] from a fit to the experimental width of the $N^{*}$ decay into $N \pi^{+} \pi^{-}$and $N \pi^{0} \pi^{0}$, $c_{1}^{*}=-7.27 \mathrm{GeV}^{-1}$ and $c_{2}^{*}=0 \mathrm{GeV}^{-1}$. For the $N^{*} \sigma N$ coupling the effective vertex is $-i \Delta \tilde{H}_{\sigma N N^{*}}=i F(q) g_{\sigma N N^{*}}$ where $g_{\sigma N N^{*}}^{2} /(4 \pi)=1.33$ and $F$ a form
factor of the monopole type for the off-shell $\sigma$ with $\Lambda_{\sigma}=1.7 \mathrm{GeV}, m_{\sigma}=550$ MeV .

In the heavy baryon approximation we can put the external nucleons at rest and, thus, obtain for the elastic scattering of a pion of any charge with a nucleon of any charge

$$
\begin{align*}
(-i t) & =2\left(-2 i \frac{m_{\pi}^{2}}{f_{\pi}^{2}} c_{1}^{*}\right) \frac{i}{m_{N}-m_{N^{*}}}\left(i F_{\sigma}\left(q_{\sigma}\right) g_{\sigma N N^{*}}\right) \\
& \times \frac{i}{-m_{\sigma}^{2}}\left(i F_{\sigma}\left(q_{\sigma}\right) g_{\sigma N N}\right)\left(\rho_{p}+\rho_{n}\right) \tag{3.73}
\end{align*}
$$

where the $\sigma N N$ coupling is the same as in the Bonn model [164] with $g_{\sigma N N}^{2} /(4 \pi)=5.69$. The contribution in Eq. (3.73) already contains the sum of the two diagrams on the left hand side of Fig. 3.24. The isoscalar modification is, thus,

$$
\begin{equation*}
b_{0}=2 \frac{c_{1}^{*} m_{\pi}^{2} g_{\sigma N N^{*}} g_{\sigma N N} m_{N}}{2 \pi f_{\pi}^{2} m_{\sigma}^{2}\left(m_{N}-m_{N^{*}}\right)\left(m_{\pi}+m_{N}\right)} \rho \simeq 0.38 m_{\pi}^{-4} \rho \tag{3.74}
\end{equation*}
$$

At normal nuclear matter density $\rho=\rho_{0}$ this leads to an isoscalar of $b_{0}=$ $0.188 m_{\pi}^{-1}$ which implies attraction. The result from Eq. (3.74) is huge compared to the isoscalar from the model of $\pi N$ interaction from chapter 2 [1] of $b_{c}=-0.0336 m_{\pi}^{-1}$, see Tab. 3.3. We can use the $b_{0}$ from Eq. (3.74) and calculate $\Pi_{S}$ from Eq. (3.1). Then, already at tree level, one obtains the unrealistically large attraction of $\operatorname{Re} \Pi_{S}=-24900 \mathrm{MeV}^{2}$.

However, by turning the pion line around in the diagram of Fig. 3.24 we have implicitly changed the kinematics at which the above couplings such as $g_{\sigma N N^{*}}$ have been determined. The $N^{*}(1440)$ is now off-shell by around 500 $\mathrm{MeV}\left(E=m_{N}\right)$ which induces unknown theoretical errors in the calculation.

Instead of the set $\left(c_{1}^{*}=-7.27 \mathrm{GeV}^{-1}, c_{2}^{*}=0 \mathrm{GeV}^{-1}\right)$ one can consider the results from Ref. [146] which use the combination $c_{1}^{*}+c_{2}^{*}=(-1.56 \pm$ $3.35) \mathrm{GeV}^{-1}$ from Ref. [163] and then apply a resonance saturation hypothesis for the $c^{*}$ to be saturated by scalar meson exchange. Then the combination $c_{1}^{*}-c_{2}^{*}$ can be disentangled and our result with these values would change from $\operatorname{Re} \Pi_{S}=-24900 \mathrm{MeV}^{2}$ to $(-710 \pm 1600) \mathrm{MeV}^{2}$ (compare to Figs. 3.20, 3.21).

Obviously, with the present knowledge of the $c^{*}$ coefficients no further conclusions can be drawn for the influence of the Roper resonance on $\operatorname{Re} \Pi_{S}$.

Considering also the large uncertainties from higher order effects in density, which have been discussed in the last section, a precise determination of the $s$-wave repulsion of the pion in nuclear matter does not seem to be feasible; theoretical errors are so large that the whole range of different phenomenological fits of $\operatorname{Re} \Pi_{S}$ can be easily covered.

### 3.8 Summary and conclusions

The $s$-wave pion-nucleus optical potential has been calculated in a microscopical many-body approach including a variety of higher order corrections. The model is inspired by the well-known fact that the Ericson-Ericson rescattering piece generates a large repulsion. We have, thus, taken a chiral unitarized rescattering approach that delivers a good description of vacuum data in the vicinity of the threshold and above. Subsequently, various medium corrections have been added to the vacuum model. Whereas Pauli blocking generates repulsion, the pion polarization for intermediate pions, including $p h, \Delta h$ and short-range correlations, is responsible for a moderate attraction. The model has been formulated for asymmetric nuclear matter and allows for isospin breaking from the use of physical masses. However, the effects from isospin breaking have been found small.

For the Weinberg-Tomozawa term and the isoscalar contribution from the NLO chiral Lagrangian, several in-medium vertex corrections, some of them novel, have been included. E.g., the Weinberg-Tomozawa term is increased by a factor of 1.4 at $\rho=\rho_{0}\left(1.2\right.$ at $\left.\rho=\rho_{0} / 2\right)$ which agrees well with a recent analysis on deeply bound pionic atoms.

The $\pi N$ rescattering term together with the vertex corrections at an overall order of $\rho^{2} \rho^{1 / 3}$ in density brings the theoretical repulsion close to the phenomenological fits; the energy dependence of the pion self energy provides additional repulsion in agreement with the repulsion required by experiment.

When the vertex corrections are included in the rescattering series to all orders, we observe large contributions at orders higher than $\rho^{2} \rho^{1 / 3}$ in the nuclear density, beyond any feasible systematic expansion in terms of $\rho$. This is a signal that the medium calculation, at least in the present theoretical framework, converges slowly; realizing also the additional uncertainties from
$s$-wave pion nucleus... 147
the Roper resonance, the theoretical error is larger than previously thought.

## Chapter 4

## Chiral dynamics in the $\gamma \boldsymbol{p} \rightarrow \pi^{0} \eta p$ and $\gamma \boldsymbol{p} \rightarrow \pi^{0} K^{0} \Sigma^{+}$ reactions

Using a chiral unitary approach for meson-baryon scattering in the strangeness zero sector, where the $N^{*}(1535) S_{11}$ resonance is dynamically generated, we study the reactions $\gamma p \rightarrow \pi^{0} \eta p$ and $\gamma p \rightarrow \pi^{0} K^{0} \Sigma^{+}$at photon energies at which the final states are produced close to threshold. Among several reaction mechanisms, we find the most important is the excitation of the $\Delta^{*}(1700) D_{33}$ state which subsequently decays into a pseudoscalar meson and a baryon belonging to the $\Delta(1232) P_{33}$ decuplet. Hence, the reaction provides useful information with which to test current theories of the dynamical generation of the low-lying $3 / 2^{-}$states. The first reaction is shown to lead to sizable cross sections and the $N^{*}(1535) S_{11}$ resonance shape is seen clearly in the $\eta p$ invariant mass distribution. The same dynamical model is shown to lead to much smaller cross sections at low energies in the second reaction. Predictions are made for cross sections and invariant mass distributions which can be compared with ongoing experiments at ELSA.

### 4.1 Introduction

In chapter 2 a unitarized coupled channel approach has been used for the calculation of low energy $\pi N$-scattering and a precise determination of the threshold parameters $b_{0}$ and $b_{1}$. In chapter 3 this model was employed in
the framework of many-body techniques in order to evaluate the $s$-wave pion nucleus optical potential. The unitarization via the use of the Bethe-Salpeter equation (BSE) is important because it generates a rescattering series of $\pi N$ intermediate states. In the medium calculation of chapter 3 this leads to an increase in the repulsion of the pion in $s$-state from the Pauli blocking of the intermediate nucleons.

At higher energies in $\pi N$ scattering the unitarization becomes important in a different sense: chiral perturbative calculations break down at intermediate energies as discussed in the Introduction. In unitarized chiral perturbation theory ( $\mathrm{U} \chi \mathrm{PT}$ ) not the $T$-matrix is expanded perturbatively but the interaction kernel of the BSE is matched perturbatively with chiral perturbation theory. This provided automatically the basic analytic properties of the scattering amplitude such as the physical, right hand cut required by unitarity; in turn the results of chiral perturbation theory are perturbatively recovered by expanding the unitarized amplitude in powers of momenta and masses.

We have already seen in the Introduction that at higher energies in the $\pi N s$-wave channel $S_{11}$ the unitarization leads to the occurrence of a pole in the complex plane of the c.m. energy $s^{1 / 2}$. In particular, the $N^{*}(1535)$ can be identified with this pole. In this and the following chapters we will concentrate on these "dynamically generated resonances". The great advantage and power of the concept of dynamically generating resonances is that their properties such as branching ratios, photoproduction, magnetic moments etc. can be calculated and predicted from the underlying microscopical model where the photon-, pion-, etc. couplings to the building blocks of the resonance are known.

A good example is found in the recent experiment on photoproduction of the $\Lambda(1405) S_{01}$ resonance in the $\gamma p \rightarrow K^{+} \pi \Sigma$ reaction [165], where theoretical predictions using the chiral unitary approach had been done previously [166]: The unitary coupled channel model provides a description of the $\Lambda(1405) S_{01}$ in terms of the well-known lowest order chiral interaction; the photoproduction of this resonance can then be predicted by gauging the interaction with the electromagnetic field, or in diagrammatic language, coupling the photon to all meson-baryon components and vertices of the
rescattering diagrams. This predicts different shapes and strengths for the decay channels $\pi^{-} \Sigma^{+}$and $\pi^{+} \Sigma^{-}$for the $\pi \Sigma$ invariant masses and, indeed, experiment confirms these findings [165]. Obviously, a simple resonance model with Breit-Wigner amplitudes can not deliver this extra information about the internal structure of resonances.

In the present paper we adopt and extend the ideas of [166] and study the analogous reaction $\gamma p \rightarrow \pi^{0} \eta p$ where the $\eta p$ final state can form the $N^{*}(1535) S_{11}$ resonance. This reaction is currently being analyzed at ELSA [167]

Some of the reaction mechanisms in our model are described as a twostep process: In the initial photoproduction, two mesons are generated, one of which is the final $\pi^{0}$. The final state interaction of the other meson with the proton is then responsible for the $\eta$ production. For this interaction chiral Lagrangians in $S U(3)$ representation involving only mesons and baryons are used. In addition, the contributions from explicit baryonic resonance exchange such as $\Delta(1232) P_{33}, N^{*}(1520) D_{13}$, and $\Delta^{*}(1700) D_{33}$, which have been found essential for the two meson photoproduction, e.g., in the Valencia model [168-170], will be included. The $\Delta^{*}(1700) D_{33}$ resonance, as recent studies show $[43,44]$, qualifies as dynamically generated through the interaction of the $0^{-}$meson octet and the $3 / 2^{+}$baryon decuplet. In this picture it is possible [44] to obtain the coupling of the $\Delta^{*}(1700) D_{33}$ to the $\eta \Delta(1232) P_{33}$ and $K \Sigma^{*}(1385) P_{13}$ for which experimental information does not yet exist.

In Sec. 4.2 the model for the dynamical generation of the $N^{*}(1535) S_{11}$ resonance in $\pi N \rightarrow \pi N$ scattering will be briefly reviewed with special emphasis on the $\pi N \rightarrow \eta N$ transition. Subsequently, we study the one-meson photoproduction $\gamma p \rightarrow \eta p$ in Sec. 4.3. This allows for a simultaneous description of the existing data for these three different reactions within the chiral model. In Sec. 4.4 we predict observables for the photoproduction of $\pi^{0} \eta p$ in the final state.

At the same time we also study the $\gamma p \rightarrow \pi^{0} K^{0} \Sigma^{+}$reaction and make predictions for its cross section, taking advantage of the fact that it appears naturally within the coupled channels formalism of the $\gamma p \rightarrow \pi^{0} \eta p$ reaction and leads to a further test of consistency of the ideas explored here.

In this section we have referred to all resonances that will enter the eval-
uation of our amplitudes. In what follows for shortness of notation we will omit the description in terms of $L_{2 I, 2 J}$.

### 4.2 The $N^{*}(1535)$ in meson-baryon scattering

Before turning to the photoproduction reactions of the next sections, let us recall the properties of the $N^{*}(1535)$ in the meson-baryon sector, where this resonance shows up clearly in the spin isospin $(S=1 / 2, I=1 / 2)$ channel. In the past, this resonance has been proposed to be dynamically generated $[40,41,45,48]$ rather than being a genuine three-quark state. The model of Ref. [45] provides an accurate description of the elastic and quasielastic $\pi N$ scattering in the $S_{11}$ channel. Within the coupled channel approach in the $S U(3)$ representation of Ref. [45], not only the $\pi N$ final state is accessible, but also $K \Sigma, K \Lambda$, and $\eta N$ in a natural way.

In the case of the present reactions, we are interested in the $\eta p$ interaction which will manifest the $N^{\star}(1535)$ resonant character. This interaction was studied in detail in Ref. [45] for the charge $Q=0$, strangeness zero sector. In the present study we work in the charge $Q=+1$ sector, which requires the simultaneous consideration of the coupled channels

$$
\begin{equation*}
\pi^{0} p, \pi^{+} n, \eta p, K^{+} \Sigma^{0}, K^{+} \Lambda, K^{0} \Sigma^{+} \tag{4.1}
\end{equation*}
$$

We will subsequently refer to these channels as one through six in the order given above. In this section we derive the necessary modifications of the coupled channels in the $Q=+1$ sector. The theoretical framework of the photoproduction mechanisms is found in subsequent sections. In Ref. [45] the transition in the Weinberg-Tomozawa term are given for the charge $C=0$ sector. The $C_{i j}$ coefficients for the channels with charge +1 needed here are straightforward calculated from the Lagrangian from Eq. (2.4) and shown in Tab. 4.1. The amplitudes after unitarization are given in matrix form [45] by means of the Bethe-Salpeter equation

$$
\begin{equation*}
T(\sqrt{s})=[1-V(\sqrt{s}) G(\sqrt{s})]^{-1} V(\sqrt{s}) \tag{4.2}
\end{equation*}
$$

with $V$ obtained from Eq. (2.6). We are only interested in the s-wave mesonbaryon interaction to generate the $S_{11}$ amplitude; the projection of $V$ into

Table 4.1: $C_{i j}$ coefficients for the six channels. The matrix is symmetric.

|  | $\pi^{0} p$ | $\pi^{+} n$ | $\eta p$ | $K^{+} \Sigma^{0}$ | $K^{+} \Lambda$ | $K^{0} \Sigma^{+}$ |
| :--- | :--- | :--- | :--- | :--- | :--- | :--- |
| $\pi^{0} p$ | 0 | $\sqrt{2}$ | 0 | $-\frac{1}{2}$ | $-\frac{\sqrt{3}}{2}$ | $\frac{1}{\sqrt{2}}$ |
| $\pi^{+} n$ |  | 1 | 0 | $\frac{1}{\sqrt{2}}$ | $-\sqrt{\frac{3}{2}}$ | 0 |
| $\eta p$ |  |  | 0 | $-\frac{\sqrt{3}}{2}$ | $-\frac{3}{2}$ | $-\sqrt{\frac{3}{2}}$ |
| $K^{+} \Sigma^{0}$ |  |  |  | 0 | 0 | $\sqrt{2}$ |
| $K^{+} \Lambda$ |  |  |  |  | 0 | 0 |
| $K^{0} \Sigma^{+}$ |  |  |  |  |  | 1 |

this partial wave is given in Eq. (2.7), and $G$ is the meson-baryon loop function in dimensional regularization from Eq. (2.8). In the following we denote by $T^{(i j)}$ the matrix elements of $T$ with the channel ordering of Eq. (4.1).

A second modification of the model of Ref. [45] with respect to other approaches concerns the $\pi N \rightarrow \pi \pi N$ channel. This channel was important to obtain a good description of the $I=3 / 2$ amplitude but it has only a small influence in the $I=1 / 2$ channel. It increases the width by about $10 \%$ and changes the position of the $N^{\star}(1535)$ by about 10 MeV . In the charge +1 sector this channel can be included by a change of the potential according to $V_{\pi N, \pi N} \rightarrow V_{\pi N, \pi N}+\delta V$ as in Ref. [1] and reads:

$$
\begin{aligned}
\delta V\left(\pi^{0} p \rightarrow \pi^{0} p\right)= & {\left[\left(-\frac{\sqrt{2}}{3} v_{31}-\frac{1}{3 \sqrt{2}} v_{11}\right)^{2}+\left(\frac{1}{3} v_{31}-\frac{1}{3} v_{11}\right)^{2}\right] G_{\pi \pi N} } \\
\delta V\left(\pi^{0} p \rightarrow \pi^{+} n\right)= & {\left[\left(-\frac{\sqrt{2}}{3} v_{31}-\frac{1}{3 \sqrt{2}} v_{11}\right)\left(\frac{1}{3} v_{31}-\frac{1}{3} v_{11}\right)\right.} \\
& \left.+\left(\frac{1}{3} v_{31}-\frac{1}{3} v_{11}\right)\left(-\frac{1}{3 \sqrt{2}} v_{31}-\frac{\sqrt{2}}{3} v_{11}\right)\right] G_{\pi \pi N}
\end{aligned}
$$




Figure 4.1: The $S_{11}$ partial wave in $\pi N \rightarrow \eta N$. Dots: Analysis from Ref. [113]. Solid line: full model from Ref. [45]. Dashed line: model from Ref. [45] without $t$ channel vector exchange and $\pi \pi N$ channel.
$\delta V\left(\pi^{+} n \rightarrow \pi^{+} n\right)=\left[\left(\frac{1}{3} v_{31}-\frac{1}{3} v_{11}\right)^{2}+\left(-\frac{1}{3 \sqrt{2}} v_{31}-\frac{\sqrt{2}}{3} v_{11}\right)^{2}\right] G_{\pi \pi N}$
with the isospin classification and conventions as in Ref. [45]; $G_{\pi \pi N}$ being the $\pi \pi N$ loop function that incorporates the two-pion relative momentum squared. Analytic expressions for $v_{11}$ and $v_{31}$ are found in Ref. [45].

The $\pi N \rightarrow \eta N$ production cross section has been calculated in Ref. [45] and was found to be quantitatively correct at the peak position, although somewhat too narrow at higher energies. The question is whether this is due to higher partial waves that enter at larger energies and are not part of the calculation, or due to a too narrow $N^{*}(1535)$ of the model. This can now be answered because an $S_{11}$ partial wave analysis has become available [113]. In Fig. 4.1 this analysis is compared to the model of Ref. [45]. With the solid line, the full model is indicated, and with the dashed line the model before introducing the vector exchange in the $t$-channel and the $\pi N \rightarrow \pi \pi N$ channel (details in Ref. [45]). We will refer to this second one as a "reduced" model of ref. [45] in what follows. Although we prefer the full model, as form factors and $\pi \pi N$ production certainly play an important role, we take the differences between the models in this work as an indication of the theoretical uncertainties.

In Fig. 4.1, the energies close to threshold and in particular the strength are well described by the dynamically generated resonance. The position of
the resonance in the analysis [113] is at slightly higher energies than predicted by the model and the width is considerably larger. This might be due to the contribution of the $N^{*}(1650)$ resonance which is near the $N^{*}(1535)$ in the $S_{11}$ channel and has been found to contribute to the reaction in other work [35]. Note, however, that in the same reference the total cross section above $s^{1 / 2}$ around 1650 MeV is dominated by heavier resonances from other partial waves such as the $P_{13}(1720)$ and $D_{13}(1520)$. It is also worth noting that in some variants of the chiral models with additional input to the one used here, one can account for the $N^{*}(1650)$ contribution to the $S_{11}$ amplitude [171].

In the present approach we restrict ourselves to the model for the $N^{*}(1535)$ from Ref. [45] as the behavior near the $\eta p$ threshold is well described in that work including the strength at the maximum of the cross section.

### 4.2.1 $\eta N$ scattering length and effective range

After having reviewed the main ingredients of the model for $\pi N$ scattering, we calculate scattering lengths and effective ranges for all six of the coupled channels and compare, where possible, to other theoretical approaches.

Our $T$-matrix of $s$-wave meson-baryon scattering in the absence of inelasticity is normalized as

$$
\begin{equation*}
-\frac{M}{4 \pi \sqrt{s}} T=\frac{e^{2 i \delta}-1}{2 i k}=\frac{1}{k \cot \delta-i k} \equiv f \tag{4.4}
\end{equation*}
$$

where $M$ is is the baryon mass and $k$ the CM three momentum. Scattering length $a$ and effective range $r_{0}$ are defined as

$$
\begin{equation*}
k \cot \delta=\frac{1}{a}+\frac{1}{2} r_{0} k^{2} \tag{4.5}
\end{equation*}
$$

which leads to the expressions

$$
\begin{align*}
a & =-\frac{1}{4 \pi} \frac{M}{m+M} T(m+M), \\
r_{0} & =\lim _{k \rightarrow 0}\left[\frac{2}{k^{2}}\left(-\frac{4 \pi \sqrt{s}}{M} T^{-1}(\sqrt{s})+i k-\frac{1}{a}\right)\right] \tag{4.6}
\end{align*}
$$

where $m$ is the meson mass and $T=T(\sqrt{s})$ depends only on the CM energy $\sqrt{s}$. Note the sign of $a$ in the definition of Eq. (4.5) which is chosen in

Table 4.2: Scattering lengths $a$ and effective range parameters $r_{0}$ for the particle channels in the model of Ref. [45].

| Channel | Re $a[\mathrm{fm}]$ | $\operatorname{Im} a[\mathrm{fm}]$ | $\operatorname{Re} r_{0}[\mathrm{fm}]$ | $\operatorname{Im} r_{0}[\mathrm{fm}]$ |
| :--- | :--- | :--- | :--- | :--- |
| $K^{+} \Sigma^{-} \rightarrow K^{+} \Sigma^{-}$ | -0.29 | +0.087 | +0.58 | -1.50 |
| $K^{0} \Sigma^{0} \rightarrow K^{0} \Sigma^{0}$ | -0.21 | +0.067 | -2.25 | -0.36 |
| $K^{0} \Lambda \rightarrow K^{0} \Lambda$ | -0.15 | +0.17 | +0.74 | -3.21 |
| $\pi^{-} p \rightarrow \pi^{-} p$ | +0.080 | +0.003 | -14.7 | -22.3 |
| $\pi^{0} n \rightarrow \pi^{0} n$ | -0.023 | 0 | -31.4 | 0 |
| $\eta n \rightarrow \eta n$ | +0.27 | +0.24 | -7.26 | +6.59 |

the way to have the same sign as in Ref. [45]. The phase shifts are given by complex numbers because the imaginary part of the "elastic" amplitudes such as $\eta n \rightarrow \eta n$ is not zero. The imaginary part describes the loss of particle flux into other coupled channels which are physically open. E.g., for the $\eta n \rightarrow \eta n$ transition, rescattering into the $\pi N$ channels is possible, whereas close to the $\eta N$ threshold rescattering into $K \Sigma$ or $K \Lambda$ channels can only be virtual and does not create imaginary parts in the $\eta n \rightarrow \eta n$ transition.

The scattering lengths and effective ranges for the six coupled channels are given in Tab. 4.2 for the net charge zero sector. As the model is using particle channels rather than isospin channels, there are non-trivial threshold effects, if two channels are close together in mass. In particular, this affects the channel lower in mass of the two close-by channels, which in the charge zero sector are $\pi^{0} n$ and $K^{0} \Sigma^{0}$. In those cases the effective range approximation is of no much use. Note that the $\pi^{-} p \rightarrow \pi^{-} p$ scattering length has a tiny imaginary part. This is due to isospin breaking from different masses so that the $\pi^{-} p$ threshold is higher than the $\pi^{0} n$ one.

In Fig. 4.2, the amplitude for $\eta n \rightarrow \eta n$ is shown, together with the low energy expansion using scattering length and effective range from above. The value from Tab. 4.2 for the $\pi^{-} p \rightarrow \pi^{-} p$ transition can be compared to experimental data from Eq. (2.41) which is 0.123 fm . The agreement is qualitative but not too good because the present model is optimized for the


Figure 4.2: Real (left) and imaginary (right) part of the scattering amplitude $f$ as a function of the CM energy $s^{1 / 2}$ for $\eta n \rightarrow \eta n$. Solid lines: Present model. Dashed lines: Effective range expansion, containing scattering length $a$ and effective range $r_{0}$.
energy region around the $N^{*}$ (1535) and we have found many other necessary ingredients for $\pi N$ scattering at low energies in the detailed study from chapter 2. The same applies for $\pi^{0} n \rightarrow \pi^{0} n$ which is proportional to the isoscalar which has been found to be a very sensitive quantity in chapter 2 and whose description requires terms from the NLO chiral Lagrangian which are not considered here. However, these results have to be seen in the light of the fact that the free parameters of the model have been fitted to $\pi N$ around the $N^{*}(1535)$ energies. The relatively good coincidence with $\pi^{-} p \rightarrow \pi^{-} p$, 450 MeV below the $N^{*}(1535)$, appears then as an unexpected success of the model.

The values from Tab. 4.2 other than $\pi N$ can be compared to the theoretical ones of Ref. [172], $a_{\eta N}=0.43+i 0.21 \mathrm{fm}, a_{K \Lambda}=0.26+i 0.10 \mathrm{fm}$, $a_{K \Sigma}^{1 / 2}=-0.15+i 0.09 \mathrm{fm}, a_{K \Sigma}^{3 / 2}=-0.13+i 0.04 \mathrm{fm}$. The present $\eta N$ scattering length has a similar imaginary part than in Ref. [172] whereas the real part is smaller in the present study. For $K \Lambda$, the real parts differ even in sign whereas the imaginary parts are in qualitative agreement. In order to compare the results for $K \Sigma$, we make the corresponding isospin combinations from the values of Tab. 4.2 and obtain $a_{K \Sigma}^{1 / 2}=-0.12+i 0.03 \mathrm{fm}$, $a_{K \Sigma}^{3 / 2}=-0.34+i 0.10 \mathrm{fm}$. Again, the present results and results from Ref. [172] coincide qualitatively but their quantitative disagreement reflects uncertainties of theoretical models at higher energies beyond the $N^{*}(1535)$ resonance


Figure 4.3: Photoproduction of $\eta p$ via the $N^{*}(1535)$ resonance (gray blob). Kroll-Ruderman term (a) and the meson pole term (b).
at $s^{1 / 2} \sim 1.65 \mathrm{GeV}$ and beyond, where the present model shows also disagreement with $\pi N \rightarrow \pi N$ data.

This point should be kept in mind in the following study of the $\gamma p \rightarrow \pi^{0} \eta p$ reaction and in particular for the $\gamma p \rightarrow \pi^{0} K^{0} \Sigma^{+}$reaction. However, we have now a tool at hand to estimate theoretical uncertainties which is the phenomenological $\pi N \rightarrow \eta N$ transition potential from Ref. [113]. We will use this potential in the reactions with photons in the following and compare to the predictions of the microscopical model with the dynamically generated resonance. The differences will be surprisingly small.

### 4.3 Single meson photoproduction

In the previous section we have seen that the dynamically generated $N^{*}(1535)$ resonance provides the correct strength in the $\pi N \rightarrow \eta N$ transition at low energies. Here, we test the model for the reaction $\gamma p \rightarrow \eta p$ with the basic photoproduction mechanisms plotted in Fig. 4.3, which consist of the meson pole term and the Kroll-Ruderman term - included for gauge invariance followed by the rescattering of the intermediate charged meson described by the model of the last section. We shall come back to the question of gauge invariance later on in the section by looking at other, subdominant diagrams.

The baryon-baryon-meson (BBM) vertex is given by the chiral Lagrangian

$$
\begin{equation*}
\mathcal{L}_{\mathrm{BBM}}=\frac{D+F}{2}\left\langle\bar{B} \gamma^{\mu} \gamma_{5} u_{\mu} B\right\rangle+\frac{D-F}{2}\left\langle\bar{B} \gamma^{\mu} \gamma_{5} B u_{\mu}\right\rangle \tag{4.7}
\end{equation*}
$$

with the notation from Sec. 4.2. The Kroll-Ruderman term is obtained from this interaction by minimal substitution and the $\gamma M M$ couplings emerge from scalar QED. For the $i$ th meson-baryon channel from Eq. (4.1), the $T$-matrix elements read

$$
\begin{align*}
t_{K R}^{i}(\sqrt{s}) & =-\frac{\sqrt{2} i e}{f_{i}} \vec{\sigma} \vec{\epsilon}\left(a_{\mathrm{KR}}^{i} \frac{D+F}{2}+b_{\mathrm{KR}}^{i} \frac{D-F}{2}\right) T^{(i 3)}(\sqrt{s}) G(\sqrt{s}) \\
t_{M P}^{i}(\sqrt{s}) & =\frac{\sqrt{2}}{f_{i}} \vec{\sigma} \vec{\epsilon}\left(a_{\mathrm{BBM}}^{i} \frac{D+F}{2}+b_{\mathrm{BBM}}^{i} \frac{D-F}{2}\right)\left(-i e c_{\gamma M M}^{i}\right) \\
& \times T^{(i 3)}(\sqrt{s}) \tilde{G}(\sqrt{s}) \tag{4.8}
\end{align*}
$$

where

$$
\begin{align*}
G(\sqrt{s}) & =\int^{\Lambda} \frac{d^{3} \mathbf{q}}{(2 \pi)^{3}} \frac{M}{2 \omega(\mathbf{q}) E(\mathbf{q})} \frac{1}{\sqrt{s}-E(\mathbf{q})-\omega(\mathbf{q})+i \epsilon} \\
\tilde{G}(\sqrt{s}) & =-\frac{M}{2(2 \pi)^{2}} \int_{0}^{\Lambda} d q q^{2} \int_{-1}^{1} d x \frac{q^{2}\left(1-x^{2}\right)}{E(q)} \frac{1}{\sqrt{s}-\omega-E(q)+i \epsilon} \\
& \times \frac{1}{\sqrt{s}-\omega^{\prime}-k-E(q)+i \epsilon} \frac{1}{\omega \omega^{\prime}} \frac{1}{k-\omega-\omega^{\prime}+i \epsilon} \frac{1}{k+\omega+\omega^{\prime}} \\
& \times\left[k \omega^{\prime}+(E(q)-\sqrt{s})\left(\omega+\omega^{\prime}\right)+\left(\omega+\omega^{\prime}\right)^{2}\right] \tag{4.9}
\end{align*}
$$

for the Kroll-Ruderman term and the meson pole, respectively. The amplitudes $T^{(i 3)}(\sqrt{s})$ are the strong transition amplitudes from channel $i$ to the $\eta p$ channel, following the ordering of table 1. In Eq. (4.8) and throughout this study we use the Coulomb gauge $\left(\epsilon^{0}=0, \vec{\epsilon} \cdot \mathbf{k}=0\right.$, with $\mathbf{k}$ the photon threemomentum). The assignment of momenta in Eq. (4.8) is given according to Fig. 4.3, $\omega=\sqrt{q^{2}+m_{\pi}^{2}}, \omega^{\prime}=\sqrt{q^{2}+k^{2}-2 q k x+m_{\pi}^{2}}, E(q)$ the baryon energy, $\sqrt{s}=P^{0}+k^{0}$ and $G$ being the meson-baryon loop function according to Ref. [45]. The coefficients $a, b, c$ are given in Table 4.3. The cut-off in Eq. (4.8) has been chosen $\Lambda=1400 \mathrm{MeV}$. With this value, the reduced model of

Table 4.3: Isospin coefficients for the Kroll-Ruderman term ( $a_{\mathrm{KR}}^{i}, b_{\mathrm{KR}}^{i}$ ), BBM vertex $\left(a_{\mathrm{BBM}}^{i}, b_{\mathrm{BBM}}^{i}\right)$, and $\gamma M M$ vertex $\left(c_{\gamma M M}^{i}\right)$.

|  | $\pi^{0} p$ | $\pi^{+} n$ | $\eta p$ | $K^{+} \Sigma^{0}$ | $K^{+} \Lambda$ | $K^{0} \Sigma^{+}$ |
| :--- | :--- | :--- | :--- | :--- | :--- | :--- |
| $a_{\mathrm{KR}}^{i}$ | 0 | -1 | 0 | 0 | $\sqrt{\frac{2}{3}}$ | 0 |
| $b_{\mathrm{KR}}^{i}$ | 0 | 0 | 0 | $-\frac{1}{\sqrt{2}}$ | $-\frac{1}{\sqrt{6}}$ | 0 |
| $a_{\mathrm{BBM}}^{i}$ | $\frac{1}{\sqrt{2}}$ | 1 | $\frac{1}{\sqrt{6}}$ | 0 | $-\sqrt{\frac{2}{3}}$ | 0 |
| $b_{\mathrm{BBM}}^{i}$ | 0 | 0 | $-\sqrt{\frac{2}{3}}$ | $\frac{1}{\sqrt{2}}$ | $\frac{1}{\sqrt{6}}$ | 1 |
| $c_{\gamma M M}^{i}$ | 0 | -1 | 0 | -1 | -1 | 0 |

the rescattering (see comment below Eq. (4.3)) provides the same strength as the data [173] at the maximum position of the total cross section. Once the cut-off has been fixed, we continue using this value for $\Lambda$ in the following sections, for the reduced and full model.

The amplitudes are unitarized by the coupled channel approach from Ref. [45] in the final state interaction which provides at the same time the $\eta$ production. This is indicated diagrammatically in Fig. 4.3 with the gray blob. The total amplitude including the rescattering part is then given by

$$
\begin{equation*}
T_{\gamma p \rightarrow \eta p}(\sqrt{s})=\sum_{i=1}^{6} t_{K R}^{i}(\sqrt{s})+t_{M P}^{i}(\sqrt{s}) \tag{4.10}
\end{equation*}
$$

The resulting cross section is plotted in Fig. 4.4 together with the data compilation from Ref. [173]. With the solid line, the result including the full model for the $M B \rightarrow \eta p$ transition according to Sec. 4.2 is plotted (dashed line: reduced model). The diagrams from Fig. 4.3, together with the unitarization, explain quantitatively the one-meson photoproduction at low energy which indicates that these mechanisms should be included in the two-meson photoproduction reactions of the next section.

Instead of our microscopic description of the $\eta$ production, one can also insert the phenomenological $\pi N \rightarrow \eta N$ transition amplitude from Sec. 4.2


Figure 4.4: Cross section for $\gamma p \rightarrow \eta p$. Dots: data from Ref. [173]. Solid line: Prediction including the full model from Ref. [45]. Dashed line: Reduced model from Ref. [45]. Thin dotted line: Phenomenological $\pi N \rightarrow \eta N$ potential from from Ref. [113].
and Ref. [113] into the rescattering according to Fig. 4.3. The channels $K^{+} \Sigma^{0}$ and $K^{+} \Lambda$ in the first loop play an important role and should be incorporated as initial states in the $M B \rightarrow \eta p$ transition. In this case we include them by replacing $T^{(i 3)}$ in Eq. (4.10) by

$$
\begin{equation*}
T^{(i 3)}(\sqrt{s}) \rightarrow \frac{T_{\mathrm{ph}}^{(23)}(\sqrt{s})}{T^{(23)}(\sqrt{s})} T^{(i 3)}(\sqrt{s}) \tag{4.11}
\end{equation*}
$$

where $T_{\mathrm{ph}}^{(23)}$ is the phenomenological $S_{11}$ amplitude to the transition $\pi^{+} n \rightarrow$ $\eta p$. The prescription of Eq. (4.11) is the correct procedure for the $\pi^{+} p$ channel which is the dominant one, and we assume it to be valid for the other channels. We choose $\Lambda=1400 \mathrm{MeV}$ for the cut-off as before. The cross section is displayed in Fig. 4.4 with the thin dotted line and indeed shows a wider shape.

As we can see in Fig. 4.4, the description of the data is only qualitative. Given the theoretical uncertainties one should not pretend a better agreement with the data. Yet, in both theoretical calculations the distribution is too narrow, reflecting most probably the lack of the $N^{*}(1650) S_{11}$ contribution in the theoretical calculation. The uncertainties of the model for this reaction will be considered later on in the study of the $\gamma p \rightarrow \pi^{0} \eta p$ reaction in order to estimate its theoretical uncertainties.

In Ref. [174] a more elaborate chiral unitary model for $\eta$ photoproduction is constructed. The number of free parameters is larger from the inclusion of low energy constants of the NLO chiral meson-baryon Lagrangian. This additional freedom allows for a broader shape of the $N^{*}(1535)$ in the $\pi^{-} p \rightarrow \eta p$ reaction which also results in a broader shape in the photoproduction $\gamma p \rightarrow \eta p$. Data of both reactions are included in the global fit of [174]. However, for the present purposes the qualitative agreement is sufficient because we will see in subsequent sections that the final results for the two-meson photoproduction are almost indpendent of the width of the $N^{*}(1535)$.

### 4.3.1 The ratio $\sigma(\gamma n \rightarrow \eta n) / \sigma(\gamma p \rightarrow \eta p)$

The ratio of cross sections $\sigma_{n} / \sigma_{p} \equiv \sigma(\gamma n \rightarrow \eta n) / \sigma(\gamma p \rightarrow \eta p)$ of $\eta$ photoproduction on the neutron and proton, respectively, has been measured on the
deuterium at TAPS/MAMI [175]. The experimental result is displayed in the upper left panel of Fig. 4.5. In the upper right panel one finds another compilation of data including also the production on He from Ref. [176], together with a theoretical calculation from Refs. [40, 41] indicated with the dashed line.

The simple model of $\eta$-photoproduction via dynamically generated $N^{*}$ (1535) which has been developed in Sec. 4.3 can be adapted straightforward in order to calculate $\sigma_{n} / \sigma_{p}$. The strategy will be to perform a fit of the free parameters of the model to $\sigma_{p}$, then calculate $\sigma_{n}$ with these values for the parameters and take the ratio $\sigma_{n} / \sigma_{p}$. The free parameters of the model are given by the regularization of the photon loops displayed in Fig. 4.3.

For the fit we slightly modify the cut-off scheme used in Eq. (4.8): varying the cut-off in Eq. (4.8) for the loop of the Kroll-Ruderman term and the meson pole term has practically the same effect as adding a constant to the real part of the loop function ${ }^{1}$. In Ref. [45] a good fit to $\pi N$-data in $S_{11}$ and $S_{31}$ has been obtained using such a scheme: the cut-offs of the loop functions $G$ have been fixed at $\Lambda=1 \mathrm{GeV}$; a variation of the cut-off was then simulated by adding constants to the $G$. Such a scheme is convenient and we adapt it to the present problem that requires special care due to problems of gauge invariance.

Let us concentrate on the left side of the first loops with $\gamma$ in Fig. 4.3, i.e., we consider only the tree level process $\gamma N \rightarrow \pi N$ with the momenta as assigned in the figure. Whereas the Kroll-Ruderman term gives a contribution proportional to $\boldsymbol{\sigma} \cdot \boldsymbol{\epsilon}$ the corresponding term from the pole term is

$$
\begin{equation*}
\boldsymbol{\sigma} \cdot \boldsymbol{\epsilon}\left[\mathbf{q}^{2}-\frac{(\mathbf{q} \mathbf{k})^{2}}{\mathbf{k}^{2}}\right] \frac{1}{(k-q)^{2}-m^{2}+i \epsilon} . \tag{4.12}
\end{equation*}
$$

Replacing $\boldsymbol{\epsilon} \rightarrow \mathbf{k}$, with $\mathbf{k}$ the photon momentum, both terms cancel in the limit discussed in Appendix C, Eq. (C.16). Thus, gauge variance is approximately ensured. Gauge invariance is fully restored by coupling the $\gamma$ directly to the baryonic lines and the components of the dynamically generated $N^{*}(1535)$ itself as discussed in the next section 4.3.2, where it is shown

[^4]

Figure 4.5: The ratio $\sigma_{n} / \sigma_{p}$ as a function of the photon lab energy $E_{\gamma}[\mathrm{MeV}]$. Upper left: from Ref. [175] from photoproduction of the $\eta$ on the deuteron. Upper right: from [176]. The theoretical calculation indicated with "Kaiser et al." is from Refs. [40, 41]. Below, (a)-(d): from dynamically generated $N^{*}(1535)$. Solid lines: Full model. Dashed lines: Reduced model for the $N^{*}(1535)$.
that these additional diagrams are all very small. We rewrite the regularization scheme from [45], $G=a+\int^{\Lambda=1 \mathrm{GeV}}$, with the subtraction constant $a$ according to

$$
\begin{align*}
G(\sqrt{s}) & =i \int^{\Lambda=1 \mathrm{GeV}} \frac{d^{4} q}{(2 \pi)^{4}}\left(\frac{M}{E} \frac{1}{\sqrt{s}-q^{0}-E(\mathbf{q})+i \epsilon}+\frac{a}{C}\right) \\
& \times \frac{1}{\left(q^{0}\right)^{2}-\mathbf{q}^{2}-m^{2}+i \epsilon}, \quad C=\int_{0}^{1 \mathrm{GeV}} \frac{d q}{2 \pi^{2}} \frac{q^{2}}{2 \omega} \tag{4.13}
\end{align*}
$$

for the ordinary meson-baryon loop function $G$. By considering Eq. (4.12) it is obvious that the expression for the meson pole loop, that restores approximate gauge invariance, is then given by

$$
\begin{align*}
\tilde{G}^{\prime}(\sqrt{s}) & =i \int^{\Lambda=1 \mathrm{GeV}} \frac{d^{4} q}{(2 \pi)^{4}}\left[\mathbf{q}^{2}-\frac{(\mathbf{q k})^{2}}{\mathbf{k}^{2}}\right] \frac{1}{(k-q)^{2}-m^{2}+i \epsilon} \\
& \times\left(\frac{M}{E(\mathbf{q})} \frac{1}{\sqrt{s}-q^{0}-E(\mathbf{q})+i \epsilon}+\frac{a}{C}\right) \frac{1}{q^{2}-m_{\pi}^{2}+i \epsilon} \tag{4.14}
\end{align*}
$$

Calling $\delta \tilde{G}$ the term with $a / C$ in this expression, the new regularization scheme is easily implemented in the expressions from Eq. (4.9) by the replacements $G \rightarrow G+a, \tilde{G} \rightarrow \tilde{G}+\delta \tilde{G}$ where

$$
\begin{equation*}
\delta \tilde{G}=\frac{a}{C} \frac{1}{8 \pi^{2}} \int_{-1}^{1} d x \int_{0}^{\Lambda} d q q^{4}\left(1-x^{2}\right) \frac{\omega+\omega^{\prime}}{\omega \omega^{\prime}} \frac{1}{k^{0}-\omega-\omega^{\prime}} \frac{1}{k^{0}+\omega+\omega^{\prime}} \tag{4.15}
\end{equation*}
$$

where $\omega^{2}=\mathbf{q}^{2}+m^{2}, \omega^{\prime 2}=(\mathbf{k}-\mathbf{q})^{2}+m^{2}, m(M)$ is the meson (baryon) mass, $x$ is the angle between $\mathbf{k}$ and $\mathbf{q}$, and all loops have the same cut-off $\Lambda=1$ GeV .

Besides the new regularization scheme we need the photon couplings for the charge zero sector for the reaction $\gamma n \rightarrow \eta n$. The coefficients from Eq. (4.8) are straightforward derived by the use of Eq. (4.7) and given in Tab. 4.4; note the different channel ordering. For the dynamically generated $N^{*}(1535)$ itself, the $C_{i j}$ coefficients in the charge zero sector are given in Ref. [45] in the same ordering.

We have performed various fits to the data [173] for $\gamma p \rightarrow \eta p$ allowing for two different subtraction constants $a_{\pi N}$ and $a_{K \Sigma}=a_{K \Lambda}$ in the photon

Table 4.4: Coefficients for the photoproduction for charge zero; for charge +1 see Tab. 4.3.

|  | $K^{+} \Sigma^{-}$ | $K^{0} \Sigma^{0}$ | $K^{0} \Lambda$ | $\pi^{-} p$ | $\pi^{0} n$ | $\eta n$ |
| :--- | :--- | :--- | :--- | :--- | :--- | :--- |
| $a_{\mathrm{KR}}^{i}$ | 0 | 0 | 0 | 1 | 0 | 0 |
| $b_{\mathrm{KR}}^{i}$ | -1 | 0 | 0 | 0 | 0 | 0 |
| $a_{\mathrm{BBM}}^{i}$ | 0 | 0 | $-\sqrt{2 / 3}$ | 1 | $-1 / \sqrt{2}$ | $1 / \sqrt{6}$ |
| $b_{\mathrm{BBM}}^{i}$ | 1 | $-1 / \sqrt{2}$ | $1 / \sqrt{6}$ | 0 | 0 | $-\sqrt{2 / 3}$ |
| $c_{\gamma M M}^{i}$ | -1 | 0 | 0 | 1 | 0 | 0 |

loops. In the reaction $\gamma p \rightarrow \eta p$ the photon can couple to the initial $\pi^{+} n$, $K^{+} \Sigma^{0}$, or $K^{+} \Lambda$ whereas for $\gamma n \rightarrow \eta n$ it can only couple to $K^{+} \Sigma^{-}$and $\pi^{-} p$. Thus, it makes no sense to allow for an individual $a_{K \Lambda}$ and we have set $a_{K \Sigma}=a_{K \Lambda}$. Once the subtraction constants are fixed, the cross section $\sigma_{n}$ for the reaction $\gamma p \rightarrow \eta n$ and subsequently $\sigma_{n} / \sigma_{p}$ is calculated. Note that we allow now for two different subtraction constants in contrast to Sec. 4.3 where only one cut-off was used for the regularization of all initial $\pi N$-, $K \Sigma$ - and $K \Lambda$-loops (see the discussion following Eq. (4.9)). This additional freedom is necessary as there are higher order corrections that affect photon loops with $\pi N$ differently than photon loops with $K \Sigma, K \Lambda$, in analogy to the model for the $N^{*}(1535)$ itself, where different subtraction constants are used for the channels $\pi N, K \Sigma, K \Lambda$, and $\eta N$.

We have performed the fits using different ranges of energy and limits for the subtraction constants, in order to check the stability of the results. For the fit parameters, see Tab. 4.5 and for the ratio $\sigma_{n} / \sigma_{p}$ see Fig. 4.5. The values in brackets in Tab. 4.5 for the subtraction constants correspond to the reduced model for the $N^{*}(1535)$ whereas the values without brackets are for the full model. In the following we summarize some observations for the different fits:

- Fit (a): The results in Fig. 4.5 show a decrease of the ratio at higher lab

Table 4.5: Fit parameters for $\gamma p \rightarrow \eta p$. The values in brackets correspond to the reduced model for the $N^{*}(1535)$.

| Fit | Fit range $\sqrt{s}[\mathrm{GeV}]$ | $a_{\pi N}[\mathrm{MeV}]$ | $a_{K \Sigma(\Lambda)}[\mathrm{MeV}]$ | Remarks |
| :--- | :--- | :--- | :--- | :--- |
| (a) | $1.488-1.573$ | $60.9(47.7)$ | $44.9(25.5)$ |  |
| (b) | $1.488-1.573$ | $-27.5(-26.8)$ | $-9.3(0.2)$ | $\left\|a_{\pi N}\right\| \leq 40 \mathrm{MeV}$ |
| (c) | $1.488-1.613$ | $40(40)$ | $19.9(20.6)$ | $\left\|a_{\pi N}\right\| \leq 40 \mathrm{MeV}$, |
|  |  |  |  | $\mathrm{BG} 2 \mu \mathrm{~b}$ |
| (d) | $1.488-1.573$ | $60.8(47.9)$ | $45.1(25.5)$ | isospin limit |

photon energies $E_{\gamma}=\left(s-M^{2}\right) /(2 M)$ that follows the same trend as the experimental result from the upper left plot. The reduced and the full model differ at higher photon energies which shows the high sensitivity of $\sigma_{n} / \sigma_{p}$ to the model of the $N^{*}(1535)$. Two ingredient have been removed in the reduced model: The $\pi \pi N$-loops which play a minor role for the $S_{11}$-channel in which the $N^{*}(1535)$ appears dynamically generated. Second, the form factors for the Weinberg-Tomozawa term, that serve in the original model of Ref. [45] to reduce strength at $\pi N$ energies below the $N^{*}$ (1535)-resonance, have been removed. These form factors are quite strong at the higher energies considered here and might lead to a distortion of the result for $\sigma_{n} / \sigma_{p}$. The reduced model might be more reliable in this case.

- Fit (b): In this fit the range for $a_{\pi N}$ has been restricted to be less than 40 MeV . As the values for $a_{\pi N}$ and $a_{K \Sigma(\Lambda)}$ in Tab. 4.5 show there is obviously another $\chi^{2}$-minimum for the combination of the two parameters because the values are quite different from Fit (a). However, the ratio follows again the same trend as the data; this should be compared to the theoretical calculation from [40, 41] that is plotted with the dashed line in the upper right panel in Fig. 4.5.
- Fit (c): For this fit, the range of $a_{\pi N}$ is again restricted. The fitted data ranges now further up in energy (up to $\sqrt{s}=1.613 \mathrm{MeV}$ ). Additionally, we have subtracted a constant background of $2 \mu \mathrm{~b}$ from the experimental cross section [173] for $\gamma p \rightarrow \eta p$. This has been motivated by the results from [177], Fig. 3, that shows a background of around $2 \mu \mathrm{~b}$ on top of the fitted $1 / 2^{-}$baryonic resonance contribution. However, this additional potential background does not change our results for $\sigma_{n} / \sigma_{p}$ much as we have also verified for the other fits. The inclusion of higher energy data in the fit, however, has the effect that $\sigma_{n} / \sigma_{p}$ for the full model does not decrease so much at higher energies as compared to the other fits and the experimental results. In fact, the agreement with experiment is remarkable.
- Fit (d): In this fit the isospin limit for different members of the same multiplet has been taken, meaning that, e.g., all pions have the same averaged mass; the same limit is taken for kaons and $\Sigma$-baryons. Compared to fit (a) which is performed under the same conditions but with different masses for charged and uncharged pions, kaons, $\Sigma$ 's, there are substantial changes visible, particularly at the lower photon energies. From this it becomes obvious that $\sigma_{n} / \sigma_{p}$ is extremely sensitive to isospin breaking effects.
- We can check the numerical code by taking the isospin limit and allowing the initial photon to couple only to a $\pi N$ loop (no coupling to initial loops of $K \Sigma, K \Lambda$ ). In this case we expect $\sigma_{n} / \sigma_{p}=1$ for all energies, and indeed this is the case. Note that this check also shows that the more complicated structures like $\pi \pi N$-loops from Eq. (4.3) and all other ingredients of the chiral unitary model for the $N^{*}$ (1535) have been implemented correctly.
- This limit (isospin limit plus photon coupling to $\pi N$ only) corresponds to a $N^{*}(1535)$-exchange model without any knowledge of the internal structure of this resonance. The outcome of $\sigma_{n} / \sigma_{p}=1$ for all energies is, therefore, the benchmark for all models for the $N^{*}(1535)$; in almost all fits the present model follows the experimental data quite better.

Although in this section we have allowed for two different subtraction constants for different photon loops, we will proceed with the the result from Sec. 4.3 in the following, for simplicity. This means that we return to the cut-off scheme and use $\Lambda=1.4 \mathrm{GeV}$ for all channels as discussed following Eq. (4.9). The final results for the $\gamma p \rightarrow \pi^{0} \eta p$ and related reactions do not depend much on this choice, first because there are other, much more dominant, contributions and, second, because the cut-off scheme with $\Lambda=1.4$ GeV for photon loops delivers a good fit for $\gamma p \rightarrow \eta p$ at low energies as we have seen in Fig. 4.4.

### 4.3.2 Some remarks on gauge invariance and chiral symmetry

At this point we would like to make some general comments concerning basic symmetries and the degree to which they are respected in our approach, as for instance chiral symmetry or gauge invariance.

In our approach we are using chiral Lagrangians which are used as the kernel of the Bethe Salpeter equation and which are chiral symmetric up to mass terms which explicitly break the symmetry. The unitarization does not break this symmetry of the underlying theory since it is respected in chiral perturbation theory ( $\chi \mathrm{PT}$ ), and a perfect matching with $\chi$ PT to any order can be obtained with the approach that we use, as shown in Ref. [24].

Tests of symmetries can be better done in field theoretical approaches that use, for instance, dimensional regularization for the loops. Although dimensional regularization is used here in the loops for meson baryon scattering, we have preferred to use a cut off for the first loop involving the photon and do some fine tuning to fit the data. Then, we use this cut off (which is well within reasonable values) for the other loops that we will find later on. The cut off method is also easier and more transparent when dealing with particles with a finite width as it will be our case. The use of this cut off scheme or the dimensional regularization are in practice identical, given the matching between the two loop functions done in Section 2 of Appendix A of Ref. [42]. There, one finds that the dimensional regularization formula and the one with cut off have the same analytical properties (the log-terms) and
are numerically equivalent for values of the cut off reasonably larger than the on shell momentum of the states of the loop, which is a condition respected in our calculations. By fine tuning the subtraction constant in dimensional regularization, or fine tuning the cut off, one can make the two expressions identical at one energy and practically equal in a wide range of energies, sufficient for studies like the present one. Of particular relevance is the explicit appearance of the log-terms in the cut off scheme which preserve all the analytical properties of the scattering amplitude.

The equivalence of the schemes would also guarantee that gauge invariance is preserved with the cut off scheme if it is also the case in dimensional regularization. This of course requires that a full set of Feynman diagrams is chosen which guarantees gauge invariance. At this point we can clearly state that the set of diagrams chosen in Fig. 4.3 is not gauge invariant. Some terms are missing, which we describe below, and which are omitted because from previous studies we know they are negligible for low energy photons [178]. Since the energy of the photon is not so small here, it is worth retaking the discussion which we do below.

The issue of gauge invariance for pairs of interacting particles has received certain attention [179-182], but for the purpose of the present paper we can quote directly the work of [183] which proves that when using the Bethe Salpeter equation with the kernel of the Weinberg-Tomozawa term, as we do here, gauge invariance is automatically satisfied when the coupling of the photon is made not only to the external legs and vertices, but also to the vertices and intermediate particle propagators of the internal structure of the Bethe-Salpeter equation.

A complete set of diagrams fulfilling gauge invariance requires in addition to the diagrams shown in Fig. 4.3 (a), (b), other diagrams where the photon couples to the baryon lines, vertices, or the internal meson lines from rescattering. We plot such diagrams in Fig. 4.6. All of them vanish in the heavy baryon approximation. This is easy to see. In diagram (a), Fig. 4.6, the first loop to the left (think for the moment about a $\pi^{+} n$ loop) contains a $p$-wave vertex of the $\boldsymbol{\sigma} \cdot \mathbf{q}$ type and an $s$-wave vertex, and vanishes in any case. Diagram (b) contains a $p$-wave and an $s$-wave vertex in the loop plus a $\gamma n n$ vertex proportional to $\boldsymbol{\sigma} \times \mathbf{k}$. In the baryon propagators one momen-
(a)


Figure 4.6: Photon coupling besides the diagrams from Fig. 4.3. The photon can also couple to the external baryon (a), internal baryon of the first loop (b), and components of the rescattering (c)-(e).
tum is $\mathbf{q}$ and the other one $\mathbf{q}+\mathbf{k}$ and the integral does not vanish. However, the contribution is of the order $\left(\frac{k}{2 M_{p}}\right)^{2}$ or $5 \%$. The term (c) has the same property, a $p$-wave and an $s$-wave vertex in the first loop to the left, and only the fact that the propagator depends on $\mathbf{k}+\mathbf{q}$ renders a small contribution (remember we are performing a nonrelativistic calculation by taking $\sigma \cdot \mathbf{q}$ for the Yukawa vertices, but this is more than sufficient for the estimates we do). In diagram (d) the $M M B B \gamma$ vertex is of the type $(\boldsymbol{\sigma} \times \mathbf{q}) \cdot \boldsymbol{\epsilon}$ (see Sec. 4.4.1), hence once again we have the same situation as before for the first loop to the left. Finally, in diagram (e) the photon is coupled to the internal meson line of the rescattering. In this case both the loop of the photon as well as the first one to the left contain just one $p$-wave coupling and the diagram is doubly suppressed.

In order to know more precisely how small are the diagrams in our particular case we perform the calculation of one of them, diagram (b), explicitly. By assuming a $\pi^{+} n$ in the loop to the left in diagram (b), we obtain for the loop

$$
\begin{align*}
\tilde{t}^{(b)} & =\frac{\mu_{n}}{2 M} \boldsymbol{\sigma} \cdot \boldsymbol{\epsilon} e \sqrt{2} \frac{D+F}{2 f_{\pi}} \int \frac{d^{3} \mathbf{q}}{(2 \pi)^{3}} \frac{1}{2 \omega(\mathbf{q})} \frac{M}{E_{n}(\mathbf{q})} \frac{M}{E_{N}(\mathbf{k}+\mathbf{q})} \\
& \times \frac{1}{\sqrt{s}-\omega(\mathbf{q})-k-E_{N}(\mathbf{k}+\mathbf{q})+i \epsilon} \frac{1}{\sqrt{s}-\omega(\mathbf{q})-k-E_{N}(\mathbf{q})+i \epsilon} \mathbf{q} \cdot \mathbf{k} \tag{4.16}
\end{align*}
$$

while the equivalent loop function for the Kroll-Ruderman term would be

$$
\begin{align*}
\tilde{t}^{(\mathrm{KR})} & =-\boldsymbol{\sigma} \cdot \boldsymbol{\epsilon} e \sqrt{2} \frac{D+F}{2 f_{\pi}} \\
& \times \int \frac{d^{3} \mathbf{q}}{(2 \pi)^{3}} \frac{1}{2 \omega(\mathbf{q})} \frac{M}{E_{n}(\mathbf{q})} \frac{1}{\sqrt{s}-\omega(\mathbf{q})-k-E_{N}(\mathbf{q})+i \epsilon} \tag{4.17}
\end{align*}
$$

where $\mu_{n}$ is the neutron magnetic moment. The explicit evaluation of the terms $\tilde{t}^{(b)}$ and $\tilde{t}^{(\mathrm{KR})}$ indicates that when the two terms are added coherently there is a change of $6 \%$ in $|t|^{2}$ with respect to the Kroll-Rudermann term alone. If we add now the term with $\pi^{0} p$ in the intermediate state of the first loop and project over $I=1 / 2$ to match with $\eta N$ in the final state, the contribution of the magnetic part is proportional to $2 \mu_{n}+\mu_{p}$ instead of $2 \mu_{n}$ with the $\pi^{+} n$ state alone, and the contribution becomes of the order of $2 \%$.

The convection term $e\left(\mathbf{p}+\mathbf{p}^{\prime}\right) \cdot \boldsymbol{\sigma} /(2 M)\left(p, p^{\prime}\right.$ nucleon momenta) of the $\gamma p p$ coupling (not present for the neutron) leads to an equally small contribution.

The exercise tells us how small is the contribution that vanishes exactly in the heavy baryon limit. Since we do not aim at a precision of better than $20 \%$, these terms are negligible for us and hence are not further considered.

### 4.4 Eta pion photoproduction

Having reviewed the single $\eta$ production in the meson-baryon sector and having applied the model to the single $\eta$ photo production we turn now to the more complex reaction $\gamma p \rightarrow \pi^{0} \eta p$. The reaction will be discussed in three steps: In the first part, the participating hadrons will be only mesons and baryons with their chiral interaction in $S U(3)$. In the second part the contributions from explicit baryonic resonances will also be taken into account as they are known to play an important role, e.g., in the two pion photoproduction [168-170]. Finally, the decay channels of the $\Delta^{*}(1700)$ into $\eta \Delta(1232)$ and $K \Sigma^{*}(1385)$ will be included.

### 4.4.1 Contact interaction and anomalous magnetic moment

One of the important features of the models for reactions that produce dynamically generated resonances is that the Lagrangians do not involve explicitly the resonance degrees of freedom. Thus, the coupling of photons and mesons is due to the more elementary components, in this case the mesons and baryons, which are the building blocks of the coupled channels and which lead to the resonance through their interactions.

We follow the formalism of Ref. [166] for the $\gamma p \rightarrow K^{+} \pi \Sigma$ reaction where the $\Lambda(1405)$ resonance is clearly visible in the $\pi \Sigma$ invariant mass distribution. The derivative coupling in the meson vertex of Eq. (2.6) leads to a $\gamma M M B B$ contact vertex through minimal coupling, see Fig. 4.7 (c), and guarantees approximate gauge invariance together with the meson pole terms of Fig. 4.7 (a),(b). Note that additional couplings to the initial or intermediate baryon

(a)

(b)

(c)

Figure 4.7: Photon interaction with mesons and a baryon. The straight dashed line symbolizes an outgoing meson and the curved line the meson in a loop of the final state interaction.
or vertices of the rescattering series are required to ensure gauge invariance; these terms are small as we have discussed in Sec. 4.3.2.

The contact term of Fig. 4.7 (c) is easily generated and assuming the reaction $\gamma M_{i} B_{i} \rightarrow M_{j} B_{j}$ the amplitude is given by

$$
\begin{equation*}
V_{i j}^{(\gamma)}=C_{i j} \frac{e}{4 f_{i} f_{j}}\left(Q_{i}+Q_{j}\right) \bar{u}\left(p^{\prime}\right) \gamma^{\mu} u(p) \epsilon_{\mu} \tag{4.18}
\end{equation*}
$$

with $Q_{i}, Q_{j}$ the meson charges. In the Coulomb gauge this becomes

$$
\begin{equation*}
V_{i j}^{(\gamma)}=-C_{i j} \frac{e}{4 f_{i} f_{j}} i \frac{\vec{\sigma} \times \mathbf{q}}{2 M_{p}} \vec{\epsilon}\left(Q_{i}+Q_{j}\right) \tag{4.19}
\end{equation*}
$$

in the $\gamma p$ CM frame. Since the initial channel $i$ is $\pi^{0} p$, or channel number 1 in the order of the channels from Sec. 4.2, we obtain

$$
\begin{equation*}
V_{1 j}^{(\gamma)}=-C_{1 j} \frac{e}{4 f_{1} f_{j}} i \frac{\vec{\sigma} \times \mathbf{q}}{2 M_{p}} \vec{\epsilon} Q_{j} . \tag{4.20}
\end{equation*}
$$

It was shown in Ref. [166] that the meson pole terms of Fig. 4.7 (a), (b) are small compared to the amplitude of Eq. (4.20) for energies where the final particles are relatively close to threshold, as is the case here, both at the tree level or when the photon couples to the mesons within loops. The coupling of the photon to the baryon components was also small and will be neglected here, as was done in Ref. [166].

Before we proceed to unitarize the amplitude, it is worth looking at the structure of Eq. (4.20) which contains the ordinary magnetic moment of the proton. It is logical to think that a realistic amplitude should contain also
the anomalous part of the magnetic moment. This is indeed the case if one considers the effective Lagrangians given in Ref. [184]

$$
\begin{align*}
\mathcal{L}= & -\frac{i}{4 M_{p}} b_{6}^{F}\left\langle\bar{B}\left[S^{\mu}, S^{\nu}\right]\left[F_{\mu \nu}^{+}, B\right]\right\rangle \\
& -\frac{i}{4 M_{p}} b_{6}^{D}\left\langle\bar{B}\left[S^{\mu}, S^{\nu}\right]\left\{F_{\mu \nu}^{+}, B\right\}\right\rangle \tag{4.21}
\end{align*}
$$

with

$$
\begin{align*}
& F_{\mu \nu}^{+}=-e\left(u^{\dagger} Q F_{\mu \nu} u+u Q F_{\mu \nu} u^{\dagger}\right), \\
& F_{\mu \nu}=\partial_{\mu} A_{\nu}-\partial_{\nu} A_{\mu} \tag{4.22}
\end{align*}
$$

with $M_{p}$ the proton mass and $A_{\mu}$ the electromagnetic field. The operator $Q$ in Eq. (4.22) is the quark charge matrix $Q=\operatorname{diag}(2,-1,-1) / 3$ and $S^{\mu}$ is the spin matrix which in the rest frame becomes $(0, \vec{\sigma} / 2)$. In Ref. [185] the Lagrangians of Eq. (4.21) were used to determine the magnetic moment of the $\Lambda(1405)$. In the Coulomb gauge one has for an incoming photon

$$
\begin{equation*}
\left[S^{\mu}, S^{\nu}\right] F_{\mu \nu} \rightarrow(\vec{\sigma} \times \mathbf{q}) \vec{\epsilon} \tag{4.23}
\end{equation*}
$$

and, thus, the vertex from the Lagrangian of Eq. (4.21) can be written as

$$
\begin{align*}
& \mathcal{L} \rightarrow e \frac{\vec{\sigma} \times \mathbf{q}}{2 M_{p}} \vec{\epsilon}\left(\frac{i}{2} b_{6}^{F}\left\langle\bar{B}\left[\left(u^{\dagger} Q u+u Q u^{\dagger}\right), B\right]\right\rangle\right. \\
&\left.+\frac{i}{2} b_{6}^{D}\left\langle\bar{B}\left\{\left(u^{\dagger} Q u+u Q u^{\dagger}\right), B\right\}\right\rangle\right) . \tag{4.24}
\end{align*}
$$

Expanding the terms up to two meson fields leads to contact vertices with the same structure as Eq. (4.19). Taking $u=1$ in Eq. (4.24), and hence with no meson fields, provides the full magnetic moments of the octet of baryons from where one obtains the values of the coefficients [184, 185]

$$
b_{6}^{D}=2.40, \quad b_{6}^{F}=1.82
$$

It is easy to see [185] that by setting $b_{6}^{D}=0, b_{6}^{F}=1$, one obtains the ordinary magnetic moments of the baryons without the anomalous contribution. Similarly, taking the same values of $b_{6}^{D}, b_{6}^{F}$ one obtains Eq. (4.19) for the

Table 4.6: $X_{1 j}$ and $Y_{1 j}$ coefficients for the anomalous magnetic moment.

|  | $\pi^{0} p$ | $\pi^{+} n$ | $\eta p$ | $K^{+} \Sigma^{0}$ | $K^{+} \Lambda$ | $K^{0} \Sigma^{+}$ |
| :--- | :--- | :--- | :--- | :--- | :--- | :--- |
| $X_{1 j}$ | 0 | $\sqrt{2}$ | 0 | $\frac{1}{2}$ | $-\frac{1}{2 \sqrt{3}}$ | 0 |
| $Y_{1 j}$ | 0 | $\sqrt{2}$ | 0 | $-\frac{1}{2}$ | $-\frac{\sqrt{3}}{2}$ | 0 |

vertices $\gamma M M B B$. This is easily seen by explicitly evaluating the matrix elements of Eq. (4.24) which lead to the amplitude

$$
\begin{equation*}
-i t_{i j}^{\gamma}=-\frac{e}{2 M_{p}}(\vec{\sigma} \times \mathbf{q}) \vec{\epsilon} \frac{1}{2 f_{i} f_{j}}\left[X_{i j} b_{6}^{D}+Y_{i j} b_{6}^{F}\right] \tag{4.25}
\end{equation*}
$$

where the coefficients $X_{i j}$ and $Y_{i j}$ are given in Table 4.6. The combination of the $Y_{1 j}$ in Table 4.6 and the $C_{1 j}$ of Table 4.1 shows the identity of Eq. (4.25) and Eq. (4.20) for the case of $b_{6}^{D}=0, b_{6}^{F}=1$.

## Unitarization

For the amplitude $\gamma p \rightarrow \pi^{0} \eta p$, the first thing to realize is that at tree level the amplitude is zero with the interactions from Eqs. (4.20) and (4.25). It is the unitarization and the coupled channel procedure that renders this amplitude finite and sizable. The unitarization procedure with the coupled channels allows the intermediate channels with charged mesons of Eq. (4.1) to be formed, even if some of them are not physically open. The scattering of these states leads finally to $\eta p$. Diagrammatically, this is depicted in Fig. 4.8 which implicitly assumes the unitarization is implemented via the use of the Bethe-Salpeter equation (4.2) which generates the diagrams of Fig. 4.8.

Since $\eta p$ is channel 3 in our list of coupled channels, our final amplitude reads

$$
\begin{equation*}
T_{\gamma p \rightarrow \pi^{0} \eta p}=-i \sum_{j}\left(b_{6}^{D} X_{1 j}+b_{6}^{F} Y_{1 j}\right) \frac{e}{4 f_{1} f_{j}} \frac{\vec{\sigma} \times \mathbf{q}}{2 M_{p}} \vec{\epsilon} G_{j}(z) T^{(j 3)}(z), \tag{4.26}
\end{equation*}
$$

where $G_{j}$ is the meson-baryon loop function which is obtained in Ref. [45] using dimensional regularization, and the $T^{(j 3)}$ are the ordinary scattering
$\pi^{0} \eta$ photoproduction...


Figure 4.8: Unitarization of the transition amplitude for $\eta p$ production. The possible states (see table 4.1) for one and two loops are indicated.
matrices of the $\eta p$ and coupled channels from Eq. (4.2). The invariant kinematical argument $z$ is given by the invariant mass $M_{I}(\eta p)$ of the $\eta p$ system,

$$
\begin{equation*}
z=M_{I} \tag{4.27}
\end{equation*}
$$

or, alternatively,

$$
\begin{equation*}
z=\left(s+m_{\pi}^{2}-2 \sqrt{s} p_{\pi}^{0}\right)^{1 / 2} \tag{4.28}
\end{equation*}
$$

with $p_{\pi}^{0}=\left(\mathbf{p}_{\pi}^{2}+m_{\pi}^{2}\right)^{1 / 2}$, when the amplitude is expressed in terms of the invariant mass $M_{I}\left(\pi^{0} p\right)$ of the $\pi^{0} p$ system.

One might also question why we do not unitarize the other $\pi^{0}$ with the $\eta$ or the proton. The reason has to do with the chosen kinematics. By being close to threshold the $\pi^{0}$ has a small momentum and is far from the region of the $a_{0}(980)$ resonance that could be created interacting with the $\eta$. The generation of the $\pi^{0} p$ invariant masses in the $\Delta(1232)$ region in the phase space that we investigate is more likely. However, as one can see from Fig. 5 an extra loop of the $\pi^{0}$ and $p$ lines produces a $\Delta(1232)$ which would involve an $s$-wave vertex and a p-wave vertex. This would vanish in the loop integration in the limit of large baryon masses. Later, we shall consider other diagrams in which the $\Delta(1232)$ is explicitly produced.

### 4.4.2 Kroll-Ruderman and meson pole term

Next, we take into account diagrams which involve the $\gamma N \rightarrow \eta N$ amplitude which has been discussed in Sec. 4.3, and which are shown in Fig. 4.9. With


Figure 4.9: Kroll-Ruderman term and meson pole term as sub-processes in the $\pi^{0} \eta$ production.


Figure 4.10: Pion emission from inside the first meson-baryon loop. Diagram (d) is required by gauge invariance.

Eqs. (4.8) and (4.10) the amplitude for the diagrams in Fig. 4.9 is given by

$$
\begin{align*}
\left(-i T_{\gamma p \rightarrow \pi^{0} \eta p}\right) & =\frac{D+F}{2 f_{\pi}} \frac{M}{E_{N}\left(\mathbf{k}+\mathbf{p}_{\pi}\right)} \frac{i}{E_{N}(\mathbf{k})-p_{\pi}^{0}-E_{N}\left(\mathbf{k}+\mathbf{p}_{\pi}\right)} \\
& \times\left(-i T_{\gamma p \rightarrow \eta p}(z)\right)\left(-\vec{\sigma} \cdot \mathbf{p}_{\pi}\right) \tag{4.29}
\end{align*}
$$

where $z$ takes the values given by Eq. (4.27) or (4.28).

## Intermediate pion emission

In addition to the diagrams considered above, there are additional diagrams in which the $\pi^{0}$ is produced inside the first meson-baryon loop as displayed in Fig. 4.10. The amplitude for the channel $i$ for the sum of diagram (c) and
(d) is given by

$$
\begin{align*}
& t_{(c)+(d)}^{i}(\sqrt{s})= \\
& -\frac{e}{2 f_{\pi} f_{i}}\left(\vec{\sigma} \cdot \mathbf{p}_{\pi}\right)(\vec{\sigma} \vec{\epsilon})\left[a_{i}(D+F)+b_{i}(D-F)\right]\left[a_{i}^{\prime}(D+F)+b_{i}^{\prime}(D-F)\right] \\
& \times T^{(i 3)}(z) \int \frac{d^{3} \mathbf{q}}{(2 \pi)^{3}} \frac{1}{2 \omega} \frac{M}{E_{M}(q)} \frac{1}{\sqrt{s}-\omega-E_{M}(q)+i \epsilon} \frac{M^{\prime}}{E_{M^{\prime}}\left(\mathbf{q}+\mathbf{p}_{\pi}\right)} \\
& \times \frac{1}{\sqrt{s}-\omega-p_{\pi}^{0}-E_{M^{\prime}}\left(\mathbf{q}+\mathbf{p}_{\pi}\right)+i \epsilon}\left(1-\frac{\vec{q}_{\mathrm{on}}^{2}}{3 q_{\mathrm{on}}^{0} k^{0}}\right) F_{\pi}(q-k) \tag{4.30}
\end{align*}
$$

where the index $i$ stands for our standard ordering of the channels in Eq. (4.1) and the only non-zero values of the $a_{i}, a_{i}^{\prime}, b_{i}, b_{i}^{\prime}$ are: $a_{2}=-\frac{1}{\sqrt{2}}, a_{2}^{\prime}=$ $-1, a_{4}=b_{4}=\frac{1}{\sqrt{6}}, a_{4}^{\prime}=\sqrt{\frac{2}{3}}, b_{4}^{\prime}=-\frac{1}{\sqrt{6}}, a_{5}=b_{5}=\frac{1}{\sqrt{6}}, b_{5}^{\prime}=-\frac{1}{\sqrt{2}}$. Note that channel two has the external $\pi^{0}$ coupled to $n, n$ to the left and right in the diagram, channel four has the $\pi^{0}$ coupled to the $\Lambda, \Sigma$ to the left and right, and channel five has the $\pi^{0}$ coupled to the $\Sigma, \Lambda$ to the left and right. In the equation the variable $P-q$ refers to the baryon on the left (M) of the emitted $\pi^{0}$ and the variable $P-q-p_{\pi}$ to the right ( $M^{\prime}$ ) of the emitted $\pi^{0}$ as shown in Fig. 4.10. The contribution of the terms in Fig. 4.10 is therefore given by the sum of Eq. (4.30) for the three non-vanishing channels.

In Eq. (4.30) we introduce the ordinary meson-baryon form factor $F_{\pi}$ of monopole type with $\Lambda=1.25 \mathrm{GeV}$ as used in the two pion photoproduction [168]. It appears naturally in the meson pole term of Fig. 4.10 and, as done in Ref. [168], it is also included in Fig. 4.10(c) (Kroll-Ruderman term) for reasons of gauge invariance. This form factor does not change much the results and it is approximated by taking the $q^{0}$ variable on shell and making an angle average of the $\vec{q}$ momentum. This is done to avoid fictitious poles in the $q^{0}$ integrations.

The meson pole term (d) in Fig. 4.10 is small and an approximation can be made for the intermediate pion at $q-k$ which is far off-shell. This concerns terms with mixed scalar products of the form $\mathbf{k} \cdot \mathbf{q}$ that give only a small contribution when integrating over $\mathbf{q}$ in Eq. (4.30). Additionally, we have set in this term $q \equiv q_{\text {on }}$, the on-shell momentum of the other meson at $q$. This is, considering the kinematics, a good approximation.

One can also have the pion emission from the final proton. However, this would imply having the $\pi N \rightarrow \eta N$ amplitude away from the $N^{*}(1535)$



Figure 4.11: Terms with $\Delta^{*}(1700), N^{*}(1520)$, and $\Delta(1232)$. The diagram on the right is the $\Delta$-Kroll-Ruderman term. The latter diagram implies also a meson pole contribution, required by gauge invariance, which is not separately plotted but is included in the calculation.
resonance at a value $M_{I}=\sqrt{s}$ where the $\pi N \rightarrow \eta N$ amplitude would only provide a background term above the $N^{*}(1535)$ resonance. Once again the set of diagrams considered leads to small cross sections compared to the dominant terms to be considered later in the paper so further refinements are unnecessary.

### 4.4.3 Baryonic resonances in $\eta \pi^{0}$ production

In the present study, the $\eta \pi^{0}$ production is described as a two-step process: The first step consists in the photoproduction of two mesons and a baryon; the second step describes the subsequent transitions of meson-baryon $\rightarrow \eta p$ via the dynamically generated $N^{*}(1535)$ resonance. In particular, the first stage contains two pion photoproduction. For this part it is known that baryonic resonances such as $\Delta^{\prime}$ 's and $N^{*}$ 's can play an important role [168$170,186]$. For this reason we include the relevant mechanisms from Ref. [168] adapted to the present context. Fig. 4.11 shows the processes that are taken into account. The $s$-wave character of the $N^{*}(1535)$ (gray blob in Fig. 4.11) discards all those processes from Ref. [168] where both pions couple in $p$-wave to the baryons, because the $\pi N$ loop function involving odd powers of $\vec{q}$ in the integral is zero in the heavy baryon limit. For the remaining processes, some contributions to the $\pi^{+} \pi^{0}$ and $\pi^{0} \pi^{0}$ cross sections are small as, e.g., from the Roper resonance. Finally, one is left with the $\Delta^{*}(1700) \Delta, N^{*}(1520) \Delta$, and $\Delta$-Kroll-Ruderman terms from Fig. 4.11. The latter implies also a pole
term which is required by gauge invariance in the same way as in Fig. 4.10 (d). Since many resonances appear in this section we refer the reader to the notation used in the Introduction.

The amplitudes for diagrams (e) and (f) in Fig. 4.11 are given by

$$
\begin{align*}
T_{\gamma p \rightarrow \pi^{0} \eta p} & =\sum_{i=1,2} T^{(i, 3)}(z) \frac{1}{(2 \pi)^{2}} \int_{0}^{\Lambda} d q \int_{-1}^{1} d x t_{\Delta}^{i} \frac{q^{2}}{2 \omega} \frac{M}{E} \\
& \times \frac{1}{\sqrt{s}-\omega-p_{\pi}^{0}-E+i \epsilon} \frac{1}{\sqrt{s_{\Delta}}-M_{\Delta}+i \frac{\Gamma\left(\sqrt{s_{\Delta}}\right)}{2}} \tag{4.31}
\end{align*}
$$

with the sum only over the first two channels according to Eq. (4.1) and $\Lambda=1400 \mathrm{MeV}$ as in Sec. 4.3. The meson energy $\omega$, baryon energy $E$ and energy of the $\Delta(1232)$ read

$$
\begin{align*}
\omega^{2} & =m^{2}+q^{2}, \\
E^{2} & =M^{2}+q^{2}+p_{\pi}^{2}+2 q p_{\pi} x, \\
s_{\Delta} & =(\sqrt{s}-\omega)^{2}-q^{2} \tag{4.32}
\end{align*}
$$

with meson mass $m$, baryon mass $M$, and $p_{\pi}=\left|\mathbf{p}_{\pi}\right|$ with $p_{\pi}^{0}=\sqrt{p_{\pi}^{2}+m_{\pi}^{2}}$. The argument $z$ is given by Eq. (4.27) or (4.28).

Using the notation from Ref. [168] the amplitudes $t_{\Delta}^{i}$, which can depend on the loop momentum, are given by:

$$
\begin{align*}
& t_{\Delta}^{1}=t_{\gamma p \rightarrow)^{*}}^{\Delta^{*}(1700)}+t_{\gamma p \rightarrow \pi^{0}+\pi^{0} n}^{N^{*}(1520)}+t_{\gamma p \rightarrow \pi^{+} \pi^{0} n}^{\Delta-\mathrm{KR}},  \tag{4.33}\\
& t_{\Delta}^{2}=t_{\gamma p \rightarrow \pi^{0} \pi^{0} n}^{\Delta_{n}^{*}(1700)}+t_{\gamma p \rightarrow \pi^{0} \pi^{0} n}^{N^{*}(1520)}+t_{\gamma p \rightarrow \pi^{0} \pi^{0} p}^{\Delta-\mathrm{KR}} \tag{4.34}
\end{align*}
$$

where

$$
\begin{align*}
t_{\gamma p \rightarrow \pi^{+} \pi^{0} n}^{\Delta^{*}(170)}= & -i \frac{2}{\sqrt{3}} \frac{f_{\Delta N \pi}^{*}}{m_{\pi}} \vec{S} \cdot \mathbf{p}_{\pi}\left(\tilde{f}_{\Delta^{*} \Delta \pi}+\frac{1}{3} \frac{\tilde{g}_{\Delta^{*} \Delta \pi}}{m_{\pi}^{2}} \vec{q}^{2}\right) G_{\Delta^{*}}(\sqrt{s}) \\
& {\left[g_{1}^{\prime} \frac{\vec{S}^{\dagger} \cdot \mathbf{k}}{2 M}(\vec{\sigma} \times \mathbf{k}) \vec{\epsilon}-i \vec{S}^{\dagger} \cdot \vec{\epsilon}\left(g_{1}^{\prime}\left(k^{0}+\frac{\mathbf{k}^{2}}{2 M}\right)+g_{2}^{\prime} \sqrt{s} k^{0}\right)\right], }  \tag{4.35}\\
t_{\gamma p \rightarrow \pi^{+} \pi^{0} n}^{N^{*}(1520)}= & -i \frac{\sqrt{2}}{3} \frac{f_{\Delta N \pi}^{*}}{m_{\pi}} \vec{S} \cdot \mathbf{p}_{\pi}\left(\tilde{f}_{N^{*^{\prime}} \Delta \pi}+\frac{1}{3} \frac{\tilde{g}_{N^{*^{\prime}} \Delta \pi}}{m_{\pi}^{2}} \vec{q}^{2}\right) G_{N^{*^{\prime}}}(\sqrt{s}) \\
& {\left[g_{1} \frac{\vec{S}^{\dagger} \cdot \mathbf{k}}{2 M}(\vec{\sigma} \times \mathbf{k}) \vec{\epsilon}-i \vec{S}^{\dagger} \cdot \vec{\epsilon}\left(g_{1}\left(k^{0}+\frac{\mathbf{k}^{2}}{2 M}\right)+g_{2} \sqrt{s} k^{0}\right)\right], } \tag{4.36}
\end{align*}
$$

$t_{\gamma p \rightarrow \pi^{+} \pi^{0} n}^{\Delta-\mathrm{KR}}=\frac{e \sqrt{2}}{9}\left(\frac{f_{\Delta N \pi}^{*}}{m_{\pi}}\right)^{2}\left(2 \mathbf{p}_{\pi}-i\left(\vec{\sigma} \times \mathbf{p}_{\pi}\right)\right) \cdot \vec{\epsilon} F_{\pi}\left(q_{o n}-k\right)\left(1-\frac{1}{3} \frac{\vec{q}_{\mathrm{on}}^{2}}{q_{\mathrm{on}}^{0} k^{0}}\right)$,

$$
\begin{gather*}
t_{\gamma p \rightarrow \pi^{0} \pi^{0} n}^{\Delta^{*}(1700)}=\frac{1}{2 \sqrt{2}} t_{\gamma p \rightarrow \pi^{+} \pi^{0} n}^{\Delta^{*}(1700)},  \tag{4.38}\\
t_{\gamma p \rightarrow \pi^{0} \pi^{0} n}^{N^{*}(1520)}=\sqrt{2} t_{\gamma p \rightarrow \pi^{+} \pi^{0} n}^{N^{*}(1520)},  \tag{4.39}\\
t_{\gamma p \rightarrow \pi^{0} \pi^{0} p}^{\Delta-\mathrm{KR}}=0 .
\end{gather*}
$$

We have already projected out the $s$-wave parts of the $\Delta^{*}(1700) \Delta \pi$ and $N^{*}(1520) \Delta \pi$ transitions that come from the term $\tilde{g}_{N^{*} \Delta \pi} / m_{\pi}^{2} \vec{S}^{\dagger} \cdot \vec{p} \pi \vec{S} \cdot \vec{p}_{\pi}$, see Ref. [168]. The vector $\mathbf{p}_{\pi}$ depends implicitly on the invariant mass which will be specified later, Eqs. (4.64) or (4.66). The amplitudes in Eqs. (4.35)-(4.40) are formulated for real photons, which is the case we are considering here. The meson pole diagram related to the $\Delta$-Kroll-Ruderman term has been included in the last factor of Eq. (4.37) by making the same approximation as in Eq. (4.30) for the intermediate off-shell pion. The pion form factor $F_{\pi}$ (see Ref. [168]) has to be inserted since the intermediate pion in the meson pole term is far off-shell. For the $\Delta^{*}$ propagator,

$$
\begin{equation*}
G_{\Delta^{*}}(\sqrt{s})=\frac{1}{\sqrt{s}-M_{\Delta^{*}}+i \frac{\Gamma(\sqrt{s})}{2}}, \tag{4.41}
\end{equation*}
$$

the (momentum-dependent) width according to its main decay channels has been taken into account: For $\Delta^{*} \rightarrow \pi N$ in $d$-wave, $\Delta^{*} \rightarrow N \rho(N \pi \pi)$, and $\Delta^{*} \rightarrow \Delta \pi(N \pi \pi)$ we obtain in a similar way as in Ref. [168]

$$
\begin{aligned}
& \Gamma_{\Delta^{*} \rightarrow N \pi}(\sqrt{s})=\Gamma_{\Delta^{*} \rightarrow N \pi}\left(M_{\Delta^{*}}\right) \frac{q_{\mathrm{CM}}(\sqrt{s})^{5}}{q_{\mathrm{CM}}\left(M_{\Delta^{*}}\right)^{5}}, \\
& \Gamma_{\Delta^{*} \rightarrow N \rho[\pi \pi]}(\sqrt{s})=\frac{M_{N}}{6(2 \pi)^{3}} \frac{m_{\Delta^{*}}}{\sqrt{s}} g_{\rho}^{2} f_{\rho}^{2} \\
& \times \int d \omega_{1} d \omega_{2}\left|D_{\rho}\left(q_{1}+q_{2}\right)\right|^{2}\left(\mathbf{q}_{1}-\mathbf{q}_{2}\right)^{2} \Theta(1-|A|), \\
& A=\frac{\left(\sqrt{s}-\omega_{1}-\omega_{2}\right)^{2}-M_{N}^{2}-\mathbf{q}_{1}^{2}-\mathbf{q}_{2}^{2}}{2\left|\mathbf{q}_{1}\right|\left|\mathbf{q}_{2}\right|}
\end{aligned}
$$

$$
\begin{align*}
\Gamma_{\Delta^{*} \rightarrow \Delta \pi[N \pi \pi]} & =\frac{15}{16 \pi^{2}} \int d M_{I} \frac{M_{I} k\left(M_{I}\right)}{4 \pi \sqrt{s}} \frac{\Gamma_{\Delta \rightarrow N \pi}\left(M_{I}\right)\left(\left|A_{s}\right|^{2}+\left|A_{d}\right|^{2}\right)}{\left(M_{I}-M_{\Delta}\right)^{2}+\left(\frac{\Gamma_{\Delta \rightarrow N \pi}\left(M_{I}\right)}{2}\right)^{2}} \\
& \times \Theta\left(\sqrt{s}-M_{I}-m_{\pi}\right) . \tag{4.42}
\end{align*}
$$

Here, $q_{\mathrm{CM}}(\sqrt{s})$ is the CM momentum of the pion and the nucleon and $\Gamma_{\Delta^{*} \rightarrow N \pi}\left(M_{\Delta^{*}}\right)$ is determined through the branching ratio into that channel. For the $s$-wave decay into $N \rho, g_{\rho}=2.6$ is the $\Delta^{*} N \rho$ coupling, also determined through the branching ratio. Furthermore, $f_{\rho}=6.14$ is the $\rho \pi \pi$ coupling, $q_{i}=\left(\omega_{i}, \mathbf{q}_{i}\right), i=1,2$ the four-momentum of the outgoing pions, and $D_{\rho}$ the $\rho$ propagator incorporating the $\rho$ width. For the decay into $\Delta \pi$, the finite width of the $\Delta, \Gamma_{\Delta \rightarrow N \pi}$, has been taken into account by performing the convolution. For the partial amplitudes $A_{s}$ and $A_{d}$ of the $\Delta^{*}$ decay into $\Delta \pi$ in $s$ and $d$-wave, see Ref. [168]. The $N^{*}(1520)$ propagator is dressed in a similar way with the analytic expressions given in Ref. [168].

There is an additional relevant channel given in the $\operatorname{PDB}$ [57]: the $\rho N$ decay in $d$-wave is assigned a branching ratio that can reach $30 \%$. At threshold and close above the $\Delta^{*}(1700)$ mass this results in a small contribution. The $\rho N$ channel is physically closed. However, when going to higher energies, this channel can contribute. We have not taken the $\rho N d$-wave channel into account in this chapter but will discuss the consequences in chapter 5 .

### 4.4.4 $S U(3)$ couplings of the $\Delta^{*}(1700)$

In Ref. [44] the rescattering of the $0^{-}$meson octet with the $3 / 2^{+}$baryon decuplet leads to a set of dynamically generated resonances, one of which has been identified with the $\Delta^{*}(1700)$. The advantage of such a microscopic model is that couplings of the resonance to decay channels are predicted which have not yet been determined experimentally.

## $s$-wave channels

Following Ref. [44], we briefly recall how the $\Delta^{*}(1700)$ appears as a dynamically generated resonance in the $s$-wave interaction of the $3 / 2^{+}$baryon decuplet with the $0^{-}$meson octet. The lowest order term of the chiral

Lagrangian relevant for the interaction is given by [66] (we use the metric $\left.g^{\mu \nu}=\operatorname{diag}(1,-1,-1,-1)\right)$

$$
\begin{equation*}
\mathcal{L}=-i \bar{T}^{\mu} \mathcal{D} T_{\mu} \tag{4.43}
\end{equation*}
$$

where $T_{a b c}^{\mu}$ is the spin decuplet field and $D^{\nu}$ the covariant derivative given by

$$
\begin{equation*}
\mathcal{D}^{\nu} T_{a b c}^{\mu}=\partial^{\nu} T_{a b c}^{\mu}+\left(\Gamma^{\nu}\right)_{a}^{d} T_{d b c}^{\mu}+\left(\Gamma^{\nu}\right)_{b}^{d} T_{a d c}^{\mu}+\left(\Gamma^{\nu}\right)_{c}^{d} T_{a b d}^{\mu} \tag{4.44}
\end{equation*}
$$

where $\mu$ is the Lorentz index, $a, b, c$ are the $S U(3)$ indices, and $\Gamma^{\nu}$ the vector current. The vector current $\Gamma^{\nu}$ is given by

$$
\begin{equation*}
\Gamma^{\nu}=\frac{1}{2}\left(\xi \partial^{\nu} \xi^{\dagger}+\xi^{\dagger} \partial^{\nu} \xi\right) \tag{4.45}
\end{equation*}
$$

with

$$
\begin{equation*}
\xi^{2}=U=e^{i \sqrt{2} \Phi / f} \tag{4.46}
\end{equation*}
$$

where $\Phi$ is the ordinary $3 \times 3$ matrix of fields for the pseudoscalar mesons [10] and $f=93 \mathrm{MeV}$. We shall only study the $s$-wave part of the baryon meson interaction which allows for some technical simplifications. For the RaritaSchwinger fields $T_{\mu}$ we take the representation $T u_{\mu}$ from ref. [187, 188] with the Rarita-Schwinger spinor $u_{\mu}$ given by

$$
\begin{equation*}
u_{\mu}=\sum_{\lambda, s} \mathcal{C}\left(1 \frac{1}{2} \frac{3}{2} ; \lambda s s_{\Delta}\right) e_{\mu}(p, \lambda) u(p, s) \tag{4.47}
\end{equation*}
$$

with $e_{\mu}=(0, \hat{e})$ in the particle rest frame, $\hat{e}$ the spherical representation of the unit vector $(\lambda=0, \pm 1), \mathcal{C}$ the Clebsch Gordan coefficients and $u(p, s)$ the ordinary Dirac spinors $\left(s= \pm \frac{1}{2}\right)$. Then, Eq. (4.43) involves the Dirac matrix elements

$$
\begin{equation*}
\bar{u}\left(p^{\prime}, s^{\prime}\right) \gamma^{\nu} u(p, s)=\delta^{\nu 0} \delta_{s s^{\prime}}+\mathcal{O}(|\vec{p}| / M) \tag{4.48}
\end{equation*}
$$

which for the $S$-wave interaction can be very accurately substituted by the non-relativistic approximation $\delta^{\nu 0} \delta_{s s^{\prime}}$ as done in Ref. [22] and related works. The remaining combination of the spinors $u_{\mu} u^{\mu}$ involves

$$
\begin{align*}
& \sum_{\lambda^{\prime}, s^{\prime}} \sum_{\lambda, s} \mathcal{C}\left(1 \frac{1}{2} \frac{3}{2} ; \lambda^{\prime} s^{\prime} s_{\Delta}\right) e_{\mu}^{*}\left(p^{\prime}, \lambda^{\prime}\right) \mathcal{C}\left(1 \frac{1}{2} \frac{3}{2} ; \lambda s s_{\Delta}\right) e^{\mu}(p, \lambda) \delta_{s s^{\prime}} \\
& =-1+\mathcal{O}\left(|\vec{p}|^{2} / M^{2}\right) \tag{4.49}
\end{align*}
$$

Consistently with the non-relativistic approximations done, and the free part of the Lagrangian of Eq. (4.43) the Rarita Schwinger propagator undressed from the spinors is the one of an ordinary non-relativistic particle in quantum mechanics. These approximations make the formalism analogous to that of $[22,189]$ regarding the meson baryon loops and the general treatment. For another example of the formalism see Appendix C. The interaction Lagrangian for decuplet-meson interaction can then be written in terms of the matrix

$$
\begin{equation*}
(\bar{T} \cdot T)_{a d}=\sum_{b, c} \bar{T}^{a b c} T_{d b c} \tag{4.50}
\end{equation*}
$$

as

$$
\begin{equation*}
\mathcal{L}=3 i \operatorname{Tr}\left\{\bar{T} \cdot T \Gamma^{0 T}\right\} \tag{4.51}
\end{equation*}
$$

where $\Gamma^{0 T}$ is the transposed matrix of $\Gamma^{0}$, with $\Gamma^{\nu}$ given, up to two meson fields, by

$$
\begin{equation*}
\Gamma^{\nu}=\frac{1}{4 f^{2}}\left(\Phi \partial^{\nu} \Phi-\partial^{\nu} \Phi \Phi\right) \tag{4.52}
\end{equation*}
$$

Let us recall the identification of the $S U(3)$ components of $T$ to the physical states [190, 191]:

$$
\begin{align*}
& T^{\mu}=T_{a d e} u^{\mu}, \quad \bar{T}_{\mu}=\bar{T}^{a d e} \bar{u}_{\mu}, \\
& T_{111}=\Delta^{++}, T_{112}=\frac{\Delta^{+}}{\sqrt{3}}, T_{122}=\frac{\Delta^{0}}{\sqrt{3}}, T_{222}=\Delta^{-}, T_{113}=\frac{\Sigma^{*+}}{\sqrt{3}}, \\
& T_{123}=\frac{\Sigma^{* 0}}{\sqrt{6}}, T_{223}=\frac{\Sigma^{*-}}{\sqrt{3}}, T_{133}=\frac{\Xi^{* 0}}{\sqrt{3}}, T_{233}=\frac{\Xi^{*-}}{\sqrt{3}}, T_{333}=\Omega^{-} . \tag{4.53}
\end{align*}
$$

The phase convention that we follow implies the phases for the isospin states, $\left|\pi^{+}\right\rangle=-|1,1\rangle,\left|K^{-}\right\rangle=-|1 / 2,-1 / 2\rangle,\left|\Sigma^{+}\right\rangle=-|1,1\rangle$.

In Ref. [44] the expansion of the Lagrangian is done up to two mesons of incoming (outgoing) momentum $k\left(k^{\prime}\right)$ which leads to an interaction kernel of the form

$$
\begin{equation*}
V_{i j}=-\frac{1}{4 f^{2}} C_{i j}\left(k^{0}+k^{\prime 0}\right) . \tag{4.54}
\end{equation*}
$$



Figure 4.12: Rescattering scheme with the coupled channels for the $\Delta^{*}(1700)$.
for the $s$-wave transition amplitudes, similar as in Ref. [22].
The matrix $V$ is then used as the kernel of the usual Bethe-Salpeter equation to obtain the unitary transition matrix [22]. This results in the matrix equation

$$
\begin{equation*}
T=(1-V G)^{-1} V \tag{4.55}
\end{equation*}
$$

where $G$ is a diagonal matrix representing the meson-baryon loop function given in Ref. [61]. The loop function contains an undetermined subtraction constant, which accounts for terms from higher order chiral Lagrangians that make it finite. In Ref. [61] the value of this constant has been fixed to $a_{i}=-2$ for a renormalization scale of $\mu=700 \mathrm{MeV}$.

In Fig. 4.12 the rescattering scheme with the coupled channels for the $\Delta^{*}(1700), \pi \Delta(1232), \eta \Delta(1232)$, and $K \Sigma^{*}(1385)$ is shown. In principle, a more realistic description of the $(I, S)=(3,3)$ with negative parity would require the inclusion of more channels such as $\pi N$ in $d$-wave or $\rho N$ in $s$ and $d$-wave. This is in principle feasible; for this study, however, we follow a semi-phenomenological approach taking the width of the $\Delta^{*}(1700)$ and its coupling to the photon from data analyses; in chapter 7 we will, however, see that the coupling to the photon is well reproduced using the present coupled channel approach.

### 4.4.5 Processes with $\Delta^{*} \eta \Delta$ and $\Delta^{*} K \Sigma^{*}$ couplings

The analytic continuation of the unitarized amplitude from Eq. (4.55) to the complex plane provides at the pole position the isospin $3 / 2$ couplings of the resonance to $\eta \Delta$ and $K \Sigma^{*}$. Identifying the pole with the $\Delta^{*}(1700)$


Figure 4.13: Left side: Coupling of the dynamically generated $\Delta^{*}(1700)$ to $K \Sigma^{*}$ and $\eta \Delta$. The loop is given by $\eta \Delta^{+} p, K^{+} \Sigma^{* 0} \Lambda$, or $K^{0} \Sigma^{*+} \Sigma^{+}$. Right side: $\Sigma^{*}$ Kroll-Ruderman term.
we can incorporate the model from Ref. [44] in the present study in the diagrammatic way as indicated on the left hand side of Fig. 4.13, with the $\gamma p \Delta^{*}$ coupling from Ref. [168]. This procedure can be regarded as a first step towards the incorporation of dynamically generated $3 / 2^{-}$resonances in the two meson photoproduction. In further studies, the initial $\gamma p \rightarrow \Delta^{*}$ process could be included in the microscopic model of Ref. [44] in a similar way as was done here for the $\gamma p N^{*}(1535)$ coupling in Sec. 4.3. However, phenomenologically, the procedure followed here is reliable. Indeed, the $\gamma p \Delta^{*}$ coupling can be directly taken from the dynamical model by coupling the photon to the mesons, baryons, and vertices that constitute the $\Delta^{*}(1700)$ in the picture of dynamical generation. This gives a very similar value as the phenomenological one from [168] as we will see in chapter 7.

The $\Delta^{*}(1700) \Delta \eta$ and $\Delta^{*}(1700) \Sigma^{*} K$ couplings from Ref. [44] are given up to a global sign by $g_{\eta}=1.7-i 1.4$ and $g_{K}=3.3+i 0.7$, respectively. However, in Ref. [44] the coupling to $\Delta \pi$ is also given, $g_{\Delta}=0.5+i 0.8$. The sign of the real part and the order of magnitude agree with the empirical analysis of the $\Delta^{*}(1700) \rightarrow \pi \Delta$ decay that we are using thus far [168]; hence, we take for $g_{\eta}$ and $g_{K}$ the values quoted above. We note that the cross section is almost independent of the global sign, whereas there are some minor differences in the invariant mass spectra.

Having included the $\Sigma^{*}$ in the $\Delta^{*}$ decay it is straightforward to consider also the corresponding $\Sigma^{*}$-Kroll-Ruderman term given on the right side of Fig. 4.13. This term, together with the other ones from this section, allows
for an extension of the model to higher energies, where the intermediate $\Delta(1232)$ from the processes of Sec. 4.4 .3 is off-shell but the $\Sigma^{*}(1385)$ is on-shell.

For the baryon decuplet baryon octet meson octet vertices, and the corresponding Kroll-Ruderman vertex, we take the effective Lagrangian from Ref. [190],

$$
\begin{align*}
\mathcal{L} & =\mathcal{C}\left(\bar{T}_{\mu} A^{\mu} B+\bar{B} A_{\mu} T^{\mu}\right) \\
& =\mathcal{C}\left(\sum_{a, b, c, c, e}^{1, \cdots, 3} \epsilon_{a b c} \bar{T}^{a d e} \bar{u}_{\mu} A_{d}^{b, \mu} B_{e}^{c}+\sum_{a, b, c, d, e}^{1, \cdots, 3} \epsilon^{a b c} \bar{B}_{c}^{e} A_{b, \mu}^{d} T_{a d e} u^{\mu}\right) \tag{4.56}
\end{align*}
$$

with the same phase conventions as in Eq. (4.43) and the spin and flavor structure as given in Ref. [44] and Eq. (4.53). The same phase convention is also used in Ref. [44]. This allows us to relate all the couplings to the one of $\Delta \pi N$. Up to a different phase, these factors agree with those used in Ref. [192]. In Appendix C the explicit matrix elements for all transitions from Eq. (4.56) are given for completeness.

The corresponding amplitudes for the diagrams in Fig. 4.13 read now:

$$
\begin{align*}
t_{\eta \Delta^{+} p}^{(3)} & =-\sqrt{\frac{2}{3}} g_{\eta} \frac{f_{\Delta N \pi}^{*}}{m_{\pi}} G_{\Delta^{*}}(\sqrt{s}) \vec{S} \cdot \mathbf{p}_{\pi} \\
& \times\left[-i g_{1}^{\prime} \frac{\vec{S}^{\dagger} \cdot \mathbf{k}}{2 M}(\vec{\sigma} \times \mathbf{k}) \vec{\epsilon}-\vec{S}^{\dagger} \cdot \vec{\epsilon}\left(g_{1}^{\prime}\left(k^{0}+\frac{\mathbf{k}^{2}}{2 M}\right)+g_{2}^{\prime} \sqrt{s} k^{0}\right)\right]  \tag{4.57}\\
t_{K^{+\Sigma^{*} 0} \Lambda}^{(5)} & =1.15 \sqrt{\frac{24}{25}} g_{K} \frac{D+F}{2 f_{\pi}} G_{\Delta^{*}}(\sqrt{s}) \vec{S} \cdot \mathbf{p}_{\pi} \\
& \times\left[-i g_{1}^{\prime} \frac{\vec{S}^{\dagger} \cdot \mathbf{k}}{2 M}(\vec{\sigma} \times \mathbf{k}) \cdot \vec{\epsilon}-\vec{S}^{\dagger} \cdot \vec{\epsilon}\left(g_{1}^{\prime}\left(k^{0}+\frac{\mathbf{k}^{2}}{2 M}\right)+g_{2}^{\prime} \sqrt{s} k^{0}\right)\right] \tag{4.58}
\end{align*}
$$

$$
\begin{align*}
t_{K^{0} \Sigma^{*+\Sigma^{+}}}^{(6)} & =\frac{2}{5} \frac{D+F}{2 f_{\pi}} g_{K} G_{\Delta^{*}}(\sqrt{s}) \vec{S} \cdot \mathbf{p}_{\pi} \\
& \times\left[-i g_{1}^{\prime} \frac{\vec{S}^{\dagger} \cdot \mathbf{k}}{2 M}(\vec{\sigma} \times \mathbf{k}) \cdot \vec{\epsilon}-\vec{S}^{\dagger} \cdot \vec{\epsilon}\left(g_{1}^{\prime}\left(k^{0}+\frac{\mathbf{k}^{2}}{2 M}\right)+g_{2}^{\prime} \sqrt{s} k^{0}\right)\right] \tag{4.59}
\end{align*}
$$

$$
\begin{equation*}
t_{\Sigma^{*}-\mathrm{KR}}^{(5)}=-1.15 e \frac{4 \sqrt{3}}{25}\left(\frac{D+F}{2 f_{\pi}}\right)^{2}\left(2 \mathbf{p}_{\pi}-i\left(\vec{\sigma} \times \mathbf{p}_{\pi}\right)\right) \cdot \vec{\epsilon} \tag{4.60}
\end{equation*}
$$

with $g_{\eta}$ and $g_{K}$ given in Ref. [44]. In order to obtain the full amplitudes, $T_{\gamma p \rightarrow \pi^{0} \eta p}$, these t's have to be inserted as $t_{\Delta}^{i}$ Eq. (4.31) but the sum over index $i$ goes now from three to six. The lower index for the amplitudes in Eqs. (4.57) - (4.60) indicates the particles in the loop to be considered in the evaluation of Eq. (4.31). The upper index indicates the channel number $i$ and therefore which $T^{(i 3)}$ has to be chosen in Eq. (4.31). For the amplitudes from Eq. (4.58), (4.59), and (4.60), the $\Delta(1232)$ propagator in Eq. (4.31) has to be replaced with the $\Sigma^{*}(1385)$ one. The latter is defined in the same way as the $\Delta$ (1232) propagator and we take a momentum-dependent width with $\Gamma_{\text {rest }}=36 \mathrm{MeV}$ assuming the dominant $p$-wave decay of the $\Sigma^{*}$ into $p \Lambda$. The numerical factor of 1.15 appearing in Eqs. (4.58) and (4.60) is a phenomenological correction factor from the $\operatorname{SU}(3) \Sigma^{*} \pi \Lambda$ coupling in order to provide the empirical $\Sigma^{*} \rightarrow \pi \Lambda$ partial decay width.

### 4.4.6 Tree level contribution from the $\Delta^{*}(1700) \rightarrow \eta \Delta$ decay

So far we have always considered processes emitting a pion first and then producing an $\eta$ from an intermediate excited state, the $N^{*}(1535)$. This is schematically shown in Fig. 4.14 on the right hand side. The $X^{*}$ can be a $\Delta^{*}(1700), N^{*}(1520), \cdots$. However, one can also reverse the order and emit the $\eta$ first, as indicated in Fig. 4.14. Indeed, in the $\pi^{0} \eta p$ production, the $\Delta^{*} \Delta \eta$ coupling together with the subsequent $\Delta \rightarrow \pi^{0} p$ decay provides such a term as shown in Fig. 4.15. The contribution for this reaction is simply given by

$$
\begin{equation*}
T_{\gamma p \rightarrow \pi^{0} \eta p}^{\mathrm{BG}}=t_{\eta \Delta+p}^{(3)} G_{\Delta}\left(z^{\prime}\right) \tag{4.61}
\end{equation*}
$$

from Eq. (4.57). The invariant argument $z^{\prime}$ for this amplitude differs from $z$ of the former processes,

$$
\begin{equation*}
z^{\prime}=\left(s+m_{\eta}^{2}-2 \sqrt{s} p_{\eta}^{0}\right)^{1 / 2}, \quad z^{\prime}=M_{I} \tag{4.62}
\end{equation*}
$$

with $p_{\eta}^{0}=\left(\mathbf{p}_{\eta}^{2}+m_{\eta}^{2}\right)^{1 / 2}$ depending on whether the amplitude is parametrized in terms of $M_{I}(\eta p)$ or $M_{I}\left(\pi^{0} p\right)$, respectively. We have explicitly tested that


Figure 4.14: Decay scheme for the two-meson photoproduction


Figure 4.15: Tree level process from the decay of the $\Delta^{*}(1700)$ to $\eta \Delta(1232)$.
recoil corrections for the $\Delta(1232)$ decay $\vec{S} \cdot \mathbf{p}_{\pi}$ in Eq. (4.61), in the way they are applied in Ref. [168], are negligible.

### 4.5 Results

In this section, invariant mass spectra $M_{I}(\eta p)$ and $M_{I}\left(\pi^{0} p\right)$ for the reaction $\gamma p \rightarrow \pi^{0} \eta p$ are predicted, together with the total cross section for this reaction. The corresponding observables for the $\pi^{0} K^{0} \Sigma^{+}$final state are also given. These observables can be directly compared to ongoing experiments at the ELSA facility [167].

We evaluate the phase space integrals for the invariant mass distribution of $\eta p$ in the $\eta p \mathrm{CM}$ system,

$$
\begin{align*}
\frac{d \sigma}{d M_{I}(\eta p)} & =\frac{1}{4(2 \pi)^{5}} \frac{M_{p} M_{i}}{s-M_{p}^{2}} \frac{\tilde{p}_{\eta} p_{\pi}}{\sqrt{s}} \int_{0}^{2 \pi} d \phi_{\pi} \int_{-1}^{1} d \cos \theta_{\pi} \\
& \times \int_{0}^{2 \pi} d \tilde{\phi} \int_{-1}^{1} d \cos \tilde{\theta} \overline{\sum \sum}\left|T_{\gamma p \rightarrow \pi^{0} \eta p}\right|^{2} \tag{4.63}
\end{align*}
$$

with $\tilde{p}_{\eta}$ the modulus of the momentum $\overrightarrow{\tilde{p}}_{\eta}$ of the $\eta$ in the $\eta p$ rest frame $\tilde{p}_{\eta}=\lambda^{1 / 2}\left(M_{I}^{2}, m_{\eta}^{2}, M_{p}^{2}\right) /\left(2 M_{I}\right)$ in terms of the ordinary Källen function where the direction of $\overrightarrow{\tilde{p}}_{\eta}$ is given by $\tilde{\phi}$ and $\tilde{\theta}$. This vector is connected to $\vec{p}_{\eta}$ in the $\gamma p$ rest frame by the boost

$$
\begin{equation*}
\vec{p}_{\eta}=\left[\left(\frac{\sqrt{s}-\omega_{\pi}}{M_{I}}-1\right)\left(-\frac{\vec{p}_{\eta} \vec{p}_{\pi}}{\vec{p}_{\pi}^{2}}\right)+\frac{\tilde{p}_{\eta}^{0}}{M_{I}}\right]\left(-\vec{p}_{\pi}\right)+\overrightarrow{\tilde{p}}_{\eta} \tag{4.64}
\end{equation*}
$$

where $\tilde{p}_{\eta}^{0}=\sqrt{\overrightarrow{\tilde{p}}_{\eta}^{2}+m_{\eta}^{2}}$ and the $\pi^{0}$ three momentum in the $\gamma p$ CM frame is given by the modulus $p_{\pi}=\lambda^{1 / 2}\left(s, M_{I}^{2}, m_{\pi}^{2}\right) /(2 \sqrt{s})$ and the two angles $\phi_{\pi}, \theta_{\pi}$. Furthermore, $\omega_{\pi}$ is the pion energy in the $\gamma p$ CM frame. In Eq. (4.63), $M_{p}, M_{i}$ are proton mass and mass of the final baryon, in the present case also a proton ( $i=3$ with the channel ordering from Eq. (4.1)). Eqs. (4.63) and (4.64) are a generalization of the corresponding expression in Ref. [166] as in the present case the amplitude depends explicitly on the angles of the particles.


Figure 4.16: Invariant mass at $E_{\gamma}=1.2 \mathrm{GeV}$. Dotted line: Contact interaction from Fig. 4.8 including the anomalous magnetic moment. Dashed dotted line: Meson pole plus Kroll-Ruderman term from Fig. 4.9. Double dashed dotted line: $\Delta^{*}(1700) K \Sigma^{*}$ transitions from Fig. 4.13, see Eqs. (4.58), (4.59). Solid line: Intermediate pion emission from Fig. 4.10.

The individual numerical contributions from the various processes from Sec. 4.4 are shown in Figs. 4.16, 4.17, and 4.18. We have chosen here a lab energy for the photon of $E_{\gamma}=1.2 \mathrm{GeV}$ so that the allowed invariant mass range is wide enough to distinguish the $N^{*}(1535)$ from pure phase space. On the other hand, this energy is low enough, so that unknown contributions from heavier resonances than the $\Delta^{*}(1700)$ should be small. All contributions contain the resonant structure of the $N^{*}(1535)$ in the final state interaction, except the background term from Eq. (4.61). Although the shape of this contribution is similar to the resonant part, this is a combined effect of phase space and the intermediate $\Delta(1232)$ that becomes less off-shell at lower invariant masses for $E_{\gamma}=1.2 \mathrm{GeV}$.

The individual contributions shown in Figs. 4.16, 4.17, and 4.18 are evaluated using the full model for the $N^{*}(1535)$ from Sec. 4.2 with the coherent sum indicated in Fig. 4.18, solid line. The coherent sum using the reduced model is displayed with the dashed line in Fig. 4.18. We take the difference between the two curves as an indication of the theoretical uncertainty as in the previous sections.


Figure 4.17: Invariant mass at $E_{\gamma}=1.2 \mathrm{GeV}$. Processes with explicit resonances. Dotted line: $\Delta^{*}(1700) \pi \Delta$ contribution from Fig. 4.11 (e), see Eqs. (4.35) and (4.38). Solid line: Contribution from $N^{*}(1520) \pi \Delta$ in Fig. 4.11 (e), see Eqs. (4.36) and (4.39). Dashed dotted line: $\Delta$-Kroll-Ruderman term from Fig. 4.11 (f), see Eq. (4.37). Double dashed dotted line: $\Sigma^{*}$-KrollRuderman term from Fig. 4.13, see Eq. (4.60).


Figure 4.18: Invariant mass at $E_{\gamma}=1.2 \mathrm{GeV}$. Dashed dotted line: $\Delta^{*}(1700) \eta \Delta$ transition and $\eta p \rightarrow \eta p$ rescattering from Fig. 4.13, see Eq. (4.57). Dotted line: Tree level process with $\Delta^{*}(1700) \eta \Delta$ transition but no rescattering, Fig. 4.15, see Eq. (4.61). Solid line: Coherent sum of all contributions, full model for the $N^{*}(1535)$. Dashed line: Coherent sum of all contributions, reduced model for $N^{*}(1535)$ from Sec. 4.2 (no vector particles in $t$-channel, no $\pi \pi N$ channel).



Figure 4.19: Modulus of the amplitude (full model for $N^{*}(1535)$ ) for the reaction $M_{i} B_{i} \rightarrow \eta p$ (left side) and $M_{i} B_{i} \rightarrow \pi^{0} p$ (right side). As these amplitudes appear squared in invariant mass spectra and total cross sections, they serve as a useful tool to distinguish dominant processes for the $\pi^{0} \eta p$ final state. Initial states: Dotted lines: $\pi^{0} p$, Dashed lines: $\pi^{+} n$, Solid lines: $\eta p$, Dashed dotted line: $K^{+} \Sigma^{0}$, Double dashed dotted lines: $K^{+} \Lambda$, Triple dashed dotted lines: $K^{0} \Sigma^{+}$.

The first thing to note is that the peak position of the $N^{*}(1535)$ is lowered by some 20 MeV due to the interference of the dynamically generated resonance with the background term from Fig. 4.15. A width of 93 MeV for the $N^{*}(1535)$ has been extracted in Ref. [45]. In the invariant mass spectra the $N^{*}(1535)$ exhibits a considerably smaller width. This is for two reasons: First, the $N^{*}(1535)$ is situated close to the $\eta p$ threshold and the phase space cuts the lower energy tail. This is clearly visible in Fig. 4.19: As the phase space factors in Eq. (4.63) are smooth functions around the $N^{\star}(1535)$ resonance, the shape of the curves in Figs. 4.16, 4.17, and 4.18 reflects the $N^{\star}(1535)$ resonance seen through a $|T|^{2}$ matrix involving the coupled channels.

The second reason for the narrow $N^{\star}(1535)$ is that at higher invariant mass the amplitude for the resonance is suppressed by the initial photoproduction mechanism: A closer inspection of the dominant resonant contributions as, e.g., from Eq. (4.57) shows that the $\Delta(1232)$ propagator in Eq. (4.31) of the first loop becomes more and more off-shell at higher $\eta p$ invariant masses which leads to a suppression of the spectrum for this kinematics. This effect is in fact so pronounced that the shape of the invariant mass dis-


Figure 4.20: Phenomenological potential for the $M B \rightarrow \eta p$ transition at $E_{\gamma}=1.2 \mathrm{GeV}$. With solid and dashed line, full and reduced model for the $N^{*}(1535)$ as in Fig. 4.18. Dotted line: Phenomenological potential in the meson-baryon $\rightarrow \eta p$ final state interaction for the diagrams from Figs. 4.8, 4.9, 4.10, 4.11, and 4.13.
tribution hardly changes if the $M B \rightarrow \eta p$ theoretical transition amplitudes are replaced in the scattering diagrams by the phenomenological ones given by Eq. (4.11). This is clearly seen in Fig. 4.20.

The transitions $\left|T^{(i 3)}\right|$ and $\left|T^{(i 1)}\right|$ in Fig. 4.19 explain the size of some of the contributions in Figs. 4.16, 4.17, and 4.18 as they appear squared in invariant mass spectra and cross section. E.g., the $\pi N \rightarrow \eta p$ transitions on the left side of Fig. 4.19 are small which explains why the $\Delta$-KrollRuderman term and the $N^{*}(1525) \pi \Delta, \Delta^{*}(1700) \pi \Delta$ transitions from Eqs. (4.35)-(4.40) contribute little, opposite to what was found in the two-pion photoproduction [168]. In contrast, the diagrams using $S U(3)$ Lagrangians without explicit resonances from Figs. 4.8, 4.9, and 4.10 contain $K \Lambda$ and $K \Sigma$ channels in the first loop so that the contributions are larger. The by far largest contributions in the rescattering part (Fig. 4.18) comes from the $\Delta^{*}(1700) \rightarrow \eta \Delta$ decay with the subsequent unitarization of $\eta p$. Indeed, the $\eta p \rightarrow \eta p$ scattering amplitude is very large as Fig. 4.19 shows. Additionally,


Figure 4.21: Selection of diagrams with $\pi^{0} p$ being the final state of the rescattering instead of $\eta p$. These diagrams are suppressed.
the loop for this reaction in Fig. 4.13 contains a $\eta$ instead of a $\pi$, and the particles in the loop can be simultaneously on-shell, whereas for the $\pi \Delta N$ loop at least one particle is always further off-shell.

The diagrams with $\Sigma^{*}$ (1385) in the first loop are relatively large (Fig. 4.16) due to the large $\Delta^{*}(1700) K \Sigma^{*}$ coupling and the large $K \Sigma \rightarrow \eta p$ and $K \Lambda \rightarrow \eta p$ transitions from Fig. 4.19. However, the $\Sigma^{*}(1385)$ is off-shell at $E_{\gamma}=1.2 \mathrm{GeV}$ and the contribution can not become as big as the loop from the $\Delta^{*}(1700) \rightarrow \Delta \eta$ decay. Therefore, diagrams with a $\Delta^{*}(1700) K \Sigma^{*}$ coupling become more important at higher energies. From Fig. 4.19, right side, we can also directly read off that additional diagrams like those displayed in Fig. 4.21 which use $T^{(i 1)}$ instead of $T^{(i 3)}$ are small compared to their counterparts from Sec. 4.4.

### 4.5.1 Extension to higher energies

In Fig. 4.22 the results for the invariant mass distribution are shown for higher values of the incoming $\gamma$ momentum, $q_{\text {lab }}=1.2-1.7 \mathrm{GeV}$. The resonant shape of the $N^{*}$ (1535) is not modified if a bigger photon energy is chosen, only the size decreases slightly as the intermediate $\Delta^{*}(1700)$ of the dominant processes becomes off-shell. At higher incident photon energies, the peak of the $N^{*}(1535)$ moves back to its original position around 1520-1540


Figure 4.22: Invariant mass spectrum $\frac{d \sigma}{d M_{I}(\eta p)}\left[\mu \mathrm{beV}^{-1}\right]$ as a function of $M_{I}(\eta p)[\mathrm{MeV}]$ for various photon lab energies $E_{\gamma}$. Solid and dashed lines: Full and reduced model for the $N^{*}(1535)$, respectively.


Figure 4.23: Integrated cross section for the $\gamma p \rightarrow \pi^{0} \eta p$ reaction. Solid line: Full model for the $N^{*}(1535)$. Dashed line: Reduced model (see Sec. 4.2). Dashed dotted line: Phenomenological potential for the $M B \rightarrow \eta p$ transition (only available up to $E_{\gamma} \sim 1.2 \mathrm{GeV}$ ).

MeV (see, e.g., Fig. 4.1) as the interference of the dynamically generated $N^{*}(1535)$ with the tree level process from Fig. 4.15 becomes weaker.

A second maximum appears for $E_{\gamma}>\sim 1.5 \mathrm{GeV}$ and moves to higher invariant masses with increasing photon energy. This can be traced back to be a reflection of the $\Delta(1232)$ resonance in the tree level process from Fig. 4.15 which is on-shell around the position of the second peak. When predicting this double hump structure in the $\eta p$ invariant mass, one has to keep in mind that our model for the dynamically generated $N^{*}(1535)$ resonance underpredicts the width of this resonance (see, e.g., Figs. 4.1, 4.4). Furthermore, there are unknown contributions from resonances heavier than the $\Delta^{*}(1700)$ about which little is known and which can fill up the space in invariant mass between the two humps. As a result, we expect a separation of the two maxima not at $E_{\gamma}=1.5 \mathrm{GeV}$ as Fig. 4.22 suggests but at higher energies. Nevertheless, the tree level process from Fig. 4.15 contributes so strongly to the coherent sum that the double hump structure should be qualitatively visible in experiment.

In Fig. 4.23 the integrated cross section is shown. There is a steep rise
below $E_{\gamma}=1.2 \mathrm{GeV}$ simply due to growing phase space. Above that, the cross section grows slower and finally saturates. At high photon energies, the tree level process and the dynamically generated $N^{*}(1535)$ are almost completely separated in invariant mass (see Fig. 4.22) and we do not expect a further rise beyond 1.7 GeV within our approach, as the particles involved in the various processes become more and more off-shell. However, the narrow $N^{*}(1535)$ width of our model, together with unknown contributions from resonances heavier than the $\Delta^{*}(1700)$, lead to uncertainties at high photon energies which are hard to control.

As we have already seen in Fig. 4.20, the use of the wider phenomenological potential increases the cross section slightly (dashed dotted line), but it is remarkable how insensitive the cross section is to the actual width of the $N^{*}(1535)$, regarding the large difference in width between the results using the phenomenological potential or microscopic theory which we have seen for the $\gamma p \rightarrow \eta p$ reaction in Fig. 4.4.

### 4.5.2 The $\pi^{0} p$ invariant mass

In the discussion of the last section we have seen that the $\Delta(1232)$ plays a prominent role in the $\pi^{0} \eta p$ photoproduction. For completeness, the invariant mass spectra for the $\pi^{0} p$ particle pair is given, which should show a signal of the $\Delta(1232)$. The phase space integrals are evaluated in the $\pi^{0} p$ rest frame and lead - similar to the expression in Eq. (4.63) - to the invariant mass distribution for $M_{I}\left(\pi^{0} p\right)$ :

$$
\begin{align*}
\frac{d \sigma}{d M_{I}\left(\pi^{0} p\right)} & =\frac{1}{4(2 \pi)^{5}} \frac{M_{p} M_{i}}{s-M_{p}^{2}} \frac{\tilde{p}_{\pi} p_{\eta}}{\sqrt{s}} \int_{0}^{2 \pi} d \phi_{\eta} \int_{-1}^{1} d \cos \theta_{\eta} \\
& \times \int_{0}^{2 \pi} d \tilde{\phi} \int_{-1}^{1} d \cos \tilde{\theta} \overline{\sum \sum\left|T_{\gamma p \rightarrow \eta \pi^{0} p}\right|^{2}} \tag{4.65}
\end{align*}
$$

with $M_{i}=M_{p}$ and with $\tilde{p}_{\pi}$ the modulus of the momentum $\overrightarrow{\tilde{p}}_{\pi}$ of the $\pi^{0}$ in the $\pi^{0} p$ rest frame $\tilde{p}_{\pi}=\lambda^{1 / 2}\left(M_{I}^{2}, m_{\pi}^{2}, M_{p}^{2}\right) /\left(2 M_{I}\right)$ where the direction of $\overrightarrow{\tilde{p}}_{\pi}$ is given by $\tilde{\phi}$ and $\tilde{\theta}$. This vector is connected to $\vec{p}_{\pi}$ in the $\gamma p$ rest frame by
the boost

$$
\begin{equation*}
\vec{p}_{\pi}=\left[\left(\frac{\sqrt{s}-\omega_{\eta}}{M_{I}}-1\right)\left(-\frac{\vec{p}_{\pi} \vec{p}_{\eta}}{\vec{p}_{\eta}^{2}}\right)+\frac{\tilde{p}_{\pi}^{0}}{M_{I}}\right]\left(-\vec{p}_{\eta}\right)+\overrightarrow{\tilde{p}}_{\pi} \tag{4.66}
\end{equation*}
$$

where $\tilde{p}_{\pi}^{0}=\sqrt{\overrightarrow{\tilde{p}}_{\pi}^{2}+m_{\pi}^{2}}$ and the $\eta$ three momentum in the $\gamma p$ CM frame is given by the modulus $p_{\eta}=\lambda^{1 / 2}\left(s, M_{I}^{2}, m_{\eta}^{2}\right) /(2 \sqrt{s})$ and the two angles $\phi_{\eta}, \theta_{\eta}$. Note that the invariant arguments for the solution of the Bethe-Salpeter equation (4.2) have changed compared to the case when the amplitude is expressed in terms of the $\eta p$ invariant mass, see Eqs. (4.27), (4.28), and (4.62).

The invariant mass distribution including all processes from this study is plotted in Fig. 4.24. We have checked explicitly for the individual processes and for the coherent sum of all processes that the integration over $M_{I}\left(\pi^{0} p\right)$ in Eq. (4.65) leads to the same values for the cross section as when integrating over the $\eta p$ invariant mass distribution from Eq. (4.63). In the plot for $E_{\gamma}=$ 1.2 GeV , the dotted line indicates the negligible effect of recoil corrections for the tree level process from Fig. 4.15 as described below Eq. (4.62).

In Fig. 4.24 we observe at $E_{\gamma}=1.2 \mathrm{GeV}$ a shift of strength towards higher invariant masses compared to the pure phase space (gray line) obtained by setting $T=$ const in Eq. (4.65). This is caused by the low energy tail of the $\Delta(1232)$ from the tree level process from Fig. 4.15, indicated with the dasheddotted line. Indeed, at higher photon energies, the intermediate $\Delta$ (1232) in this process shows up as a shoulder at $E_{\gamma}=1.5 \mathrm{GeV}$, and as a clear peak beyond. Additionally, there is a shift of strength towards higher invariant masses that results in a maximum which moves with energy, as it becomes apparent at $E_{\gamma}=1.5 \mathrm{GeV}$. This is a reflection of the $N^{*}(1535)$ resonance that becomes on-shell around these invariant masses, in full analogy to the reflection of the $\Delta(1232)$ resonance in the $\eta p$ invariant mass spectra in Fig. 4.22. As we have already argued in Sec. 4.5.1, the separation of the two peaks might happen at higher values of the incident photon energy but should be qualitatively visible in experiment.

At this point we would like to make some comments concerning the accuracy of our results. If one looks at the results obtained for the $\gamma p \rightarrow \eta p$ cross section in Fig. 4.4 we can see that except in the low energy regime close


Figure 4.24: Invariant mass spectrum $\frac{d \sigma}{d M_{I}\left(\pi^{0} p\right)}\left[\mu \mathrm{b} \mathrm{GeV}^{-1}\right]$ as a function of $M_{I}\left(\pi^{0} p\right)[\mathrm{MeV}]$ for various photon lab energies $E_{\gamma}$. Solid lines: Full model for the $N^{*}(1535)$. Gray lines: Phase space only ( $T=$ const). Dashed dotted lines: Only tree level process from Fig. 4.15, see Eq. (4.61). Dotted line in plot for $E_{\gamma}=1.2 \mathrm{GeV}$ : Effect when including recoil corrections for the $\Delta \pi^{0} p$ vertex for the tree level diagram in Fig. 4.15.
to threshold, we have large discrepancies between the three options and also with experiment. The agreement can be considered just qualitatively. It is clear that some background and the contribution of the $N^{*}(1650) S_{11}$ might be missing and that in any case the theoretical uncertainties from different acceptable options are as big as $20-25 \%$ at some energies. We should not expect better agreement with experiment in the $\gamma p \rightarrow \pi^{0} \eta p$ reaction which requires the $\gamma p \rightarrow \eta p$ amplitude in some terms (see Fig. 4.9). However, as we have discussed, these terms give a small contribution to the total amplitude, since the largest contribution comes from the tree level diagram of Fig. 4.15 and its unitarization in Fig. 4.13. Thus, at the end, the uncertainties in the result for the $\eta p$ invariant mass distribution, as seen in Fig. 4.20, are smaller than those of Fig. 4.4. Furthermore, when one integrates over the $\eta p$ invariant mass distribution, the uncertainties in the calculation in the total cross section are relatively small, although we would not claim a precision of better than $20 \%$ considering all the different sources that enter the calculation. Given the complexity of the model, such an uncertainty is not easy to decrease at the present time, but it is more than acceptable for this first model of the reaction.

### 4.5.3 The reaction $\gamma p \rightarrow \pi^{0} K^{0} \Sigma^{+}$

The $\gamma p \rightarrow \pi^{0} K^{0} \Sigma^{+}$reaction is calculated in a similar way as in the last sections for the $\pi^{0} \eta p$ final state as the coupled channel formalism for the $N^{*}(1535)$ contains the $K^{0} \Sigma^{+}$final state in a natural way. This is indicated in Fig. 4.25 that shows the various processes considered for the $\gamma p \rightarrow \pi^{0} K^{0} \Sigma^{+}$ and which are closely related to the contributions for the $\pi^{0} \eta p$ final state discussed in the last sections. There is, however, a different tree level diagram as displayed in Fig. 4.26 with the amplitude

$$
\begin{align*}
T_{\gamma p \rightarrow \pi^{0} K^{0} \Sigma^{+}}^{\mathrm{BG}} & =\frac{2}{5} \frac{D+F}{2 f_{\pi}} g_{K} G_{\Sigma^{*}}\left(z^{\prime \prime}\right) G_{\Delta^{*}}(\sqrt{s}) \vec{S} \cdot \mathbf{p}_{\pi} \\
& \times\left[-i g_{1}^{\prime} \frac{\vec{S}^{\dagger} \cdot \mathbf{k}}{2 M}(\vec{\sigma} \times \mathbf{k}) \cdot \vec{\epsilon}-\vec{S}^{\dagger} \cdot \vec{\epsilon}\left(g_{1}^{\prime}\left(k^{0}+\frac{\mathbf{k}^{2}}{2 M}\right)+g_{2}^{\prime} \sqrt{s} k^{0}\right)\right] \tag{4.67}
\end{align*}
$$



$$
\begin{aligned}
& (M, B)=(\pi N),(K \Sigma),(K \Lambda) \\
& \left(m, X^{*}, B\right)=(\pi \Delta N),(\eta \Delta N),\left(K \Sigma^{*} \Sigma\right),\left(K \Sigma^{*} \Lambda\right)
\end{aligned}
$$

Figure 4.25: The production of the $K^{0} \Sigma^{+}$final state from the dynamically generated $N^{*}(1535)$ (dashed blob).


Figure 4.26: Tree level contribution for the $\pi^{0} K^{0} \Sigma^{+}$final state.



Figure 4.27: Invariant mass spectrum for the reaction $\gamma p \rightarrow \pi^{0} K^{0} \Sigma^{+}$for two photon lab energies $E_{\gamma}$. Solid lines: full model for the $N^{*}(1535)$. Dashed lines: reduced model for the $N^{*}(1535)$. Dotted lines: Only tree level process from Fig. 4.26.
where

$$
\begin{equation*}
z^{\prime \prime}=\left(s+m_{K^{0}}^{2}-2 \sqrt{s} p_{K^{0}}^{0}\right)^{1 / 2} \tag{4.68}
\end{equation*}
$$

in analogy to Eq. (4.62), when expressing the amplitude in terms of the $K^{0} \Sigma^{+}$ invariant mass. The $\Sigma^{*}(1385)$ propagator $G_{\Sigma^{*}}$ has been given its width as explained below Eq. (4.60). Note that $T_{\gamma p \rightarrow \pi^{0} K^{0} \Sigma^{+}}^{B G}=t_{K^{0} \Sigma^{*+} \Sigma^{+}}^{(6)} G_{\Sigma^{*}}\left(z^{\prime \prime}\right)$ with $t_{K^{0} \Sigma^{*+\Sigma^{+}}}^{(6)}$ from Eq. (4.59) in analogy to Eq. (4.61). For the contributions with rescattering, we simply have to choose the $(i, 6)$ channel instead of the $(i, 3)$ channel in $T^{(i j)}$ from Eq. (4.2), in the ordering of the channels from Eq. (4.1). This means the replacement $T^{(j 3)} \rightarrow T^{(j 6)}$ in Eq. (4.26) and accordingly for the rest of the contributions. The invariant mass distribution is obtained from a similar formula as Eq. (4.63) with $M_{i}=M_{\Sigma^{+}}$and is plotted in Fig. 4.27.

The tree level contribution from Fig. 4.26, dotted line, dominates the spectrum. The reaction is situated at much higher energies than the $\gamma p \rightarrow$ $\pi^{0} \eta p$ process and the dynamically generated $N^{*}(1535)$ is off-shell for the entire invariant mass range. Thus, the rescattering part appears as a uniform background. The tree level process shows a pronounced maximum which moves with the incident photon energy. This situation is analogous to the moving peak of the $\Delta(1232)$ in Fig. 4.22 and reflects the $\Sigma^{*}(1385)$ which is on-shell around the peak position. The full and reduced model for the final


Figure 4.28: Integrated cross section for the $\gamma p \rightarrow \pi^{0} K^{0} \Sigma^{+}$reaction. Solid line: Full model for the $N^{*}(1535)$. Dashed line: Reduced model for the $N^{*}(1535)$. Dotted line: Contribution from the tree level diagram in Fig. 4.26.
state interaction (see Sec. 4.2) differ considerably in Fig. 4.27 at $E_{\gamma}=1.7$ GeV (solid versus dashed line). This is due to the fact that the model for the $N^{*}(1535)$ becomes uncertain at these high energies as it differs from the $\pi N \rightarrow \pi N$ partial wave analysis from Ref. [113] above $\sqrt{s} \sim 1600 \mathrm{MeV}$. At lower energies, the differences are smaller. In any case, in the energy range studied here, the dominant term is provided by the tree level diagram of Fig. 4.26.

The integrated cross section is displayed in Fig. 4.28. Comparing with Fig. 4.23 it becomes obvious that the production of $\pi^{0} K^{0} \Sigma^{+}$is highly suppressed. This is a combined effect of the $\Delta^{*}(1700)$ and the $N^{*}(1535)$ being off shell which we quantify below. Both reactions are compared at an energy of 150 MeV above their respective thresholds, where both cross sections have become significantly different from zero. For the resulting photon energies of $E_{\gamma}=1202 \mathrm{MeV}$ and 1603 MeV for the $\pi^{0} \eta p$ and $\pi^{0} K^{0} \Sigma^{+}$ final states, respectively, we obtain $\sigma_{\pi^{0}{ }_{\eta p}} / \sigma_{\pi^{0} K^{0} \Sigma^{+}}=10.9$. First, this ratio is an effect of the positive interference between the dynamically generated $N^{*}(1535)$ with the large contribution of the tree level term from




Figure 4.29: Distribution $\frac{d \sigma}{d \cos \theta d M_{I}}$ in $\left[\mu \mathrm{b} \mathrm{GeV}{ }^{-1}\right]$ for the $\Sigma^{+}$, for $E_{\gamma}=1.5$ GeV at $M_{I}\left(K^{0} \Sigma^{+}\right)=1690,1720$, and 1780 MeV (from left to right).

Fig. 4.15. Calculating the cross section by using this latter term only, $\sigma_{\pi^{0} \eta p}(1202 \mathrm{MeV})$ decreases by a factor 2.0. Second, and more important, the $\Delta^{*}(1700)$ propagator from Eq. (4.41) is off shell for the higher photon energy, $\left|G_{\Delta^{*}}(1202 \mathrm{MeV})\right|^{2} /\left|G_{\Delta^{*}}(1603 \mathrm{MeV})\right|^{2}=5.4$. Multiplying these two factors, one obtains 10.8 which clarifies the origin of the factor 10.9 quoted above.

Turning the argument around, if the experiment sees a factor 10 suppression of the $\pi^{0} K^{0} \Sigma^{+}$final state, compared to $\pi^{0} \eta p$, this can be easily explained by the dominant role of the $\Delta^{*}(1700)$ found in the present study.

Other resonances beyond the $\Delta^{*}(1700)$ can contribute at these high energies, and their omission produces uncertainties in the calculated cross section. However, assuming that we have included the relevant mechanisms in the present model, the suppression of the $\pi^{0} K^{0} \Sigma^{+}$versus the $\pi^{0} \eta p$ final state is such a strong effect that it should be visible in experiment.

## Angular distribution of the $\Sigma^{+}$in the $\gamma p \rightarrow \pi^{0} K^{0} \Sigma^{+}$reaction

In the ELSA/Bonn experiment the angular distribution of the $\Sigma^{+}$with respect to the incoming photon is also measured. We provide predictions from the present model for this observable. For this, we consider only the reaction at tree level from Fig. 4.26 with the contribution in Eq. (4.67). In the last section we have seen that this is the dominant term. The other, much smaller diagrams all rely on the unitarized meson-baryon $\rightarrow K^{0} \Sigma^{+}$amplitude which shows large theoretical uncertainties at these high energies.




Figure 4.30: Distribution $\frac{d \sigma}{d \cos \theta}$ in $[\mu \mathrm{b}]$ for the $\Sigma^{+}$, for $E_{\gamma}=1.4,1.5$, and 1.7 GeV (from left to right).

The results for the angular distribution in the $\gamma p$ CM frame are displayed in Fig. 4.29 and 4.30. In the first figure, we choose a photon lab energy of $E_{\gamma}=1.5 \mathrm{GeV}$ and plot $d \sigma /\left(d \cos \theta d M_{I}\right)$ for three different values of the $K^{0} \Sigma^{+}$invariant mass. In Fig. 4.30, $d \sigma / d \cos \theta$ is shown for three different lab photon energies. In both cases, one can see that integrating over $\cos \theta$ leads to the invariant masses and cross sections of Figs. 4.27 and 4.28 , respectively.

In all cases, we observe a nearly flat distribution with some very shallow minimum at $\cos \theta=0$, meaning a $\Sigma^{+}$perpendicular to the photon. Preliminary results from ELSA ${ }^{2}$ confirm the predictions of the angular distributions.

### 4.6 Conclusions

In this paper we have studied the reactions $\gamma p \rightarrow \pi^{0} \eta p$ and $\gamma p \rightarrow \pi^{0} K^{0} \Sigma^{+}$ making use of a chiral unitary framework which considers the interaction of mesons and baryons in coupled channels and dynamically generates the $N^{*}(1535)$. This resonance appears from the $s$-wave rescattering of $\eta N$ and coupled channels. We have used general chiral Lagrangians for the photoproduction mechanisms and have shown that even if at tree level the amplitudes for these reactions are zero, the unitarization in coupled channels renders the cross sections finite by coupling the photon to intermediate charged meson channels that lead to the $\eta p$ and $K^{0} \Sigma^{+}$in the final state through multiple scattering of the coupled channels.

[^5]The theoretical framework has been complemented by other ingredients, considering explicit excitation of resonances, whose couplings to photons are taken from experiment.

The interaction of the meson octet with the baryon decuplet leads to a set of dynamically generated resonances, one of which has been identified with the $\Delta^{*}(1700)$. The decay of this resonance into $\eta \Delta$ and $K \Sigma^{*}$, followed by the unitarization, or in other words, the $\Delta^{*}(1700) \rightarrow \pi^{0} N^{*}(1535)$ decay, provides in fact the dominant contribution to the $N^{*}(1535)$ peak in the invariant mass spectrum. A similar term provides also a tree level process which leads, together with the $N^{*}$ (1535), to a characteristic double hump structure in the $\eta p$ and $\pi^{0} p$ invariant mass at higher photon energies.

A virtue of this approach, concerning the $\eta p$ spectrum around the $N^{*}(1535)$, and a test of the nature of this resonance as a dynamically generated object, is that one can make predictions about cross sections for the production of the resonance without introducing the resonance explicitly into the formalism; only its components in the $\left(0^{-}, 1 / 2^{+}\right)$and $\left(0^{-}, 3 / 2^{+}\right)$meson-baryon base are what matters, together with the coupling of the photons to these components and their interaction in a coupled channel formalism. The reactions studied here also probe decay channels of the $\Delta^{*}(1700) \rightarrow \eta \Delta(1232), \Delta^{*}(1700) \rightarrow$ $K \Sigma^{*}(1385)$ or transitions like $\eta p \rightarrow N^{*}(1535) \rightarrow \eta p$ which are predicted by the model and not measured yet.

We have made predictions for the cross sections and for invariant mass distributions in the case of the $\gamma p \rightarrow \pi^{0} \eta p$ reaction. For the second reaction under study, $\gamma p \rightarrow \pi^{0} K^{0} \Sigma^{+}$, we could see that in the regions not too far from threshold of the $\gamma p \rightarrow \pi^{0} K^{0} \Sigma^{+}$reaction, the cross section for the latter one was much smaller than for the first reaction.

The measurement of both cross sections is being performed at the ELSA/ Bonn Laboratory and hence the predictions are both interesting and opportune and can help us gain a better insight in the nature of some resonances, particularly the $N^{*}(1535)$ and the $\Delta^{*}(1700)$ in the present case.

## Chapter 5

## Clues to the nature of the $\Delta^{*}(1700)$ resonance from pionand photon-induced reactions

The $\pi^{-} p \rightarrow K^{0} \pi^{0} \Lambda, \pi^{+} p \rightarrow K^{+} \pi^{+} \Lambda, K^{+} \bar{K}^{0} p, K^{+} \pi^{+} \Sigma^{0}, K^{+} \pi^{0} \Sigma^{+}$, and $\eta \pi^{+} p$ reactions are studied [6], in which the basic dynamics is given by the excitation of the $\Delta^{*}(1700)$ resonance which subsequently decays into $K \Sigma^{*}(1385)$ or $\Delta(1232) \eta$. In a similar way we also study the $\gamma p \rightarrow K^{0} \pi^{+} \Lambda, K^{+} \pi^{-} \Sigma^{+}$, $K^{+} \pi^{+} \Sigma^{-}, K^{0} \pi^{0} \Sigma^{+}$, and $\eta \pi^{0} p$ related reactions. The cross sections are proportional to the square of the coupling of $\Delta^{*}(1700)$ to $\Sigma^{*} K(\Delta \eta)$ for which there is no experimental information but which is provided in the context of coupled channels chiral unitary theory where the $\Delta^{*}(1700)$ is dynamically generated. Within present theoretical and experimental uncertainties one can claim a global qualitative agreement between theory and experiment. We provide a list of items which need to be improved in order to make further progress along these lines.

### 5.1 Introduction

Recent work has extended the number of dynamically generated resonances to the low lying $3 / 2^{-}$resonances which appear from the interaction of the octet of pseudoscalar mesons (M) with the decuplet of baryons ( $B^{*}$ ) [43, 44].

One of the $3 / 2^{-}$resonance which appears in this scheme is the $\Delta^{*}(1700)$, and in [44] the couplings of the resonance to the coupled channels $\Delta \pi, \Sigma^{*} K$, and $\Delta \eta$ were calculated. The couplings $g_{i}$ are 1.0, 3.4, and 2.2 , respectively, for these channels. It is interesting to note the large strength of the coupling to the $\Sigma^{*} K$ channel. Due to this, it was found in chapter 4 (Ref. [3]) that the $\gamma p \rightarrow K^{0} \pi^{0} \Sigma^{+}$reaction was dominated by the mechanisms shown in Fig. 4.26 where the $\Delta^{*}(1700)$ is excited by the photon and decays into $K^{0} \Sigma^{*+}$ and the $\Sigma^{*+}$ subsequently decays into $\pi^{0} \Sigma^{+}$. Since the cross section for this reaction is proportional to $\left(g_{\Delta^{*} K \Sigma^{*}}\right)^{2}$, the agreement of the predicted cross section with experiment would provide support for the coupling provided by the theory with the assumption of the $\Delta^{*}(1700)$ as a dynamically generated resonance. The experiment for this reaction has been performed at ELSA and is presently under analysis. Preliminary results presented in the NSTAR05 workshop [193] agree with the theoretical predictions.

We would like to stress the fact that the couplings predicted by the theory based on the dynamical nature of the $\Delta^{*}(1700)$ resonance are by no means trivial. Indeed, should we assume that the $\Delta^{*}(1700)$ belongs to an $S U(3)$ decuplet as suggested in the PDG (table 14.5) [57] it is easy to see that the couplings to the $\Delta \pi, \Sigma^{*} K, \Delta \eta$ states in $I=3 / 2$ are proportional to $\sqrt{5 / 8}$, $\sqrt{1 / 4}, \sqrt{1 / 8}$, respectively. The squares of these coefficients are proportional to $1,2 / 5,1 / 5$, respectively, compared to the squares of the coefficients of the dynamically generated resonance, $1,11.56,4.84$. It was noted in [44] that the strength of the $\Delta \pi$ coupling of the dynamically generated model was consistent with the experimental branching ratio of the $\Delta^{*}(1700)(33 \%$ from theory versus $25-50 \%$ branching ratio into $(\pi \Delta)_{s}$ from [57]). Hence, this means that the dynamically generated model produces considerable strength for the $\Sigma^{*} K$ and $\Delta \eta$ channels in absolute terms. With respect to the decuplet assumption of the PDG one obtains factors 27.5 and 24 larger for the square of the couplings to $\Sigma^{*} K$ and $\Delta \eta$, respectively. These large couplings indicate that, even if the $\Delta^{*}(1700)$ resonance is somewhat sub-threshold for the $\pi N \rightarrow$ $\Sigma^{*} K$ and $\pi N \rightarrow \Delta \eta$ reactions, the combination of these couplings and the $\Delta^{*}(1700)$ width $(\sim 300 \mathrm{MeV})$ should make this resonance play an important role in those reactions close to their thresholds.

It is clear that ultimately it is the consistency of the predictions of the
theoretical models that builds up support for the theory. Hence, it is straightforward to suggest an additional reaction with a similar mechanism as in Fig. 4.26 but rather with the $\Sigma^{*} \rightarrow \pi \Lambda$ decay. Since the branching ratio for $\pi \Lambda$ decay of the $\Sigma^{*}$ is $88 \%$, the cross section would be reasonably larger than for $\gamma p \rightarrow K^{0} \pi^{0} \Sigma^{+}$and one would have extra tests for the model.

Additional tests can be also done with the related reactions, $\pi^{-} p \rightarrow$ $K^{0} \pi^{0} \Lambda$ and all the other pion-induced reactions mentioned in the abstract, for which some data on cross sections are already available [58,194-197]. It is quite interesting to recall that in the theoretical model studied in [198], which was based on the excitation of the $\Lambda(1405)$, the $K^{0} K^{-} p, K^{0} \bar{K}^{0} n, K^{0} \pi^{+} \Sigma^{-}$, and $K^{0} \pi^{-} \Sigma^{+}$channels were reproduced within $25 \%$, while the cross section for the $K^{0} \pi^{0} \Lambda$ channel predicted was $6 \mu \mathrm{~b}$ compared to the $104 \pm 8 \mu \mathrm{~b}$ of the experiment. The lack of the $\Sigma^{*}(1385)$ resonance in the model of [198], which relied upon the final state interaction of the particles to generate dynamically the resonances, did not allow one to make a realistic approach for the $K^{0} \pi^{0} \Lambda$ final state, which in [58] was shown to be dominated by $K^{0} \Sigma^{* 0}$ production. The $\Sigma^{*}(1385)$, as all the other elements of the decuplet of the $\Delta(1232)$, does not qualify as a dynamically generated resonance, and is indeed a building block to generate other resonances like the $\Lambda^{*}(1520)$ or $\Delta^{*}(1700)$.

While at the time of [198] the information on the $\Delta^{*}(1700) \rightarrow K \Sigma^{*}$ coupling was not available, the works of [43, 44], and particularly [44] where the coupling is evaluated, have opened the door to tackle this reaction and this is one of the aims of the present work.

The data for the $\pi^{-} p \rightarrow K^{0} \pi^{0} \Lambda$ reaction from [58] is at $\sqrt{s}=2020$ MeV which is about 320 MeV above the $\Delta^{*}(1700)$ peak, potentially too far away to claim dominance of the $\Delta^{*}(1700)$, but there are also data at lower energies $[194,195]$ around $\sqrt{s}=1930-1980 \mathrm{MeV}$. On the other hand there are data $[196,197]$ for other reactions, $\pi^{+} p \rightarrow K^{+} \pi^{+} \Lambda, \pi^{+} p \rightarrow K^{+} \pi^{+} \Sigma^{0}$, $\pi^{+} p \rightarrow K^{+} \pi^{0} \Sigma^{+}$, and $\pi^{+} p \rightarrow \eta \pi^{+} p$ at energies around $\sqrt{s}=1800 \mathrm{MeV}$ which allow us to make a more direct comparison with the theoretical predictions. In the next section we study these reactions. Similarly, there are also some data for the $\gamma p \rightarrow K^{0} \pi^{+} \Lambda, K^{+} \pi^{-} \Sigma^{+}, K^{+} \pi^{+} \Sigma^{-}$reactions [200-202] and we shall also address them in the same context.

### 5.2 The model for the $\pi p \rightarrow K \pi \Lambda, K \pi \Sigma, K \bar{K} N$, $\eta \pi N$ reactions

In chapter 4 ([3]) on the $\gamma p \rightarrow K^{0} \pi^{0} \Sigma^{+}$reaction, the diagram of Fig. 4.26 was evaluated together with many loop diagrams involving rescattering of the $K \Sigma$ state which are the building blocks together with $\pi N$ and others of the $N^{*}(1535)$ resonance. It was found there that the tree level diagram was dominant. For energies of $\sqrt{s}=2000 \mathrm{MeV}$, hence 300 MeV above the $\Delta^{*}(1700)$ nominal mass, the loop terms could provide a contribution of about $30 \%$ with large uncertainties. On the other hand the results obtained for the cross section find support in preliminary experimental results presented at the NSTAR05 workshop [193]. With this information at hand we have good justification to propose that the dominant mechanism for the $\pi^{-} p \rightarrow K^{0} \pi^{0} \Lambda$, $\pi^{+} p \rightarrow K^{+} \pi^{+} \Lambda, K^{+} \bar{K}^{0} p, K^{+} \pi^{+} \Sigma^{0}, K^{+} \pi^{0} \Sigma^{+}$, and $\eta \pi^{+} p$ reactions is given by the diagrams of Fig. 5.1.

All the elements of these diagrams are at hand from chapter $4([3,44])$. The new information needed here is the $\pi N$ coupling to the $\Delta^{*}(1700)$ which is not an ingredient of the building blocks in the studies of $[43,44]$ that only take into account the interaction of the octet of pseudoscalar mesons with the decuplet of baryons. Thus, we take this information from experiment by looking at the branching ratio in the PDG [57]. In spite of the larger phase space for decay into this channel the branching ratio to $\pi N$ is only $10-20 \%$.

By taking into account that the coupling of $\pi N$ to $\Delta^{*}(1700)\left(3 / 2^{-}\right)$is in $d$-wave, the structure of the $\Delta^{*} \pi N$ vertex is most conveniently written as

$$
\begin{align*}
& -i t_{\Delta^{*}(1700) \rightarrow \pi N} \\
& =-i g_{\pi N \Delta^{*}}^{(d)} \mathcal{C}(1 / 223 / 2 ; m, M-m) Y_{2, m-M}^{*}(\hat{\mathbf{k}})(-1)^{M-m} \sqrt{4 \pi} \tag{5.1}
\end{align*}
$$

as in Ref. [61] (see their Eq. (25)) to account for the $\Lambda^{*}(1520) \rightarrow \pi \Sigma$ coupling. In Eq. (5.1) the Clebsch Gordan coefficient accounts for the matrix element of the rank two spin operator needed to couple the spherical harmonic $Y_{2}$ to a scalar. The quantities $M, m$ are the third components of the spin of the $\Delta^{*}$ and the nucleon and $\mathbf{k}$ is the momentum of the pion. The $\pi N$ state is in $I=3 / 2$.


Figure 5.1: Tree level contributions for the pion-induced strangeness production via the $\Delta^{*}(1700)$.

With the parametrization of Eq. (5.1) the partial decay width of the $\Delta^{*}$ to $\pi N$ is written as

$$
\begin{equation*}
\Gamma_{\Delta^{*} \rightarrow \pi N}=\frac{\left(g_{\pi N \Delta^{*}}^{(d)}\right)^{2}}{2 \pi} \frac{M_{N}}{M_{\Delta^{*}}} k_{\pi} . \tag{5.2}
\end{equation*}
$$

By taking the width of the $\Delta^{*}, \Gamma_{\Delta^{*}}=300 \pm 100 \mathrm{MeV}$, and the $\pi N$ branching ratio of $15 \pm 5 \%$ and summing the errors in quadrature we obtain the value

$$
\begin{equation*}
g_{\pi N \Delta^{*}}^{(d)}=0.94 \pm 0.20 \tag{5.3}
\end{equation*}
$$

which will necessarily lead to uncertainties of the order of $50 \%$ in the cross section. Given the isospin decomposition of the $\pi^{-} p$ state in $I=1 / 2,3 / 2$, the coupling $g_{\pi N \Delta^{*}}$ (in $I=3 / 2$ ) has to be multiplied by $\sqrt{1 / 3}$ to account for the coupling of the $\pi^{-} p$ state to the $\Delta^{*}$ and by -1 to account for the coupling of $\pi^{+} p$ to $\Delta^{*++}$ (although irrelevant for the cross section we use the isospin phase convention $\left.\left|\pi^{+}\right\rangle=-|1,1\rangle\right)$. This means

$$
\begin{equation*}
g_{\pi^{-} p \Delta^{* 0}}^{(d)}=\sqrt{\frac{1}{3}} g_{\pi N \Delta^{*}}^{(d)}, g_{\pi^{+} p \Delta^{*++}}^{(d)}=-g_{\pi N \Delta^{*}}^{(d)} \tag{5.4}
\end{equation*}
$$

There is another point worth noting which is that due to the $d$-wave character of the $\pi N \Delta^{*}$ vertex, the coupling $g_{\pi N \Delta^{*}}$ implicitly incorporates $k_{\pi}^{2}$ for the on-shell value of the pion momentum in the $\Delta^{*} \rightarrow \pi N$ decay. When we extrapolate beyond the resonance energy, as will be the case here, we must then use

$$
\begin{equation*}
g_{\pi N \Delta^{*}} \rightarrow g_{\pi N \Delta^{*}}(\text { on shell }) \frac{B W\left(k_{\pi} R\right)}{B W\left(k_{\pi}^{\text {on }} R\right)} \tag{5.5}
\end{equation*}
$$

where $B W(\cdot)$ is the Blatt and Weisskopff penetration factor $[29,203,204]$

$$
\begin{equation*}
B W(x)=\frac{x^{2}}{\left(9+3 x^{2}+x^{4}\right)^{1 / 2}} \tag{5.6}
\end{equation*}
$$

and $R=0.4 \mathrm{fm}$ according to best fits of [204].
The other couplings needed are those of the decuplet to the meson and baryon octets. For the vertices with $\Sigma^{*}(1385)$ or $\Delta(1232)$ decay in the diagrams of Fig. 5.1 we use the chiral Lagrangian [190]

$$
\begin{equation*}
\mathcal{L}=\mathcal{C}\left(\sum_{a, b, c, d, e}^{1, \cdots, 3} \epsilon_{a b c} \bar{T}^{a d e} \bar{u}_{\mu} A_{d}^{b, \mu} B_{e}^{c}+\sum_{a, b, c, d, e}^{1, \cdots, 3} \epsilon^{a b c} \bar{B}_{c}^{e} A_{b, \mu}^{d} T_{a d e} u^{\mu}\right) . \tag{5.7}
\end{equation*}
$$

which we have already defined in Eq. (4.56) and discussed following Eq. (4.56). The Lagrangian in Eq. (4.56) allows one to relate the $\pi \Lambda \Sigma^{*}$ coupling to $\pi N \Delta$. For the vertex one finds $-i t_{B^{*} \rightarrow B \Phi}=a \mathbf{S} \cdot \mathbf{q}$ where

$$
\begin{array}{rlrl}
a_{\Sigma^{* 0} \rightarrow \pi^{0} \Lambda} & =\frac{0.82}{\sqrt{2}} \frac{f_{\pi N \Delta}}{m_{\pi}}, & a_{\Sigma^{*+} \rightarrow \pi^{+} \Lambda} & =-\frac{0.82}{\sqrt{2}} \frac{f_{\pi N \Delta}}{m_{\pi}}, \\
a_{\Sigma^{*+} \rightarrow \bar{K}^{0} p} & =\frac{2 \sqrt{6}}{5} \frac{D+F}{2 f_{\pi}}, & a_{\Sigma^{*+} \rightarrow \pi^{+} \Sigma^{0}} & =-\frac{0.78}{\sqrt{6}} \frac{f_{\pi N \Delta}}{m_{\pi}}, \\
a_{\Sigma^{*+} \rightarrow \pi^{0} \Sigma^{+}} & =\frac{0.78}{\sqrt{6}} \frac{f_{\pi N \Delta}}{m_{\pi}}, & a_{\Delta++\rightarrow \pi^{+} p}=\frac{f_{\pi N \Delta}}{m_{\pi}} \tag{5.8}
\end{array}
$$

with $\mathbf{q}$ the momentum of the outgoing meson in the $\Sigma^{*}$ or $\Delta$ rest frame and $\mathbf{S}$ the spin transition operator from $3 / 2$ to $1 / 2$ normalized as

$$
\begin{equation*}
\langle M| S_{\mu}^{\dagger}|m\rangle=\mathcal{C}(1 / 213 / 2 ; m \mu M) . \tag{5.9}
\end{equation*}
$$

For the $f_{\pi N \Delta}$ we take $f_{\pi N \Delta}=2.13$ to give the experimental $\Delta$ width. Note that $S U(3)$ symmetry, implicit in Eq. (5.8), is not exact. In order to obtain the experimental $\Sigma^{*} \rightarrow \pi \Lambda, \Sigma^{*} \rightarrow \pi \Sigma$ widths one can fit the coupling $\mathcal{C}$ from Eq. (4.56) to the branching ratios from the PDG [57]. This leads to a correction which appears as a numerical factor in Eq. (5.8). For the $\Sigma^{*} \rightarrow \bar{K} N$ decay which is physically closed we use a $S U(6)$ quark model prediction [192].

The couplings from [44] of the $\Delta^{*}(1700)$ to its $s$-wave decay channels are given for $I=3 / 2$ and counting the isospin decomposition of $K^{0} \Sigma^{* 0}, K^{+} \Sigma^{*+}$, $\eta \Delta$, we find the couplings
$g_{K^{0} \Sigma^{* 0} \Delta^{* 0}}=\sqrt{\frac{2}{3}} g_{K \Sigma^{*} \Delta^{*}}, \quad g_{K+\Sigma^{*+} \Delta^{*++}}=g_{K \Sigma^{*} \Delta^{*}}, \quad g_{\eta \Delta^{++} \Delta^{*++}}=g_{\eta \Delta \Delta^{*}}$.

Altogether, our amplitudes for the diagrams of Fig. 5.1 become

$$
\begin{align*}
-i t & =a \mathbf{S} \cdot \mathbf{q} G \frac{1}{\sqrt{s_{\Delta^{*}}}-M_{\Delta^{*}}+\frac{i \Gamma_{\Delta^{*}\left(s_{\left.\Delta^{*}\right)}\right.}^{2}}{}} g_{j} g_{i}^{(d)} \frac{B W(k R)}{B W\left(k^{\mathrm{on}} R\right)} \\
& \times \mathcal{C}(1 / 223 / 2 ; m, M-m) Y_{2, m-M}(\hat{\mathbf{k}})(-1)^{M-m} \sqrt{4 \pi} \tag{5.11}
\end{align*}
$$

with $k^{\text {on }}$ the pion momentum in the $\pi N$ decay of the $\Delta^{*}(1700)$ at rest. Depending on the process, $G=1 /\left(\sqrt{s_{B^{*}}}-M_{B^{*}}+i / 2 \Gamma_{B^{*}}\left(\sqrt{s_{B^{*}}}\right)\right)$ is the $\Sigma^{*}(1385)$
or $\Delta$ (1232) propagator; $a$ in Eq. (5.11) is given by Eq. (5.8) and $g_{j}, g_{i}^{(d)}$ by Eqs. (5.10) and (5.4), respectively.

## The width of the $\Delta^{*}(1700)$

For the momentum-dependent width of the $\Sigma^{*}(1385)$, we have taken into account the $p$-wave decays into $\pi \Lambda$ and $\pi \Sigma$ with their respective branching ratios of $88 \%$ and $12 \%$. For the width of the $\Delta^{*}(1700)$ we have included the dynamics of the decay into $\Delta^{*} \rightarrow N \rho(N \pi \pi)$ and $\Delta^{*} \rightarrow \Delta \pi(N \pi \pi)$ in the same way as in chapter 4 (Ref. [3]). We introduce a novelty with respect to chapter 4.4.3 where the $\rho N$ decay of the $N^{*}(1520), \Delta^{*}(1700)$ was considered in $s$ wave. In the case of the $\Delta^{*}(1700)$ the $d$-wave $\rho N$ decay is mentioned in the PDG [57] as existing but with an undetermined strength. We have adopted here to take a $\rho N$ strength in $s$-wave of $37 \%$, and $5 \%$ for $\rho N$ in $d$-wave after a fine tuning to the data. For energies close to the $\Delta^{*}(1700)$ only the total width matters and this is about 300 MeV in our case. For the width of the $\Delta^{*} \rightarrow \rho N$ in $d$-wave we use the second equation of Eq. (4.42) by multiplying the numerator by $B W^{2}\left(\left|\mathbf{q}_{1}-\mathbf{q}_{2}\right| R\right)$ and one coupling is adjusted to get the $5 \%$ of the branching ratio used.

We should mention here that having $\rho N$ decay in $s$-wave or $d$-wave produces large differences at energies around $\sqrt{s}=2 \mathrm{GeV}$ and beyond, but only moderate differences close to the $\Delta^{*}(1700)$ peak or $100-150 \mathrm{MeV}$ above it.

For the $\Delta^{*}(1700)$ width from decay into $\pi N$ in $d$-wave, an additional Blatt-Weisskopff factor is applied to be consistent with Eq. (5.5). With the partial width $\Gamma_{\pi N}^{0}=0.15 \times 300 \mathrm{MeV}$, the width is given by [29]

$$
\begin{equation*}
\Gamma_{\pi N}=\Gamma_{\pi N}^{0} \frac{k B W^{2}(k R) M_{\Delta^{*}}}{k^{\mathrm{on}} B W^{2}\left(k^{\mathrm{on}} R\right) \sqrt{s}} \tag{5.12}
\end{equation*}
$$

and the total width is the sum of the partial widths of the decay modes. The same is done for the $d$-wave of the $\rho N$ decay as described above. In order to
summarize, the partial widths are given by

$$
\begin{aligned}
\Gamma_{\Delta^{*} \rightarrow N \pi}(\sqrt{s}) & =\Gamma_{\Delta^{*} \rightarrow N \pi}\left(M_{\Delta^{*}}\right) \frac{q_{c . \mathrm{m}} B W^{2}\left(q_{c . \mathrm{m}} R\right) M_{\Delta^{*}}}{q_{\mathrm{c} . \mathrm{m}}^{o n} B W^{2}\left(q_{\mathrm{c}, \mathrm{~m}}^{o \mathrm{o}} R\right) \sqrt{s}} \\
\Gamma_{\Delta^{*} \rightarrow(N \rho)_{s}[\pi \pi]}(\sqrt{s}) & =\frac{M_{N}}{6(2 \pi)^{3}} \frac{m_{\Delta^{*}}}{\sqrt{s}} g_{\rho, s}^{2} f_{\rho}^{2} \\
& \times \int d \omega_{1} d \omega_{2}\left|D_{\rho}\left(q_{1}+q_{2}\right)\right|^{2}\left(\mathbf{q}_{1}-\mathbf{q}_{2}\right)^{2} \Theta(1-|A|) \\
\Gamma_{\Delta^{*} \rightarrow(N \rho)_{d}[\pi \pi]}(\sqrt{s}) & =\frac{M_{N}}{6(2 \pi)^{3}} \frac{m_{\Delta^{*}}}{\sqrt{s}} g_{\rho, d}^{2} f_{\rho}^{2} \\
& \times \int d \omega_{1} d \omega_{2}\left|D_{\rho}\left(q_{1}+q_{2}\right)\right|^{2}\left(\mathbf{q}_{1}-\mathbf{q}_{2}\right)^{2} \Theta(1-|A|) \\
& \times B W\left(\left|\mathbf{q}_{1}-\mathbf{q}_{2}\right| R\right)^{2} \\
A & =\frac{\left(\sqrt{s}-\omega_{1}-\omega_{2}\right)^{2}-M_{N}^{2}-\mathbf{q}_{1}^{2}-\mathbf{q}_{2}^{2}}{2\left|\mathbf{q}_{1}\right|\left|\mathbf{q}_{2}\right|} \\
D_{\rho}(q) & =\frac{1}{q^{2}-m_{\rho}^{2}+i m_{\rho} \Gamma_{\rho}(q)}, \\
\Gamma_{\rho}(q) & =\Gamma_{\rho}^{0}(=150 \mathrm{MeV}) \frac{m_{\rho}^{2} q_{c . m .}^{3}}{q_{c . m}^{o n} q_{0}^{2}} \\
\Gamma_{\Delta^{*} \rightarrow(\Delta \pi)_{s+d}[N \pi \pi]} & =\frac{15}{16 \pi^{2}} \int d M_{I} \frac{M_{I} k\left(M_{I}\right)}{4 \pi \sqrt{s}} \\
& \times \frac{\Gamma_{\Delta \rightarrow N \pi}\left(M_{I}\right)\left(\left|A_{s}\right|^{2}+\left.\left|A_{d}\right|\right|^{2}\right)}{\left(M_{I}-M_{\Delta}\right)^{2}+\left(\frac{\Gamma_{\Delta \rightarrow N \pi}\left(M_{I}\right)}{2}\right)^{2}} \Theta\left(\sqrt{s}-M_{I}-m_{\pi}\right) .
\end{aligned}
$$

In Fig. 5.2 the widths from the different channels discussed here are plotted together with their sum (upper plot). In the middle plot we show the partial inelastic cross sections for the channels, and the lower plot shows the shape of the resonance in $\pi N \rightarrow \pi N d$-wave scattering in the spin, isospin $(\ell, I)=(3 / 2,3 / 2)$ channel.

The inelastic partial cross sections for the entrance channel $(\pi N)^{(d)} \rightarrow$ $\Delta^{*}(1700)$ are given from each term in numerator, $\Gamma_{\pi \Delta}^{(s)}, \Gamma_{\pi \Delta}^{(d)}, \Gamma_{\rho N}^{(s)}, \Gamma_{\rho N}^{(d)}$, of

$$
\begin{equation*}
\frac{\sigma_{\mathrm{tot}}}{5}=\frac{M_{N}}{p_{\text {c.m. }} \sqrt{s}} g_{\pi N \Delta^{*}}^{2}\left(\frac{B W\left(p_{\mathrm{c} . \mathrm{m} .} R\right)}{B W\left(p_{\mathrm{c} . \mathrm{m} .}^{\mathrm{on}} R\right)}\right)^{2} \frac{\Gamma_{\pi \Delta}^{(s)}+\Gamma_{\pi \Delta}^{(d)}+\Gamma_{\rho N}^{(s)}+\Gamma_{\rho N}^{(d)}}{\left(\sqrt{s}-M_{\Delta^{*}}\right)^{2}+(\Gamma / 2)^{2}} \tag{5.13}
\end{equation*}
$$

Here, $\Gamma$ is the total width and $\sigma_{\text {tot }}$ the total cross section of $(\pi N)^{(d)} \rightarrow$


Figure 5.2: The $\Delta^{*}$ (1700) width from the different decay channels (upper). Partial inelastic cross sections for $\pi N \rightarrow \pi \pi N$ in the (3, 3)-channel (middle). Shape of the cross section in $\pi N \rightarrow \pi N$ scattering in the (3,3)-channel (lower).
$\Delta^{*}(1700)$. The c.m. momentum of the incoming pion is $p_{\text {c.m. }}$ and at $\sqrt{s}=1.7$ GeV it is $p_{\mathrm{c} . \mathrm{m} .}^{\mathrm{on}}$. The Blatt-Weisskopff factor $B W$ is put in the same way as before in Eq. (5.5). Note the factor of five that has to be multiplied with the sum of the partial cross sections in order to obtain the total cross section. This is because the partial cross sections $\sigma_{i}$ are defined as $\sigma_{\text {tot }}=(2 \ell+1) \sum_{i} \sigma_{i}$. The plot with the partial cross sections can be directly compared to Fig. 4 of [204]. The $(\pi \Delta)_{s}$ contribution is very similar, although in [204] it has a somewhat stronger shoulder below the resonance. The $(\pi \Delta)_{d}$ contribution is stronger in [204] than in Fig. 5.2. The $(\rho N)_{s}$ contribution from here, which has been determined by taking the average value of the $(\rho N)$ branching ratio given in [57], is around twice as large as the value of [204]. Note that when one adds $(\rho N)_{d}$ to this, which becomes large only at the highest energies, one obtains the same rise of the $(\rho N)$ partial cross section between 1.9 and 2 GeV needed in [204].

In the lowest plot of Fig. 5.2 we plot a quantity proportional to $B W^{4}\left(q_{\text {c.m. }}\right)\left|G_{\Delta^{*}(1700)}\right|^{2}$, with $G_{\Delta^{*}(1700)}$ being the $\Delta^{*}(1700)$ propagator. This quantity is proportional to the $\pi N \rightarrow \pi N$ cross section in the (3, 3)-channel. The curve can be compared, e.g., to Fig. 2 of [29]. The shapes are qualitatively similar, in particular the slower fall-off at higher energies which comes from the multiplication with $B W^{4}\left(q_{\text {c.m. }}\right) \simeq q_{\text {c.m. }}^{8}$ in the cross section. Measuring the width from this curve, one obtains 246 MeV . Note that the sum of the $\Gamma_{i}$ at $\sqrt{s}=1.7 \mathrm{GeV}$ gives a width of 320 MeV (see upper plot in Fig. 5.2); due to the momentum dependence of the widths, these two numbers are different in general.

## Amplitudes and invariant masses for the pion-induced reactions

Summing $|t|^{2}$ from Eq. (5.11) over the final states, the sum does not depend on the original proton polarization. We are free to choose $m=1 / 2$ in which case the amplitude of Eq. (5.11) becomes

$$
\begin{align*}
-i t & =\frac{a}{\sqrt{3}} G \frac{1}{\sqrt{s_{\Delta^{*}}}-M_{\Delta^{*}}+\frac{i \Gamma_{\Delta^{*}\left(s_{\Delta^{*}}\right)}^{2}}{}} g_{j} g_{i}^{(d)} \frac{B W\left(k_{\pi} R\right)}{B W\left(k_{\pi}^{\text {on }} R\right)} \\
& \times\left\{\begin{array}{ll}
2 q_{z} & ; m^{\prime}=+1 / 2 \\
- & \left(q_{x}+i q_{y}\right)
\end{array} ; m^{\prime}=-1 / 2 .\right. \tag{5.14}
\end{align*}
$$

In the sum of $|t|^{2}$ over the final states of $\Lambda$ with $m^{\prime}=1 / 2,-1 / 2$ the part corresponding to the curly bracket in Eq. (5.14) will become

$$
\begin{equation*}
4 q_{z}^{2}+q_{x}^{2}+q_{y}^{2}=3 q_{z}^{2}+\mathbf{q}^{2} \tag{5.15}
\end{equation*}
$$

which gives an angular distribution proportional to $\left(3 \cos ^{2} \theta+1\right)$ in the angle of the outgoing meson from the $\Sigma^{*}$ or $\Delta$ decay with respect to the initial $\pi^{-}$ direction for this initial proton polarization. When integrating over angles Eq. (5.15) can be replaced by $2 \mathbf{q}^{2}$. For the first reaction from Fig. 5.1 the $\left(\pi^{0} \Lambda\right)$ invariant mass distribution is then given by

$$
\begin{equation*}
\frac{d \sigma}{d M_{I}\left(\pi^{0} \Lambda\right)}=\frac{M_{p} M_{\Lambda}}{\lambda^{1 / 2}\left(s, m_{\pi}^{2}, M_{p}^{2}\right)} \frac{q_{\pi^{0}} q_{K^{0}}}{(2 \pi)^{3} \sqrt{s}} \bar{\sum} \sum|t|^{2} \tag{5.16}
\end{equation*}
$$

in terms of the ordinary Källen function $\lambda^{1 / 2}$ and $t$ from Eq. (5.14), $q_{K^{0}}=$ $\lambda^{1 / 2}\left(s, M_{I}^{2}, m_{K^{0}}^{2}\right) /(2 \sqrt{s}), q \equiv q_{\pi^{0}}=\lambda^{1 / 2}\left(M_{I}^{2}, m_{\pi^{0}}^{2}, M_{\Lambda}^{2}\right) /\left(2 M_{I}\right)$. Furthermore, the variables in Eq. (5.14) take the values $\sqrt{s_{\Delta^{*}}}=\sqrt{s}, \sqrt{s_{\Sigma^{*}}}=M_{I}$ and the total cross section is given by integrating over $M_{I}$ in Eq. (5.16). The generalization to other channels is straightforward by changing the masses and corresponding momenta.

### 5.3 The model for the $\gamma p \rightarrow K \pi \Lambda, K \pi \Sigma, \eta \pi p$ reactions

The photon coupling to the $\Delta^{*}(1700)$ resonance is taken from $[3,168]$, i.e. from the phenomenological decay width of this resonance into $\gamma M$ (see chapter 4). As we will see in chapter 7, there is no need for this because the scheme of dynamical generation of the $\Delta^{*}(1700)$ allows for a direct calculation of the photon decay width and in fact successfully predicts it. The processes $\gamma p \rightarrow K^{0} \pi^{+} \Lambda, K^{+} \pi^{-} \Sigma^{+}, K^{+} \pi^{+} \Sigma^{-}, K^{0} \pi^{0} \Sigma^{+}, \eta \pi^{0} p$ are given by diagrams similar to those of Fig. 5.1 with the incoming pion replaced by the photon. In view of this it is very easy to modify the pion-induced amplitudes of the previous section to write the photon-induced ones. The contributions
for the first four reactions are given by

$$
\begin{align*}
T_{\gamma p \rightarrow M M B}^{\mathrm{BG}} & =b \frac{2}{5} \frac{D+F}{2 f_{\pi}} g_{K \Sigma^{*} \Delta^{*}} G_{\Sigma^{*}}\left(\sqrt{s_{\Sigma^{*}}}\right) G_{\Delta^{*}}\left(\sqrt{s_{\Delta^{*}}}\right) \vec{S} \cdot \mathbf{p}_{\pi} \\
& \times\left[-i g_{1}^{\prime} \frac{\vec{S}^{\dagger} \cdot \mathbf{k}}{2 M}(\vec{\sigma} \times \mathbf{k}) \cdot \vec{\epsilon}-\vec{S}^{\dagger} \cdot \vec{\epsilon}\left(g_{1}^{\prime}\left(k^{0}+\frac{\mathbf{k}^{2}}{2 M}\right)+g_{2}^{\prime} \sqrt{s} k^{0}\right)\right] \tag{5.17}
\end{align*}
$$

with

$$
\begin{align*}
b_{\gamma p \rightarrow K^{0} \pi^{+} \Lambda} & =1.04 \sqrt{3}, & & b_{\gamma p \rightarrow K^{+} \pi^{-} \Sigma^{+}}=-\sqrt{2}, \\
b_{\gamma p \rightarrow K^{+} \pi^{+} \Sigma^{-}} & =\sqrt{2}, & & b_{\gamma p \rightarrow K^{0} \pi^{0} \Sigma^{+}}=1 \tag{5.18}
\end{align*}
$$

which follows from Eq. (4.67) by applying the corresponding changes in isospin factors and corrections due to $S U(3)$ breaking in the $\Sigma^{*}(1385)$ decay similar as in Eq. (5.8). Cross sections and invariant masses for the photon reactions are given by Eq. (5.16) with the corresponding changes in masses (e.g., $m_{\pi} \rightarrow 0$ in $\lambda^{1 / 2}$ ). For the $\gamma p \rightarrow \eta \pi^{0} p$ reaction we replace the factor before the brackets in Eq. (5.17) by

$$
\begin{equation*}
-\sqrt{\frac{2}{3}} g_{\eta \Delta \Delta^{*}} \frac{f_{\pi N \Delta}}{m_{\pi}} G_{\Delta^{*}}\left(\sqrt{s_{\Delta^{*}}}\right) G_{\Delta}\left(\sqrt{s_{\Delta}}\right) \tag{5.19}
\end{equation*}
$$

The $\gamma p \rightarrow \eta \pi^{0}$ and $\gamma p \rightarrow K^{0} \pi^{0} \Sigma^{+}$reactions have been derived in chapter 4. Besides the tree level amplitudes there are one-loop transitions between $\Delta^{*}(1700)$ and another dynamically generated resonance, the $N^{*}(1535)$. The latter terms are not important at the high energies in which we are currently interested because the $N^{*}(1535)$ is off-shell.

### 5.4 Results and discussion

In Figs. 5.3 to 5.8 we show the results for the pion- and photon-induced reactions. In Figs. 5.3, 5.4, and 5.5 and we show the cross sections for the $\pi^{-} p \rightarrow K^{0} \pi^{0} \Lambda, \pi^{+} p \rightarrow K^{+} \pi^{+} \Lambda, \pi^{+} p \rightarrow K^{+} \bar{K}^{0} p, \pi^{+} p \rightarrow K^{+} \pi^{+} \Sigma^{0}, \pi^{+} p \rightarrow$ $K^{+} \pi^{0} \Sigma^{+}$, and $\pi^{+} p \rightarrow \eta \pi^{+} p$ reactions. The theoretical results are plotted in terms of a band. This band corresponds to taking the $\pi N \Delta^{*}$ coupling


Figure 5.3: Total cross sections for the pion-induced reactions. Data are from [194] (triangles up), [195] (triangle down), [58] (cross), [196] (dots).


Figure 5.4: Total cross sections for the pion-induced reactions (continued). Data are from [196] (dots).


Figure 5.5: Total cross sections for the pion-induced reactions (continued). Data are from [196] (dots), [197] (diamonds). For the latter data, it is indicated that these are upper limits below $p=1.67 \mathrm{GeV}$ as in Ref. [197].


Figure 5.6: Invariant mass spectra for the $\pi^{-} p \rightarrow K^{0} \pi^{0} \Lambda$ and $\pi^{+} p \rightarrow K^{+} \pi^{+} \Lambda$ reactions. Experimental distributions (arbitrary units) are from [195] and [196], respectively.
with its uncertainties (from the experimental branching ratio ) quoted in Eq. (5.3).

Much of the data in Figs. 5.3, 5.4, and 5.5 are for energies $\sqrt{s}>1950$ MeV , which, even taking into account the 300 MeV width of the $\Delta^{*}(1700)$, are relatively far away from the $\Delta^{*}(1700)$ peak. In a situation like this it is logical to assume that other channels not related to the $\Delta^{*}(1700)$ could also play a role. Indeed, other partial waves in $\pi N$ scattering are equally important in this energy region $[31,113]$. However, there are reasons to assume that they do not couple strongly to the $K \pi \Lambda, K \pi \Sigma$ in the final state at the lower energies, as we shall discuss at the end of this section. In any case, even at the higher energies, the order of magnitude predicted for the cross section is correct.

For the $\pi^{-} p \rightarrow K^{0} \pi^{0} \Lambda$ reaction we have data around $\sqrt{s}=1930 \mathrm{MeV}$ and the theory agrees with those data. Furthermore, as we can see in [195], the $\pi^{0} \Lambda$ mass spectrum is totally dominated by the $\Sigma^{*}(1385)$ and, as shown in Fig. 5.6 (a), the theoretical predictions agree with these data. It is also very instructive to see that the angular distribution of the $\Sigma^{*}$ is practically flat $[195,205]$, as our model predicts, given the $s$-wave coupling of the $\Delta^{*}(1700)$ to $K \Sigma^{*}$.

Next we discuss the cross section for the $\pi^{+} p \rightarrow K^{+} \pi^{+} \Lambda$ reaction. The range of energies extends now from about $\sqrt{s}=1800 \mathrm{MeV}$ on. We can see
that the order of magnitude of the cross sections from the theoretical band is correct, although the theoretical prediction is more than a factor of two bigger than data at around $\sqrt{s}=1900 \mathrm{MeV}$. Yet, this apparently large difference should be viewed in perspective, which is provided by the cross section of the $\pi^{+} p \rightarrow K^{+} \pi^{+} \Sigma^{0}$ and $\pi^{+} p \rightarrow K^{+} \pi^{0} \Sigma^{+}$reactions. Indeed, given the larger coupling of the $\Sigma^{*}(1385)$ resonance to $\pi \Lambda$, with a branching ratio to this latter channel about one order of magnitude larger than for $\pi \Sigma$, the mechanism that we have should provide a cross section for the $\pi^{+} p \rightarrow K^{+} \pi^{+} \Lambda$ reaction about one order of magnitude bigger than for the $\pi^{+} p \rightarrow K^{+} \pi^{+} \Sigma^{0}$ or $\pi^{+} p \rightarrow$ $K^{+} \pi^{0} \Sigma^{+}$reactions. This is indeed the case, both in the theory and in the experiment. We can see that in the region of energies below $\sqrt{s}<1900 \mathrm{MeV}$ the agreement of the theory with the data is fine at the qualitative level for these two latter reactions. We can also see that the first data point for the $\pi^{+} p \rightarrow K^{+} \pi^{+} \Lambda$ reaction is in agreement with the theoretical prediction (see the insert in Fig. 5.3)

The invariant mass spectra for different energies in the $\pi^{+} p \rightarrow K^{+} \pi^{+} \Lambda$ reaction are shown in Fig. 5.6 (b), (c). These curves can be directly compared to the data of Ref. [196]; the $\Sigma^{*}(1385)$ dominance in both theory and experiment is apparent. Note that, as mentioned in [195], the excess of strength at the lower shoulder is partly a result of the finite experimental resolution [195, 196].

In Fig. 5.4 we also plot the cross section for the $\pi^{+} p \rightarrow K^{+} \bar{K}^{0} p$ reaction. We can see that the predicted cross sections are quite low compared with experiment. This should be expected since our mechanism is doubly suppressed there, first from having the $\Delta^{*}(1700)$ off shell, and second from also having off shell the $\Sigma^{*}(1385)$ decaying into $\bar{K} N$. It is thus not surprising that our mechanism produces these small cross sections. The $\Sigma^{*}(1385)$ is, however, not off shell for the $\pi \Sigma$ and $\pi \Lambda$ in the final state, and even at the low energies of the figure the mass distribution for these two particles is dominated by the $\Sigma^{*}(1385)$ in the theory, and this is also the case in the experiment as mentioned above.

Finally, we also show in Fig. 5.5 the cross section for the $\pi^{+} p \rightarrow \eta \pi^{+} p$ reaction. Here the mechanism is also $\Delta^{*}(1700)$ production but it decays into $\eta \Delta$ (1232) followed by $\Delta(1232) \rightarrow \pi N$. The agreement of the theory with the
data is fair for low energies, even more when we read the caution statement in the experimental paper [197] warning that the data are overestimated below $p_{\text {lab }}=1670 \mathrm{MeV}$. Once again it is worth noting that below $p_{\text {lab }}=1670$ MeV the cross sections are a factor fifty larger than for $\pi^{+} p \rightarrow K^{+} \pi^{0} \Sigma^{+}$or $\pi^{+} p \rightarrow K^{+} \pi^{+} \Sigma^{0}$. The theory is producing these large order of magnitude changes in the cross sections correctly.

### 5.4.1 Photon-induced $\Delta^{*}(1700)$ production

Next we discuss the photonuclear cross sections. In Figs. 5.7 and 5.8 we can see the cross sections for the $\gamma p \rightarrow K^{0} \pi^{+} \Lambda, \gamma p \rightarrow K^{+} \pi^{-} \Sigma^{+}, \gamma p \rightarrow K^{+} \pi^{+} \Sigma^{-}$, $\gamma p \rightarrow K^{0} \pi^{0} \Sigma^{+}$and $\gamma p \rightarrow \eta \pi^{0} p$ reactions. For the first three reactions we have some data from [200-202]. Only a few data points appear in the region below $\sqrt{s}=1900 \mathrm{MeV}$ and, furthermore, the data have large uncertainties, both in the magnitude of the cross section and in the value of $\sqrt{s}$. Nevertheless, the data are valuable in this global analysis that we are doing. In the first place we can see that within errors, the agreement of theory and experiment is fair. Yet, more significant is the ratio of more than one order of magnitude, both in the theory and experiment for the cross sections of the $\gamma p \rightarrow \eta \pi^{0} p$ and the $\gamma p \rightarrow K^{+} \pi^{+} \Sigma^{-}$or $\gamma p \rightarrow K^{+} \pi^{-} \Sigma^{+}$reactions. This follows the same trend as the $\pi^{+} p \rightarrow \eta \pi^{+} p$ and $\pi^{+} p \rightarrow K^{+} \pi^{+} \Sigma^{0}$ or $\pi^{+} p \rightarrow K^{+} \pi^{0} \Sigma^{+}$reactions discussed above, supporting also the dominance of the same mechanisms in the reactions. It is also worth noting that the dominance of the $\Sigma^{*}(1385)$ production for the $\gamma p \rightarrow K^{0} \pi^{+} \Lambda$ reaction is also mentioned in Ref. [201].

In Fig. 5.8 we plot the cross sections for the $\gamma p \rightarrow K^{0} \pi^{0} \Sigma^{+}$and $\gamma p \rightarrow$ $\eta \pi^{0} p$ reactions measured at ELSA and which are presently being analyzed [193]. These latter two reactions were studied in chapter 4 [3] with far more detail including many more mechanisms. Yet, it was this detailed study that showed the dominance of the reaction mechanism considered in this paper. We also found there that around $\sqrt{s}=2000 \mathrm{MeV}$, the extra terms could modify the cross section by about 30 percent or more. For these two reactions we have taken the full model from chapter 4 . We show now a band of values given by the uncertainties from the experimental helicity amplitudes of the $\gamma p \Delta^{*}$ transition [57] for the cross section. Furthermore, we show that both


Figure 5.7: Photoproduction of strange particles. Data are from [200] (dots), [201] (crosses), and [202] (triangles up). The latter data are the sum of $K \Lambda \pi$ and $K \Sigma \pi$ final states. For the $K \pi \Sigma$ final states, also the predictions for $K^{+} \Lambda(1405)$ production from [166] are plotted (see Sec. 5.4.3).


Figure 5.8: Update of the predictions from chapter 4 (Ref. [3]) (gray bands) compared to the results from chapter 4 (solid lines).
cross sections have been reduced by about 30 percent with respect to those in chapter 4 as a consequence of the consideration of a more realistic $\Delta(1700)$ width, larger now and closer to the experimental 300 MeV , and in addition we have also taken a small fraction of the $\rho N$ decay width in $d$-wave. These latter two cross sections are also in qualitative agreement with preliminary results for the cross sections as shown in [193].

The consideration of the different reactions, with the cross sections spanning nearly two orders of magnitude, and the global qualitative agreement found for all the different reactions, gives support to the reaction mechanism suggested here in which a $\Delta^{*}(1700)$ resonance is excited which decays later on into $K \Sigma^{*}$ or $\eta \Delta$. The predictions of the cross sections are tied to the couplings of the $\Delta^{*}(1700)$ resonance to the $K \Sigma^{*}$ or $\eta \Delta$ channels provided by the hypothesis that the $\Delta^{*}(1700)$ is a dynamically generated resonance. We showed in the Introduction that the couplings to these channels were substantially different than those provided by a simple $S U(3)$ symmetry and there is hence substantial dynamical information from the underlying chiral dynamics and coupled-channels unitarity. In this respect the global analysis of these reactions offers support to the basic idea about the nature of the $\Delta^{*}(1700)$ as a dynamically generated resonance. No doubt a more detailed theoretical analysis should consider extra terms, as done for instance in chapter 4 ([3]) and also extra mechanisms beyond 200 or 300 MeV above the $\Delta^{*}(1700)$ region. However, the global qualitative agreement, considering the large span of the different cross sections, and that a simple $S U(3)$ symmetrical consideration would produce cross sections about a factor 30 different, indicate that the agreement found here is not a trivial thing.

### 5.4.2 Other possible reaction mechanisms

At this point we would like to make some comments about other contributions to some of the reactions discussed and related ones that we mention below.

Another possible mechanism for the photon- and pion-induced reactions can come from having the two mesons in a resonant state, the $K^{*}$ (892). Yet, the reactions $\gamma p(\pi p) \rightarrow \Delta^{*}(1700) \rightarrow K^{*} \Lambda$ are not allowed by isospin conservation, and precisely the reactions with $\Lambda$ in the final state are those
with the largest cross sections. The sequence $\Delta^{*}(1700) \rightarrow K^{*} \Sigma$ is possible and this can be a source of background for the reactions with lower cross sections. Of course we could also have $\gamma p(\pi p) \rightarrow K^{*} \Lambda$ without passing through a $I=3 / 2$ channel and then this final state would also be allowed. However, note that the thresholds for the $K^{*} \Lambda$ and $K^{*} \Sigma$ production are 2007 MeV and 2089 MeV , respectively. Hence, for energies lower than these by 100 or 200 MeV this contribution should be highly suppressed. This is indeed an experimental fact for the $\pi^{-} p \rightarrow K^{0} \pi^{0} \Lambda$ reaction as noted in [195] and also for the $\gamma p \rightarrow K^{0} \pi^{+} \Lambda$ reaction [201]. Yet, in some reactions the $K^{*} \Lambda$ contamination is more visible than in others, for instance in the $\pi^{-} p \rightarrow$ $K^{+} \pi^{-} \Lambda$ reaction it is more apparent than for $\pi^{-} p \rightarrow K^{0} \pi^{0} \Lambda$ [194]. This could explain why our model is short by about $30 \%$ in the $\pi^{-} p \rightarrow K^{+} \pi^{-} \Lambda$ reaction cross section at 1930 MeV [194] (shown in Fig. 5.9) while it is good for the $\pi^{-} p \rightarrow K^{0} \pi^{0} \Lambda$ reaction.

Similar comments can be made about the $\pi^{-} p \rightarrow K^{+} \pi^{0} \Sigma^{-}$and $\pi^{-} p \rightarrow$ $K^{+} \pi^{-} \Sigma^{0}$ reactions which in our model go via $K^{+} \Sigma^{*-}$ and, hence, can have $I=1 / 2,3 / 2$ while the $\pi^{+} p \rightarrow K^{+} \pi^{+} \Sigma^{0}$ and $\pi^{+} p \rightarrow K^{+} \pi^{0} \Sigma^{+}$reactions shown in Fig. 5.3 go through $K^{+} \Sigma^{*+}$ and, hence, have only $I=3 / 2$ and larger chances to couple to the $\Delta^{*}(1700)$. The Clebsch-Gordan coefficients of $\pi^{+} p \rightarrow \Delta^{*++}$ or $\pi^{-} p \rightarrow \Delta^{* 0}$ also favor the $\pi^{+} p$ reactions and indeed we find cross sections for the $\pi^{-} p \rightarrow K^{+} \pi^{0} \Sigma^{-}$and $\pi^{-} p \rightarrow K^{+} \pi^{-} \Sigma^{0}$ reactions substantially smaller (about one order of magnitude) as shown in the two lower plots of Fig. 5.9. With such smaller cross sections it should be expected that background terms become more relevant and, hence, these reactions are not considered for our tests.

Finally, let us mention the possible contribution in the entrance channels of other resonances, apart from the $\Delta^{*}(1700)$. The $N^{*}(1700) D_{13}$ can be a candidate which would possibly affect the $\pi^{-} p$ reactions but not the $\pi^{+} p$ reactions. In any case, the smaller width of the $N^{*}(1700) D_{13}(50-150 \mathrm{MeV})$ does not give much chance for contributions at the threshold of the reactions. Other possible $N^{*}$ or $\Delta^{*}$ resonances in the regions of the energies below 2000 MeV , considering their spin and parity, can be ruled out on the basis of the $s$-wave $K \Sigma^{*}$ dominance experimentally established in [195, 205]. The only possible exception is the one-star $\Delta(1940) D_{33}$ resonance for which no $K \Sigma^{*}$


Figure 5.9: Additional pion induced reactions contaminated by $K^{*}$ (892) production and/or large $I=1 / 2$ components. Data are from [194] (triangles up), [205] (stars), [58] (cross).
or $\eta \Delta(1232)$ decay channels are reported in spite of being allowed by phase space and, hence, it is not considered here.

The experimental $s$-wave $K \Sigma^{*}$ dominance at low energies served as support to our theory, but we should note that at $\sqrt{s}=2020 \mathrm{MeV}$ in the $\pi^{-} p \rightarrow K^{0} \pi^{0} \Lambda$ reaction, the $s$-wave $K \Sigma^{*}$ dominance from the $\sqrt{s}=1930$ data points $[195,205]$ does no longer hold; in fact, the production angular distribution shows a forward peak [58] which is well described by $t$-channel $K^{*}$ exchange in the framework of the Stodolsky-Sakurai model [58]. Similarly, the forward peak for the high energies is observed in the $\pi^{+} p \rightarrow K^{+} \pi^{+} \Lambda$ reaction [196]. This does not rule out our production mechanisms since the Weinberg-Tomozawa term used to generate the $\Delta^{*}(1700)$ in the BetheSalpeter equation [44] effectively accounts for a vector meson exchange in the $t$-channel (in the limit of small momentum transfer). Improvements could be done in the theory to account explicitly for the finite momentum transfer dependence, as done in [45] but, since this only affects the larger energies, we do not consider it here.

### 5.4.3 Photoproduction of $\Lambda(1405)$

In the case of the $\gamma p \rightarrow K^{+} \pi^{-} \Sigma^{+}$and $\gamma p \rightarrow K^{+} \pi^{+} \Sigma^{-}$reactions there can be a contamination from the $K^{+} \Lambda(1405)$ production. This latter reaction was studied in [166] and is presently investigated experimentally at Spring8/Osaka [165]. We have recalculated the invariant mass distribution and reproduce the results from [166] as Fig. 5.10 shows. It is straightforward to integrate over the invariant mass in order to obtain the cross section which is plotted in Fig. 5.7 with the dashed lines for the $\gamma p \rightarrow K^{+} \pi^{-} \Sigma^{+}$ and the $\gamma p \rightarrow K^{+} \pi^{+} \Sigma^{-}$reactions. It interesting to note that the model from [166] predicts quite different shapes for the $\pi^{-} \Sigma^{+}$and the $\pi^{+} \Sigma^{-}$invariant masses, the latter being a factor of around two narrower (see Fig. 5.10). Experimentally, this has been confirmed in the reactions studied at Spring 8 [165] although the experimental result is without normalization. The cross section for the $\gamma p \rightarrow K^{+} \pi^{+} \Sigma^{-}$reaction is a factor of two smaller than for the $K^{+} \pi^{-} \Sigma^{+}$final state (Fig. 5.7). Indeed, the data shows the same trend as can be seen in Fig. 5.7. Although the cross sections from the model


Figure 5.10: Invariant mass spectrum for the $\gamma p \rightarrow K^{+}(M B)$ reaction with $\Lambda(1405)$ production. The curves reproduce results from [166].
of [166] do not coincide perfectly with data, we can at least confirm now, that [166] does not only provide correctly the narrow $\pi^{+} \Sigma^{-}$and the wider $\pi^{-} \Sigma^{+}$distribution [166] but also the correct order magnitude in the cross section. Of course, the reactions studied here via $\Delta^{*}(1700)$ will interfere with the $\Lambda(1405)$ photoproduction (see the gray bands in Fig. 5.7) as these reactions are of the same order of magnitude. Once better data is available, a study in this direction can be certainly fruitful.

Given the qualitative use made of the cross sections for the reactions $\gamma p \rightarrow K^{+} \pi^{-} \Sigma^{+}$and $\gamma p \rightarrow K^{+} \pi^{+} \Sigma^{-}$in the discussion above, and the large errors in the data, the conclusions drawn, that the photoproduction of $\Delta^{*}(1700)$ is in agreement with data, are not affected. Note also that for the $\gamma p \rightarrow K^{0} \pi^{+} \Lambda$ reaction the $\Lambda(1405)$ production is not allowed and there the dominance of the $\Sigma^{*}$ (1385) production has stronger grounds. Note also that this latter reaction has a larger cross section than the other two from our $K \Sigma^{*}(1385)$ production mechanisms which should make potential extra contributions relatively smaller.

### 5.5 Conclusions

We have looked at several pion-induced and photon-induced reactions at energies above and close to the $\Delta^{*}(1700)$ with $K \pi \Lambda, K \pi \Sigma$ and $\eta \pi N$ in the final state. We have made a theoretical model assuming that a $\Delta^{*}(1700)$ is excited and then decays via the $K \Sigma^{*}(1385)$ or $\eta \Delta(1232)$ depending on the final state. We find that in spite of exploiting the tail of the resonance, around half of the width or one width above the nominal energy of the $\Delta^{*}(1700)$, the cross sections obtained are sizable. The reason is the large couplings of the $\Delta^{*}(1700)$ to the $K \Sigma^{*}(1385)$ or $\eta \Delta(1232)$ which are provided by the theory in which the $\Delta^{*}(1700)$ is a dynamically generated resonance. We showed that the couplings squared to these channels were about 20-30 times bigger than estimated by simple $\operatorname{SU}(3)$ symmetry arguments. We could also see that the presence of the $\Sigma^{*}$ (1385) was clear in the experimental data with $K \pi \Lambda, K \pi \Sigma$ final states, with little room for background at low energies, indicating a clear dominance of the $\pi \Lambda, \pi \Sigma$ final states in the $\Sigma^{*}(1385)$ channel. Despite the admitted room for improvements in the theory, the qualitative global agreement of the different cross section with the data gives a strong support to the mechanisms proposed here and the strong couplings of the $\Delta^{*}(1700)$ to the $K \Sigma^{*}(1385)$ or $\eta \Delta(1232)$ channels. The agreement found is more significant when one realizes the large difference in magnitude of the different cross sections and the clear correlation of the theoretical predictions with the data.

The results obtained are relevant because they rely upon the $\Delta^{*}(1700)$ couplings to $K \Sigma^{*}$ and $\eta \Delta(1232)$ for which there is no experimental information, but which are provided by the theory in which the $\Delta^{*}(1700)$ is dynamically generated.

The next question arises on what could be done in the future to make a more quantitative calculation. A number of factors is needed to go forward in this direction:

1. The experimental total width of the $\Delta^{*}(1700)$ and the branching ratio to $\pi N$ and $\rho N$ should be improved. The separation of the $\rho N$ channel in s- and d- waves needs also to be performed if accurate predictions are to be done for $200-300 \mathrm{MeV}$ above the $\Delta^{*}(1700)$ resonance region.
2. Experiments at lower energies, closer to the $\Delta^{*}(1700)$ energy, for the photon- and pion-induced reactions discussed, would be most welcome. There the $\Delta^{*}(1700)$ excitation mechanism would be more dominant and one would reduce uncertainties from other possible background terms.
3. Some improvements on the generation of the $\Delta^{*}$ (1700), including extra channels to the $\pi \Delta, K \Sigma^{*}$, and $\eta \Delta$ used in Refs. [43, 44], like $\pi N$ in $d$ wave and $\rho N$ in $s$ - and $d$-waves, would be most welcome, thus helping fine tune the present coupling of $\Delta^{*}(1700)$ to $K \Sigma^{*}$ and $\eta \Delta$ provided by [44].

Awaiting progress in these directions, at the present time we could claim, that within admitted theoretical and experimental uncertainties, the present data for the large sample of pion and photon-induced reactions offer support for the large coupling of the $\Delta^{*}(1700)$ resonance to $K \Sigma^{*}$ and $\eta \Delta$ predicted by the chiral unitary approach for which there was no previous experimental information.

### 5.5.1 Addendum

After finishing and publishing the work of this chapter [3], we became aware of unpublished results from two recent Phd thesis' [206, 207]. Compared to the ABBHHM $[200,201]$ data which are used here in Fig. 5.7 the new measurements show a great improvement in accuracy. In Fig. 7.1 of [206] the new data on $\gamma p \rightarrow K^{0} \Lambda \pi^{+}$is plotted; the new results coincide even better with the present theoretical prediction than the ABBHHM data. Furthermore, there is a clear dominance of $\Sigma^{*}(1385)$ production which also here is the driving mechanism. Only above $E_{\gamma}=1.8 \mathrm{GeV}$ the production of $K^{*}(892)$ begins to contribute significantly to the cross section.

As for the differential cross section for the $K^{0} \Sigma^{*}$ (1385), displayed in Fig. 7.11 of [206] the situation is not so clear: As discussed above, we predict a flat differential cross section $d \sigma / d \Omega$ for the energies where we claim validity of the model; however, in Fig. 7.11 of [206] the results are averaged for photon energies from threshold up to $E_{\gamma}=2.029$. Most data points lie in the higher energy region, and there $t$-channel exchange becomes important.

This we have seen in an analogous example for the reaction $\pi^{-} p \rightarrow K^{0} \pi^{0} \Lambda$ : At $s^{1 / 2}=1.9 \mathrm{GeV}[195,205]$, the experimental differential cross section is flat whereas for $s^{1 / 2}=2 \mathrm{GeV}$ a forward peak is measured [58] (see also the discussion in Sec. 5.4.2). Thus, in the average over $E_{\gamma}$ the potential $s$-wave dominance at very low energies (which implies small cross sections) does not clearly appear. However, in a preliminary re-analysis [208] only the lowest energy data have been taken into account for the differential cross section. For photion energies $E_{\gamma}<1.7 \mathrm{GeV}$, the number of $\Sigma^{*}(1385)$ produced in forward and backward direction are very similar, i.e., there is no sign for a forward peak, which is in agreement with the flat distribution predicted by the present model.

For the reaction $\gamma p \rightarrow K^{+} \Sigma^{-} \pi^{+}$, Ref. [207] shows recent high accuracy data. Comparing Fig. 7.5 of [207] with the present results in Fig. 5.7 the agreement is good. Differential cross sections of $K$-production for the energy bins $1.3 \leq E_{\gamma} \leq 1.4 \mathrm{GeV}$ (and further up in energy) are provided in Figs. 7.3 and 7.4 of [207]. In the present model, the kaon and the $\Sigma^{*}(1385)$ are produced in a relative $s$-wave; indeed, the data confirms this prediction for the lowest energies close to threshold. As mentioned in Sec. 5.4.3 there is also $\Lambda(1405)$ production possible, and indeed this resonance is seen in [207], together with the $\Sigma^{*}(1395)$. The production mechanism proposed in [166], which we have reproduced in Fig. 5.10, is again in $s$-wave between the $K^{+}$and the $\Lambda(1405)$. Thus, without evaluating the coherent sum of the photoproduction of $\Sigma^{*}(1385)$ and $\Lambda(1405)$, we know that it will also be in $s$-wave, and data show indeed a flat distribution for low photon energies.

## Chapter 6

## Radiative decay of the $\Lambda(1520)$

A recently developed non-perturbative chiral approach to dynamically generate the $3 / 2^{-}$baryon resonances has been extended to investigate the radiative decays $\Lambda^{*}(1520) \rightarrow \gamma \Lambda(1116)$ and $\Lambda^{*}(1520) \rightarrow \gamma \Sigma^{0}(1193)$. We show that the $\Lambda^{*}(1520)$ decay into $\gamma \Lambda$ is an ideal test for the need of extra components of the resonance beyond those provided by the chiral approach since the largest meson-baryon components give no contribution to this decay. The case is different for $\gamma \Sigma$ decay where the theory agrees with experiment, though the large uncertainties of these data call for more precise measurements. Some estimates of the weight of the needed genuine resonance component are made.

### 6.1 Introduction

New efforts have been undertaken $[43,44]$ to investigate the low lying $3 / 2^{-}$ baryonic resonances which decay in $s$-wave into $0^{-}$mesons $(M)$ and $3 / 2^{+}$ baryons $\left(B^{*}\right)$ of the decuplet. The latter particles, the $0^{-}$mesons and $3 / 2^{+}$ baryons, provide the building blocks of the coupled channels needed in the study of the meson-baryon $s$-wave interaction in the $3 / 2^{-}$channel. A parameter free Lagrangian accounts for this interaction at lowest order and the model exhibits poles in the different isospin and strangeness channels in the complex $\sqrt{s}$-plane, which have been identified with resonances such as $\Lambda^{*}(1520), \Sigma^{*}(1670), \Delta^{*}(1700)$, etc. In the chapters 4 and 5 we have already made repeated use of one of these dynamically generated resonances from
$s$-wave interaction, the $\Delta^{*}(1700)$, and found good agreement with data in almost a dozen of different photon- and pion-induced reactions.

However, the $3 / 2^{-}$resonances have also large branching ratios for $\left(0^{-}, 1 / 2^{+}\right)$ meson-baryon (MB) decays in $d$-wave, in many cases being even larger than the $s$-wave branching ratio due to larger available phase space. For a realistic model that can serve to make reliable predictions in hadronic calculations, the $d$-wave channels corresponding to these decays should be included as has been been done recently in Ref. [61] for one of the $3 / 2^{-}$resonances from Ref. [44], the $\Lambda^{*}(1520)$. For the $M B \rightarrow M B^{*} s$-wave to $d$-wave and $M B \rightarrow M B d$-wave to $d$-wave transitions, chiral symmetry does not fix the coupling strength so that free parameters necessarily enter the model. On the other hand, this freedom allows for a good reproduction of $d$-wave experimental data for $\bar{K} N \rightarrow \bar{K} N$ and $\bar{K} N \rightarrow \pi \Sigma$ via the $\Lambda^{*}(1520)$, see Ref. [60,61]. Once the free parameters are determined by fitting to the experimental data of these reactions, the predictivity of the model can be tested for different data sets as has been done in Ref. [60] for the reactions $K^{-} p \rightarrow \pi^{0} \pi^{0} \Lambda$, $K^{-} p \rightarrow \pi^{+} \pi^{-} \Lambda, \gamma p \rightarrow K^{+} K^{-} p$, and $\pi^{-} p \rightarrow K^{0} K^{-} p$, finding in all cases good agreement with data.

In this chapter we extend the chiral coupled channel approach - without introducing new parameters - to investigate the radiative decays $\Lambda^{*}(1520) \rightarrow$ $\gamma \Lambda(1116)$ and $\Lambda^{*}(1520) \rightarrow \gamma \Sigma^{0}(1193)$ for which new experimental results exist [210]. These reactions are of particular interest because they provide further insight into the nature of the $\Lambda^{*}(1520)$ : A pure dynamically generated resonance would be made out of meson-baryon components, a genuine resonance would be made of three constituent quarks, but an admixture of the two types is possible and in the real world non-exotic resonances have both components, although, by definition, the meson-baryon components would largely dominate in what we call dynamically generated resonances. Yet, even in this case it is interesting to see if some experiments show that extra components beyond the meson-baryon ones are called for.

The radiative decay of the $\Lambda^{*}(1520)$ provides a clear example of this: in one of the decays, $\Lambda^{*}(1520) \rightarrow \gamma \Lambda(1116)$, isospin symmetry filters out the dominant channels $\pi \Sigma^{*}$ and $\pi \Sigma$ of the present approach so that a sizable fraction of the partial decay width could come from a genuine three
quark admixture. In contrast, these dominant channels add up in the isospin combination for the $\Lambda^{*}(1520) \rightarrow \gamma \Sigma^{0}(1193)$ reaction, and a match to the experimental data would point out the dominant component for this channel being the quasibound meson-baryon system in coupled channels.

This situation is opposite to the quark model picture of Ref. [211] where the decay into $\gamma \Sigma^{0}(1193)$ is suppressed. This appears as a consequence of selection rules occurring in the limit in which only strange quarks are excited to $p$-wave bag orbits. Indeed, the photon de-excitation of the strange quark with a one-body operator does not affect the isospin of the $u, d$ quarks and hence $I=1$ baryons in the final state are forbidden in this limit [211]. However, as said above, it is precisely the $\gamma \Sigma^{0}(1193)$ final state which in our hadronic interaction picture appears enhanced. We should also mention other quark models [212-215] that enlarge and complement Ref. [211], as well as algebraic models [218] where the $\Lambda^{*}(1520)$ radiative decay has been evaluated.

In the quark model of Ref. [211] it is shown that the partial decay widths of the $\Lambda^{*}$ depend sensitively on the $q^{4} \bar{q}$ admixture which would correspond to meson-baryon components and, thus, could be related to the dynamically generated $\Lambda^{*}(1520)$.

### 6.2 Formulation

Although close to threshold and thus with little phase space available, the $s$ wave channels play an important role in the scheme of dynamical generation of the $\Lambda^{*}(1520)$ from $[60,61]$. The $M B^{*}$ interaction in $S=-1$ is attractive and responsible for a pole in the complex scattering plane. For the quantum numbers of the $\Lambda^{*}(1520)$, strangeness $S=-1$ and isospin $I=0$, the relevant $s$-wave channels are $\pi \Sigma^{*}$ and $K \Xi^{*}$ with the corresponding coefficients $C_{i j}$ given in Sec. 6.2.1. The formulation for the $s$-wave channels is straightforward and carried out in the same way and with the same conventions as in Sec. 4.4.4 where we have introduced the scheme for the generation of the $\Delta^{*}(1700)$. In the following, we recall the inclusion of $d$-wave channels.

### 6.2.1 $d$-wave channels

As mentioned in the Introduction, a realistic coupled channel model for the $\Lambda^{*}$ (1520) should include also meson-baryon channels (MB) of the octet of $\pi$ with the octet of $p$ as the branching ratios into $\bar{K} N$ and $\pi \Sigma$ are large. These latter states are then automatically in a $d$-wave state. For the present study we include the $d$-wave channels following Ref. [60]. In a previous work [61] the $\Lambda^{*}(1520)$ resonance was studied within a coupled channel formalism including the $\pi \Sigma^{*}, K \Xi^{*}$ in $s$-wave and the $\bar{K} N$ and $\pi \Sigma$ in $d$-waves leading to a good reproduction of the pole position of the $\Lambda^{*}(1520)$ of the scattering amplitudes. However, the use of the pole position to get the properties of the resonance is far from being accurate as soon as a threshold is opened close to the pole position on the real axis, which is the present case with the $\pi \Sigma^{*}$ channel.

Apart from that, in the approach of Ref. [61] some matrix elements in the kernel of the Bethe-Salpeter equation were not considered. Therefore, a subsequent work [60] aimed at a more precise description of the physical processes involving the $\Lambda^{*}(1520)$ resonance. Hence, other possible tree level transition potentials in $d$-wave are introduced here following Ref. [60]: $\bar{K} N \rightarrow \bar{K} N, \bar{K} N \rightarrow \pi \Sigma$ and $\pi \Sigma \rightarrow \pi \Sigma$. For these vertices, effective transition potentials are used which are proportional to the incoming and outgoing momentum squared in order to account for the $d$-wave character of the channels which will be formalized in the following.

Consider the transition $\bar{K} N$ ( $d$-wave) to $\pi \Sigma^{*}$ ( $s$-wave) as shown in Fig. 6.1. We start with an amplitude of the form


Figure 6.1: The $\bar{K} N \rightarrow \pi \Sigma^{*}$ vertex

$$
\begin{equation*}
-i t_{\bar{K} N \rightarrow \pi \Sigma^{*}}=-i \beta_{\bar{K} N}|\mathbf{k}|^{2}\left[T^{(2) \dagger} \otimes Y_{2}(\hat{\mathbf{k}})\right]_{00} \tag{6.1}
\end{equation*}
$$

where $T^{(2) \dagger}$ is a (rank 2) spin transition operator defined by

$$
\langle 3 / 2 M| T_{\mu}^{(2) \dagger}|1 / 2 m\rangle=\mathcal{C}(1 / 223 / 2 ; m \mu M)\left\langle 3 / 2\left\|T^{(2) \dagger}\right\| 1 / 2\right\rangle
$$

$Y_{2}(\hat{\mathbf{k}})$ is the spherical harmonic coupled to $T^{(2) \dagger}$ to produce a scalar, and $\mathbf{k}$ is the momentum of the $\bar{K}$. The third component of spin of the initial nucleon and the final $\Sigma^{*}$ are denoted by $m$ and $M$ respectively as indicated in the Clebsch-Gordan coefficients. The coupling strength $\beta$ is not determined from theory and has to be fixed from experiment as has been done in Ref. [60] with the results outlined below. Choosing appropriately the reduced matrix element we obtain

$$
\begin{equation*}
-i t_{\bar{K} N \rightarrow \pi \Sigma^{*}}=-i \beta_{\bar{K} N}|\mathbf{k}|^{2} \mathcal{C}(1 / 223 / 2 ; m, M-m) Y_{2, m-M}(\hat{\mathbf{k}})(-1)^{M-m} \sqrt{4 \pi} \tag{6.2}
\end{equation*}
$$

In the same way the amplitude for $\pi \Sigma$ ( $d$-wave) to $\pi \Sigma^{*}$ ( $s$-wave) is written as
$-i t_{\pi \Sigma \rightarrow \pi \Sigma^{*}}=-i \beta_{\pi \Sigma}|\mathbf{k}|^{2} \mathcal{C}(1 / 223 / 2 ; m, M-m) Y_{2, m-M}(\hat{\mathbf{k}})(-1)^{M-m} \sqrt{4 \pi}$
and similarly for the rest of the transitions mentioned above. The angular dependence disappears in the loop integrations [61]. The loop function of the meson-baryon system in $d$-wave is strongly divergent, but an on-shell factorization can be achieved [61] using arguments from the $N / D$ method from Ref. [17] as explained in the former subsection. The on-shell factorization ensures at the same time the unitarity of the amplitude after solving the Bethe-Salpeter equation (4.55).

Denoting the $\pi \Sigma^{*}, K \Xi^{*}, \bar{K} N$, and $\pi \Sigma$ channels by $1,2,3$ and 4 , respectively, the kernel $V$ of the Bethe-Salpeter equation (4.55) is written as:

$$
V=\left(\begin{array}{cccc}
C_{11}\left(k_{1}^{0}+k_{1}^{0}\right) & C_{12}\left(k_{1}^{0}+k_{2}^{0}\right) & \gamma_{13} q_{3}^{2} & \gamma_{14} q_{4}^{2}  \tag{6.4}\\
C_{21}\left(k_{2}^{0}+k_{1}^{0}\right) & C_{22}\left(k_{2}^{0}+k_{2}^{0}\right) & 0 & 0 \\
\gamma_{13} q_{3}^{2} & 0 & \gamma_{33} q_{3}^{4} & \gamma_{34} q_{3}^{2} q_{4}^{2} \\
\gamma_{14} q_{4}^{2} & 0 & \gamma_{34} q_{3}^{2} q_{4}^{2} & \gamma_{44} q_{4}^{4}
\end{array}\right)
$$

with the on-shell CM momenta $q_{i}=\frac{1}{2 \sqrt{s}} \sqrt{\left[s-\left(M_{i}+m_{i}\right)^{2}\right]\left[s-\left(M_{i}-m_{i}\right)^{2}\right]}$, meson energy $k_{i}^{0}=\frac{s-M_{i}^{2}+m_{i}^{2}}{2 \sqrt{s}}$, and baryon(meson) masses $M_{i}\left(m_{i}\right)$. The elements $V_{11}, V_{12}, V_{21}, V_{22}$ come from the lowest order chiral Lagrangian involving the decuplet of baryons and the octet of pseudoscalar mesons as discussed in Sec. 4.4.4; see also Ref. [43, 44]. The coefficients $C_{i j}$ obtained from Eq. (4.43) are $C_{11}=\frac{-1}{f^{2}}, C_{21}=C_{12}=\frac{\sqrt{6}}{4 f^{2}}$ and $C_{22}=\frac{-3}{4 f^{2}}$, where $f$ is $1.15 f_{\pi}$, with $f_{\pi}(=93 \mathrm{MeV})$ the pion decay constant, which is an average between $f_{\pi}$ and $f_{K}$ as was used in Ref. [22] in the related problem of the dynamical generation of the $\Lambda(1405)$.

In the kernel $V$ we neglect the elements $V_{23}$ and $V_{24}$ which involve the tree level interaction of the $K \Xi^{*}$ channel with the $d$-wave channels because the $K \Xi^{*}$ threshold is far from the $\Lambda^{*}(1520)$ mass and its role in the resonance structure is far smaller than that of the $\pi \Sigma^{*}$. This is also the reason why the $K \Xi$ channel in $d$-wave is completely ignored.

Summarizing, the parameters of the model are five $d$-wave coupling strengths $\gamma_{i j}$. Additionally, the subtraction constants can be fine-tuned around their natural values of -2 and -8 for $s$-wave loops and $d$-wave loops, respectively. The fit to $\bar{K} N \rightarrow \bar{K} N$ and $\bar{K} N \rightarrow \pi \Sigma$ data has been performed in Ref. [60] and the results for the parameter values can be found there.

We have reproduced the results from [60] and show the result of the data fit in Fig. 6.2 together with a plot of the modulus of the amplitude $|T|$ in the complex plane of $\sqrt{s}$.

In the study of the radiative decay of the $\Lambda^{*}(1520)$ we will need only the coupling strengths of the resonance to its coupled channels at the resonance position [60]. The effective $s$-wave ( $d$-wave) couplings $g_{\Lambda^{*} M B^{*}}\left(g_{\Lambda^{*} M B}\right)$ are obtained by expanding the amplitude around the pole in a Laurent series. The residue is then identified with the coupling strength as described in Sec. 6.4 and we display the result for the $g$ 's in the isospin $I=0$ channel from Ref. [60] in Tab. 6.1.


Figure 6.2: The $\Lambda^{*}(1520)$ in the $\bar{K} N \rightarrow \bar{K} N$ and $\bar{K} N \rightarrow \pi \Sigma$ reaction. Results from [60] are reproduced. The dots show the data from Refs. [219, 220]. Below: The $\Lambda^{*}(1520)$ appears as a pole in the complex plane of $\sqrt{s}$.


Figure 6.3: Coupling of the photon to the $\Lambda^{*}$ (1520). Diagrams (a) and (b) show the coupling to a $\pi \Sigma^{*}$ loop, which enters together with the corresponding diagrams in the $K \Xi^{*}$ channel. The rescattering series that generates the pole of the $\Lambda^{*}(1520)$ in the complex scattering plane is symbolized by $T$. Diagrams (c) and (d) show the $\gamma$ coupling to the $d$-waves of the resonance.

Table 6.1: Coupling strength of the dynamically generated $\Lambda^{*}(1520)$ to $\left(M B^{*}\right)$ in $s$-wave and $(M B)$ in $d$-wave [60].

| $g_{\Lambda^{*} \pi \Sigma^{*}}$ | $g_{\Lambda^{*} K \Xi^{*}}$ | $g_{\Lambda^{*} \bar{K} N}$ | $g_{\Lambda^{*} \pi \Sigma}$ |
| :--- | :--- | :--- | :--- |
| 0.91 | -0.29 | -0.54 | -0.45 |

### 6.3 Radiative decay

For the radiative decay of the $\Lambda^{*}$ (1520) we study the reactions shown in Fig. 6.3 corresponding to $\gamma Y \rightarrow \pi \Sigma^{*}$. We consider in the loops all the mesonbaryon states of the coupled channels and couple the photon to the first loop as shown in Fig. 6.3. In the loop attached to the photon we can have either $\pi \Sigma^{*}$ or $K \Xi^{*}$ that couples to the $\Lambda^{*}(1520)$ in $s$-wave or $\bar{K} N, \pi \Sigma$ which couple in $d$-wave. We show in the figure with the symbol $T$ the diagrams which are accounted by the $T\left(i \rightarrow \pi \Sigma^{*}\right)$ amplitude with $i$ any of the four channels $\pi \Sigma^{*}, K \Xi^{*}, \bar{K} N, \pi \Sigma$. For the photon coupling we restrict ourselves to the Kroll-Ruderman (KR) and meson-pole (MP) coupling as shown in the figure. Formally, the photon should be also coupled to the meson and baryon components of the iteration of intermediate loops forming the $\Lambda^{*}(1520)$ but then the first loop vanishes for parity reasons ( $p$-wave and $s$ or $d$-wave in the first loop). For the same reason the coupling of the photon to the $\Lambda\left(\Sigma^{0}\right)$ initial baryon would vanish. The coupling of the $\gamma$ to the baryon in the first loop vanishes in the heavy baryon limit and is very small otherwise. A general discussion of issues of gauge invariance, chiral invariance, etc., within the context of unitarized chiral theories can be found in [3,183]. In Ref. [183] one proved that gauge invariance is preserved when the photon is coupled to internal as well as external lines and vertices. An extra discussion on this issue is given in [3]. According to these findings our present approach fulfills gauge invariance with errors of the order of $2 \%$ from the approximations done.

For the diagrams from Fig. 6.3, the $M B B^{*}$ vertices and the KrollRuderman coupling $\gamma M B B^{*}$ are needed, for which we use the Lagrangian
from Ref. [190], with the part relevant for the present reaction given by

$$
\begin{equation*}
\mathcal{L}=\mathcal{C}\left(\bar{T}_{\mu} A^{\mu} B+\bar{B} A_{\mu} T^{\mu}\right) \tag{6.5}
\end{equation*}
$$

which we have already introduced following Eq. (4.56). In Eq. (6.5), the quantity $A^{\mu}$ is proportional to the axial current. It is expanded up to one meson field,

$$
\begin{equation*}
A^{\mu}=\frac{i}{2}\left(\xi \partial^{\mu} \xi^{\dagger}-\xi^{\dagger} \partial^{\mu} \xi\right) \xrightarrow{\text { one } \Phi} \frac{\partial^{\mu} \Phi}{\sqrt{2} f_{\pi}}, \quad \xi=\exp \left(\frac{i \Phi}{\sqrt{2} f_{\pi}}\right), \tag{6.6}
\end{equation*}
$$

$\Phi, B, \bar{B}$ are the standard meson and baryon $S U(3)$ fields [10], and $f_{\pi}=93$ MeV . For the Kroll-Ruderman vertex $\gamma M B B^{*}$, we couple the photon by minimal substitution to Eq. (6.5). The coupling strength $\mathcal{C}$ is determined from the $\Delta(1232)$ decay,

$$
\begin{equation*}
\frac{\mathcal{C}}{\sqrt{2} f_{\pi}}=\frac{f_{\Delta \pi N}^{*}}{m_{\pi}} \tag{6.7}
\end{equation*}
$$

with $f_{\Delta \pi N}^{*}=2.13$. The $S U(3)$ breaking in the decuplet beyond that from the different masses is of the order of $30 \%$ as a fit of Eq. (6.5) to the partial decay widths of $\Delta(1232), \Sigma^{*}$, and $\Xi^{*}$ shows $[190,192]$. In the present study, we do not take this breaking into account in order to be consistent with the model for the dynamical generation of the $\Lambda^{*}(1520)$ where the $S U(3)$ breaking from other sources than mass differences is also neglected.

From Eq. (6.5) and from the minimal coupling with the photon, Feynman rules for $\left(\Lambda, \Sigma^{0}\right) \rightarrow M B^{*}, \gamma\left(\Lambda, \Sigma^{0}\right) \rightarrow M B^{*}$, and the ordinary $\gamma M M$ vertices are obtained where the meson momentum $\mathbf{q}$ is defined as outgoing and the photon momentum $\mathbf{k}$ as incoming,

$$
\begin{align*}
& (-i t)_{B \rightarrow M(\mathbf{q}) B^{*}}=\frac{d f_{\Delta \pi N}^{*}}{m_{\pi}} \mathbf{S}^{\dagger} \cdot \mathbf{q}, \quad(-i \mathbf{t} \cdot \boldsymbol{\epsilon})_{K R}=-\frac{e c d f_{\Delta \pi N}^{*}}{m_{\pi}} \mathbf{S}^{\dagger} \cdot \boldsymbol{\epsilon}, \\
& (-i \mathbf{t} \cdot \boldsymbol{\epsilon})_{\gamma(\mathbf{k}) M(\mathbf{q}-\mathbf{k}) \rightarrow M(\mathbf{q})}=i e c(2 \mathbf{q}-\mathbf{k}) \cdot \boldsymbol{\epsilon}, \tag{6.8}
\end{align*}
$$

with the coefficients $d$ given in Tab. 6.2. In Eq. (6.8) $e>0$ is the electron charge and $c=+1(c=-1)$ for $\pi^{+}, K^{+}\left(\pi^{-}, K^{-}\right)$and $c=0$ for processes

Table 6.2: Coefficients $d$ for the Feynman rule Eq. (6.8) with $\Lambda$ or $\Sigma^{0}$ in initial state.

|  | $\pi^{-} \Sigma^{*+}$ | $\pi^{+} \Sigma^{*-}$ | $K^{+} \Xi^{*-}$ |
| :--- | :---: | :--- | :--- |
| $d, \Lambda \rightarrow M B^{*}$ | $-\frac{1}{\sqrt{2}}$ | $\frac{1}{\sqrt{2}}$ | $\frac{1}{\sqrt{2}}$ |
| $d, \Sigma^{0} \rightarrow M B^{*}$ | $-\frac{1}{\sqrt{6}}$ | $-\frac{1}{\sqrt{6}}$ | $-\frac{1}{\sqrt{6}}$ |

with neutral mesons. The photon with the polarization $\epsilon^{\mu}$ is real and we use the Coulomb gauge $\epsilon^{0}=0, \boldsymbol{\epsilon} \cdot \mathbf{k}=0$.

For the first diagram in Fig. 6.3 in which $\pi^{-} \Sigma^{*+}, \pi^{+} \Sigma^{*-}, K^{+} \Xi^{*-}$ couple in $s$-wave to $T$, we construct the amplitude for the reactions $\gamma \Lambda \rightarrow \pi \Sigma^{*}$ and $\gamma \Sigma \rightarrow \pi \Sigma^{*}$ with isospin $I=0$. For this purpose, an isospin combination for the first loop is constructed according to

$$
\begin{align*}
\left|\pi \Sigma^{*}, I=0\right\rangle & =-\frac{1}{\sqrt{3}}\left|\pi^{+} \Sigma^{*-}\right\rangle-\frac{1}{\sqrt{3}}\left|\pi^{0} \Sigma^{* 0}\right\rangle+\frac{1}{\sqrt{3}}\left|\pi^{-} \Sigma^{*+}\right\rangle, \\
\left|K \Xi^{*}, I=0\right\rangle & =\frac{1}{\sqrt{2}}\left|K^{+} \Xi^{*-}\right\rangle-\frac{1}{\sqrt{2}}\left|K^{0} \Xi^{* 0}\right\rangle \tag{6.9}
\end{align*}
$$

with the phase conventions from above. Note that states with neutral mesons do not contribute to the loops. Using the Feynman rules from Eq. (6.8), the results are [indicating, e.g., $\pi \Sigma^{*}$ in the first loop by $\left(\pi \Sigma^{*}\right)$ ]

$$
\begin{align*}
(-i \mathbf{t} \cdot \boldsymbol{\epsilon})_{\mathrm{KR}}^{(I=0)}\left[\gamma \Lambda \rightarrow\left(\pi \Sigma^{*}\right) \xrightarrow{\Lambda^{*}} \pi \Sigma^{*}\right] & =0, \\
(-i \mathbf{t} \cdot \boldsymbol{\epsilon})_{\mathrm{KR}}^{(I=0)}\left[\gamma \Lambda \rightarrow\left(K \Xi^{*}\right) \xrightarrow{\Lambda^{*}} \pi \Sigma^{*}\right] & =-\frac{e}{2} \frac{f_{\Delta \pi N}^{*}}{m_{\pi}} G_{2} T^{(21)} \mathbf{S}^{\dagger} \cdot \boldsymbol{\epsilon}, \\
(-i \mathbf{t} \cdot \boldsymbol{\epsilon})_{\mathrm{KR}}^{(I=0)}\left[\gamma \Sigma^{0} \rightarrow\left(\pi \Sigma^{*}\right) \xrightarrow{\Lambda^{*}} \pi \Sigma^{*}\right] & =-\frac{\sqrt{2} e}{3} \frac{f_{\Delta \pi N}^{*}}{m_{\pi}} G_{1} T^{(11)} \mathbf{S}^{\dagger} \cdot \boldsymbol{\epsilon}, \\
(-i \mathbf{t} \cdot \boldsymbol{\epsilon})_{\mathrm{KR}}^{(I=0)}\left[\gamma \Sigma^{0} \rightarrow\left(K \Xi^{*}\right) \xrightarrow{\Lambda^{*}} \pi \Sigma^{*}\right] & =\frac{e}{2 \sqrt{3}} \frac{f_{\Delta \pi N}^{*}}{m_{\pi}} G_{2} T^{(21)} \mathbf{S}^{\dagger} \cdot \boldsymbol{\epsilon} \tag{6.10}
\end{align*}
$$

with $T^{(i j)}$ being the matrix element obtained from the Bethe-Salpeter equation (4.55) with the channel ordering (ij) as in Eq. (6.4). In Eq. (6.10), $G_{1}$
and $G_{2}$ are the ordinary loop functions for $\pi \Sigma^{*}$ and $K \Xi^{*}$ given by

$$
\begin{equation*}
G_{i}=\int \frac{d^{3} \mathbf{q}}{(2 \pi)^{3}} \frac{1}{2 \omega} \frac{1}{\sqrt{s}-\omega(\mathbf{q})-E(\mathbf{q})+i \epsilon} \tag{6.11}
\end{equation*}
$$

with the total CM energy $\sqrt{s}$, meson and baryon energy $\omega$ and $E$. For the regularization a cut-off $\Lambda$ is used. This cut off is determined such that the $G_{i}$ functions of Eq. (6.11) have the same value as obtained in [60] using dimensional regularization. For this purpose we match the $M B^{*}$ loop function in both regularization schemes (dimensional and cut-off) at $s^{1 / 2}=1520 \mathrm{MeV}$ which results in $\Lambda_{\pi \Sigma^{*}}=418 \mathrm{MeV}$ for the $\pi \Sigma^{*}$ channel. This value is then used as the cut-off for Eq. (6.11). For the $K \Xi^{*}$ channel such a matching is not possible at energies so far below the $K \Xi^{*}$ threshold, and we set $\Lambda_{K \Xi^{*}}=500$ MeV . In any case, the final numbers are almost independent of the value of $\Lambda_{K \Xi *}$, first, because the contribution is tiny and, second, because the cut-off dependence of the $s$-wave loops is moderate.

In order to evaluate the contribution of the meson-pole term in the second diagram of Fig. 6.3, we must project the operator $\boldsymbol{\epsilon} \cdot(2 \mathbf{q}-\mathbf{k}) \mathbf{S}^{\dagger} \cdot(\mathbf{q}-\mathbf{k})$ onto $s$-wave; for this we neglect $\mathbf{k}$ which is relatively small in the radiative decay (the numerical test keeping the $\mathbf{k}$ terms proves this to be a very good approximation). Then, we get as a projection $\mathbf{S}^{\dagger} \cdot \boldsymbol{\epsilon} \frac{2}{3} \mathbf{q}^{2}$ and we have a new loop function

$$
\begin{align*}
\tilde{G}_{i} & =i \int \frac{d^{4} q}{(2 \pi)^{4}} \frac{\mathbf{q}^{2}}{(q-k)^{2}-m_{i}^{2}+i \epsilon} \frac{1}{q^{2}-m_{i}^{2}+i \epsilon} \frac{1}{P^{0}-q^{0}-E_{i}(\mathbf{q})+i \epsilon}, \\
& =-\int \frac{d^{3} \mathbf{q}}{(2 \pi)^{3}} \frac{\mathbf{q}^{2}}{2 \omega_{i} \omega_{i}^{\prime}} \frac{1}{k+\omega_{i}+\omega_{i}^{\prime}} \frac{1}{k-\omega_{i}-\omega_{i}^{\prime}+i \epsilon} \\
& \times \frac{1}{\sqrt{s}-\omega_{i}-E_{i}(\mathbf{q})+i \epsilon} \frac{1}{\sqrt{s}-k-\omega_{i}^{\prime}-E_{i}(\mathbf{q})+i \epsilon}, \\
& {\left[\left(\omega_{i}+\omega_{i}^{\prime}\right)^{2}+\left(\omega_{i}+\omega_{i}^{\prime}\right)\left(E_{i}(\mathbf{q})-\sqrt{s}\right)+k \omega_{i}^{\prime}\right] } \tag{6.12}
\end{align*}
$$

where $\omega_{i}$ and $\omega_{i}^{\prime}$ are the energies of the mesons of mass $m_{i}$ at momentum $\mathbf{q}$ and $\mathbf{q}-\mathbf{k}$, respectively, $k$ is the energy of the on-shell photon and $E_{i}$ the energy of the decuplet baryon. For the regularization of the loop we use the same cut-offs as for Eq. (6.11) from above. The diagrams with meson-pole terms can be easily incorporated by changing $G_{i} \rightarrow G_{i}+\frac{2}{3} \tilde{G}_{i}$ in Eq. (6.10),
resulting in

$$
\begin{align*}
(-i \mathbf{t} \cdot \boldsymbol{\epsilon})_{\mathrm{KR}+\mathrm{MP}}^{(I=0)}\left[\gamma \Lambda \rightarrow\left(\pi \Sigma^{*}\right) \xrightarrow{\Lambda^{*}} \pi \Sigma^{*}\right] & =0, \\
(-i \mathbf{t} \cdot \boldsymbol{\epsilon})_{\mathrm{KR}+\mathrm{MP}}^{I=0}\left[\gamma \Lambda \rightarrow\left(K \Xi^{*}\right) \xrightarrow{\Lambda^{*}} \pi \Sigma^{*}\right] & =-\frac{e}{2} \frac{f_{\Delta \pi N}^{*}}{m_{\pi}}\left(G_{2}+\frac{2}{3} \tilde{G}_{2}\right) T^{(21)} \mathbf{S}^{\dagger} \cdot \boldsymbol{\epsilon}, \\
(-i \mathbf{t} \cdot \boldsymbol{\epsilon})_{\mathrm{KR}+\mathrm{MP}}^{(I=0)}\left[\gamma \Sigma^{0} \rightarrow\left(\pi \Sigma^{*}\right) \xrightarrow{\Lambda^{*}} \pi \Sigma^{*}\right] & =-\frac{\sqrt{2} e}{3} \frac{f_{\Delta \pi N}^{*}}{m_{\pi}}\left(G_{1}+\frac{2}{3} \tilde{G}_{1}\right) T^{(11)} \mathbf{S}^{\dagger} \cdot \boldsymbol{\epsilon}, \\
(-i \mathbf{t} \cdot \boldsymbol{\epsilon})_{\mathrm{KR}+\mathrm{MP}}^{(I=0)}\left[\gamma \Sigma^{0} \rightarrow\left(K \Xi^{*}\right) \xrightarrow{\Lambda^{*}} \pi \Sigma^{*}\right] & =\frac{e}{2 \sqrt{3}} \frac{f_{\Delta \pi N}^{*}}{m_{\pi}}\left(G_{2}+\frac{2}{3} \tilde{G}_{2}\right) T^{(21)} \mathbf{S}^{\dagger} \cdot \boldsymbol{\epsilon} . \tag{6.13}
\end{align*}
$$

### 6.3.1 Radiative decay from $d$-wave loops

The third and fourth diagram in Fig. 6.3 show the photon coupling to the $d$-wave components of the $\Lambda^{*}(1520)$. The first loop implies two $p$-wave and one $d$-wave couplings which lead to a non-trivial angular momentum structure. Note that there is no coupling of the Kroll-Ruderman type because the combination of $s$ and $d$-wave couplings vanishes by parity in the loop integration.

The $M B B$-wave coupling is obtained from the lowest order chiral meson-baryon Lagrangian [10] which leads to the Feynman rule (meson momentum poutgoing)

$$
\begin{equation*}
(-i t)=i \mathcal{L}=-\frac{\sqrt{2}}{f_{\pi}} \boldsymbol{\sigma} \cdot \mathbf{p}\left(a \frac{D+F}{2}+b \frac{D-F}{2}\right) \tag{6.14}
\end{equation*}
$$

with $a$ and $b$ given in Tab. 6.3 where only the channels including charged mesons are denoted. As in the last section, the isospin zero channel is constructed from the particle channels according to

$$
\begin{align*}
|\pi \Sigma, I=0\rangle & =-\frac{1}{\sqrt{3}}\left|\pi^{+} \Sigma^{-}\right\rangle-\frac{1}{\sqrt{3}}\left|\pi^{0} \Sigma^{0}\right\rangle-\frac{1}{\sqrt{3}}\left|\pi^{-} \Sigma^{+}\right\rangle \\
|\bar{K} N, I=0\rangle & =\frac{1}{\sqrt{2}}\left|\bar{K}^{0} n\right\rangle+\frac{1}{\sqrt{2}}\left|K^{-} p\right\rangle \tag{6.15}
\end{align*}
$$

Using the Feynman rules from Eq. (6.14) and from Eq. (6.8) for the $\gamma M M$

Table 6.3: Coefficients $a$ and $b$ for the Feynman rule Eq. (6.14), meson momentum outgoing.

|  | $\pi^{-} \Sigma^{+}$ | $\pi^{+} \Sigma^{-}$ | $K^{-} p$ |
| :--- | :--- | :--- | :--- |
| $a, \Lambda \rightarrow M B$ | $\frac{1}{\sqrt{6}}$ | $\frac{1}{\sqrt{6}}$ | $-\sqrt{\frac{2}{3}}$ |
| $b, \Lambda \rightarrow M B$ | $\frac{1}{\sqrt{6}}$ | $\frac{1}{\sqrt{6}}$ | $\frac{1}{\sqrt{6}}$ |
| $a, \Sigma^{0} \rightarrow M B$ | $-\frac{1}{\sqrt{2}}$ | $\frac{1}{\sqrt{2}}$ | 0 |
| $b, \Sigma^{0} \rightarrow M B$ | $\frac{1}{\sqrt{2}}$ | $-\frac{1}{\sqrt{2}}$ | $\frac{1}{\sqrt{2}}$ |

vertex, the amplitudes read

$$
\begin{align*}
(-i \mathbf{t} \cdot \boldsymbol{\epsilon})^{(I=0)}\left[\gamma \Lambda \rightarrow(\pi \Sigma) \xrightarrow{\Lambda^{*}} \pi \Sigma^{*}\right] & =0, \\
(-i \mathbf{t} \cdot \boldsymbol{\epsilon})^{(I=0)}\left[\gamma \Lambda \rightarrow(\bar{K} N) \xrightarrow{\Lambda^{*}} \pi \Sigma^{*}\right] & =\frac{e}{\sqrt{2} f_{\pi}}\left(\frac{D}{3}+F\right) \tilde{G}_{3}^{\prime} T^{(31)} \mathbf{S}^{\dagger} \cdot \boldsymbol{\epsilon}, \\
(-i \mathbf{t} \cdot \boldsymbol{\epsilon})^{(I=0)}\left[\gamma \Sigma^{0} \rightarrow(\pi \Sigma) \xrightarrow{\Lambda^{*}} \pi \Sigma^{*}\right] & =-\frac{4 e F}{3 f_{\pi}} \tilde{G}_{4}^{\prime} T^{(41)} \mathbf{S}^{\dagger} \cdot \boldsymbol{\epsilon}, \\
(-i \mathbf{t} \cdot \boldsymbol{\epsilon})^{(I=0)}\left[\gamma \Sigma^{0} \rightarrow(\bar{K} N) \xrightarrow{\Lambda^{*}} \pi \Sigma^{*}\right] & =\frac{e}{\sqrt{6} f_{\pi}}(F-D) \tilde{G}_{3}^{\prime} T^{(31)} \mathbf{S}^{\dagger} \cdot \boldsymbol{\epsilon} \tag{6.16}
\end{align*}
$$

with the channel ordering $i=1, \cdots, 4$ being $\pi \Sigma^{*}, K \Xi^{*}, \bar{K} N, \pi \Sigma$ as in the last sections. As above, we have chosen $\pi \Sigma^{*}$ as the final state which will become clear in Sec. 6.4 when the coupled channel scheme is matched with a formalism with explicit excitation of the resonance.

The loop function $\tilde{G}_{i}^{\prime}$ in Eq. (6.16) for the first loop is given by

$$
\begin{align*}
\tilde{G}_{i}^{\prime} & =i \int \frac{d^{4} q}{(2 \pi)^{4}} \frac{\mathbf{q}^{2}}{(q-k)^{2}-m_{i}^{2}+i \epsilon} \frac{1}{q^{2}-m_{i}^{2}+i \epsilon} \\
& \times \frac{1}{P^{0}-q^{0}-E_{i}(\mathbf{q})+i \epsilon} \frac{M}{E_{i}(\mathbf{q})}\left(\frac{\mathbf{q}^{2}}{q_{\mathrm{on}}^{2}}\right) \tag{6.17}
\end{align*}
$$

which is similar to $\tilde{G}$ from Eq. (6.12) up to a factor $M / E$ from the nonrelativistic reduction of the baryon propagator and a factor $\mathrm{q}^{2} / q_{\mathrm{on}}^{2}$. As in
the case of the $M B^{*} s$-wave loops, the divergence in Eq. (6.17) is regularized by a cut-off whose value is obtained by matching dimensional regularization and cut-off scheme of the meson-baryon $d$-wave loop at $s^{1 / 2}=1520 \mathrm{MeV}$ as explained following Eq. (6.11). With the subtraction constant from Ref. [60,61], values for the cut-off of $\Lambda_{\bar{K} N}=507 \mathrm{MeV}$ and $\Lambda_{\pi \Sigma}=558 \mathrm{MeV}$ follow. In the following subsection we present the technical details which have led to Eqs. (6.16) and (6.17), projecting the meson-pole term over $d$-waves and performing the angular integrations.

## The spin-polarization structure of $d$-wave loops

The structure of the two $p$-wave couplings of the first loop in the fourth diagram of Fig. 6.3 is given by

$$
\begin{equation*}
\epsilon^{\mu}(2 q-k)_{\mu} \boldsymbol{\sigma} \cdot(\mathbf{k}-\mathbf{q}) \tag{6.18}
\end{equation*}
$$

where the meson momentum of the $M B B$ vertex is given by $q-k$ and the two mesons in the $\gamma M M$ vertex are at momentum $q-k$ and $q$. As $\epsilon^{0}=0$ in Coulomb gauge, the spin structure takes the form $\boldsymbol{\epsilon} \cdot \boldsymbol{q} \boldsymbol{\sigma} \cdot \mathbf{q}$ (neglecting the photon momentum $\boldsymbol{k}$ which is small in the radiative decay). The $d$-wave structure obtained from $\sigma_{i} q_{i} \epsilon_{j} q_{j} \rightarrow \sigma_{i} \epsilon_{j}\left(q_{i} q_{j}-\frac{1}{3} \mathbf{q}^{2} \delta_{i j}\right)$ will combine with the $d$-wave structure $Y_{2}(\hat{\mathbf{q}})$ coming from the $\bar{K} N \rightarrow \pi \Sigma^{*}$ vertex to produce a scalar quantity after the loop integration is performed (for the second loop, we choose the $\pi \Sigma^{*}$ channel in the following, but the calculations hold for any of the four channels in the second loop).

We write

$$
\begin{equation*}
\sigma_{i} \epsilon_{j}\left(q_{i} q_{j}-\frac{1}{3} \mathbf{q}^{2} \delta_{i j}\right)=A\left[[\sigma \otimes \epsilon]_{\mu}^{2} Y_{2}(\hat{\mathbf{q}})\right]_{0}^{0} \tag{6.19}
\end{equation*}
$$

which indicates that the two vector operators $\vec{\sigma}$ and $\vec{\epsilon}$ couple to produce an operator of rank 2 which couples to the spherical harmonic $Y_{2}(\hat{\mathbf{q}})$ to produce a scalar. The right-hand side can be written as

$$
\begin{equation*}
A \sum_{\mu}(-1)^{\mu}[\sigma \otimes \epsilon]_{\mu}^{2} Y_{2,-\mu}(\hat{\mathbf{q}})=A \sum_{\mu, \alpha}(-1)^{\mu} Y_{2,-\mu}(\hat{\mathbf{q}}) \mathcal{C}(112 ; \alpha, \mu-\alpha) \sigma_{\alpha} \epsilon_{\mu-\alpha} \tag{6.20}
\end{equation*}
$$

where $\mathcal{C}$ denotes the Clebsch Gordan coefficient. To find the value of $A$ we take the matrix element of both sides of Eq. (6.19) between the states $m$ and $m^{\prime}$ so that

$$
\begin{align*}
& \langle m| \sigma_{i} \epsilon_{j}\left(q_{i} q_{j}-\frac{1}{3} \mathbf{q}^{2} \delta_{i j}\right)\left|m^{\prime}\right\rangle=A \sum_{\mu}(-1)^{\mu} Y_{2,-\mu}(\hat{\mathbf{q}}) \epsilon_{\mu-m+m^{\prime}} \\
& \times \mathcal{C}\left(112 ; m-m^{\prime}, \mu-m+m^{\prime}\right) \mathcal{C}\left(\frac{1}{2} 1 \frac{1}{2} ; m^{\prime}, m-m^{\prime}\right) \tag{6.21}
\end{align*}
$$

where we have used $\langle m| \sigma_{\alpha}\left|m^{\prime}\right\rangle=\sqrt{3} \mathcal{C}\left(\frac{1}{2} 1 \frac{1}{2} ; m^{\prime}, \alpha, m\right)$. Taking specific values of spin $1 / 2$ components, $m$ and $m^{\prime}$, we obtain

$$
\begin{equation*}
A=\sqrt{\frac{8 \pi}{15}} \mathbf{q}^{2} \tag{6.22}
\end{equation*}
$$

Following Ref. [61], we now include the $\bar{K} N \rightarrow \pi \Sigma^{*}$ vertex given by

$$
\begin{equation*}
-i t_{\bar{K} N \rightarrow \pi \Sigma^{*}}=-i \beta_{\bar{K} N}|\mathbf{q}|^{2} \mathcal{C}\left(\frac{1}{2} 2 \frac{3}{2} ; m, M-m\right) Y_{2, m-M}(\hat{\mathbf{q}})(-1)^{M-m} \sqrt{4 \pi} \tag{6.23}
\end{equation*}
$$

so that the total spin structure of the $d$-wave loop in Fig. 6.3 is essentially given by

$$
\begin{align*}
J & =\sum_{m} \int \frac{d \Omega_{q}}{4 \pi}\langle m| \sigma_{i} \epsilon_{j}\left(q_{i} q_{j}-\frac{1}{3} \mathbf{q}^{2} \delta_{i j}\right)\left|m^{\prime}\right\rangle \\
& \times \mathcal{C}\left(\frac{1}{2} 2 \frac{3}{2} ; m, M-m\right) Y_{2, m-M}(\hat{\mathbf{q}})(-1)^{M-m} \sqrt{4 \pi} \tag{6.24}
\end{align*}
$$

where we have performed an average over the angles in the integration over the loop momentum $q$. Using Eqs. (6.21) and (6.22) this can be written as

$$
\begin{align*}
J & =\sqrt{\frac{2}{3}} \mathbf{q}^{2}(-1)^{1-M+m^{\prime}} \epsilon_{m^{\prime}-M} \sum_{m} \mathcal{C}\left(\frac{1}{2} 1 \frac{1}{2} ; m^{\prime}, m-m^{\prime}\right) \\
& \times \mathcal{C}\left(\frac{1}{2} 2 \frac{3}{2} ; m, M-m\right) \mathcal{C}\left(121 ; m-m^{\prime}, M-m\right) \tag{6.25}
\end{align*}
$$

where we have used the well-known relations

$$
\int d \Omega_{q} Y_{2,-\mu}(\hat{\mathbf{q}}) Y_{2, m-M}(\hat{\mathbf{q}})=(-1)^{\mu} \delta_{\mu, m-M}
$$

and

$$
\mathcal{C}\left(112 ; m-m^{\prime}, m^{\prime}-M\right)=(-1)^{1-m+m^{\prime}} \sqrt{\frac{5}{3}} \mathcal{C}\left(121 ; m-m^{\prime}, M-m\right)
$$



Figure 6.4: Effective resonance representation of the radiative decay.

The product of three Clebsch-Gordan coefficients is then combined into a single one with a Racah coefficient, resulting in the identity

$$
\begin{align*}
& \sum_{m} \mathcal{C}\left(\frac{1}{2} 1 \frac{1}{2} ; m^{\prime}, m-m^{\prime}\right) \mathcal{C}\left(\frac{1}{2} 2 \frac{3}{2} ; m, M-m\right) \mathcal{C}\left(121 ; m-m^{\prime}, M-m\right) \\
& =-\sqrt{\frac{1}{2}} \mathcal{C}\left(\frac{1}{2} 1 \frac{3}{2} ; m^{\prime}, M-m^{\prime}\right) \tag{6.26}
\end{align*}
$$

so that we finally have

$$
\begin{equation*}
J=\frac{1}{\sqrt{3}} \mathbf{q}^{2} \mathbf{S}^{\dagger} \cdot \boldsymbol{\epsilon} \tag{6.27}
\end{equation*}
$$

The above relation implies that for practical purposes we can use for the $d$-wave projection of the two $p$-wave vertices the simple form $\frac{1}{\sqrt{3}} \mathbf{q}^{2} \mathbf{S}^{\dagger} \cdot \boldsymbol{\epsilon}$ and for the $d$-wave vertex of the $M B \rightarrow M B^{*}$ amplitude the factor $\beta_{\bar{K} N} \mathbf{q}^{2}$ and continue with the formalism exactly as in $s$-wave.

In the on-shell reduction scheme for the $d$-wave transitions in the generation of the $\Lambda^{*}$, the factor $q_{\text {on }}^{2}$ from the vertex is absorbed in the kernel $V$ as can be seen in Eq. (6.4). As we cannot perform this factorization for the first loop, we continue using the factor $\beta_{\bar{K} N} \mathbf{q}^{2}$ for the $d$-wave vertex in this loop but then have to divide by $q_{\text {on }}^{2}$ which will cancel the $q_{\text {on }}^{2}$ in $V$ or the $T$ matrix. All these factors considered, we obtain Eq. (6.16) with $\tilde{G}_{i}^{\prime}$ given in Eq. (6.17).

### 6.4 Numerical results

In the previous sections the amplitudes for the process $\gamma \Lambda \xrightarrow{\Lambda^{*}} \pi \Sigma^{*}$ and $\gamma \Sigma^{0} \xrightarrow{\Lambda^{*}} \pi \Sigma^{*}$ have been determined and are written in terms of the $T^{(i 1)}$, the unitary solution of the Bethe-Salpeter equation (4.55) for meson-baryon scattering with the transitions from channel $i$ to the $\pi \Sigma^{*}$ final state. In order
to determine the partial photon decay widths of the $\Lambda^{*}(1520)$, the $T^{(i 1)}$ is expanded around the pole in the complex scattering plane and can be written as

$$
\begin{equation*}
T^{(i 1)}=\frac{g_{i} g_{\pi \Sigma^{*}}}{\sqrt{s}-M_{\Lambda^{*}(1520)}} . \tag{6.28}
\end{equation*}
$$

The matrix elements from Eq. (6.13) and (6.16) with this replacement for $T^{(i 1)}$ is now identified with the resonant process in Fig. 6.4, which is written as

$$
\begin{equation*}
(-i \mathbf{t} \cdot \boldsymbol{\epsilon})=\left(-i g_{\Lambda^{*} \pi \Sigma^{*}}\right) \frac{i}{\sqrt{s}-M_{\Lambda^{*}}} g_{\Lambda^{*} \gamma \Lambda\left(\Sigma^{0}\right)} \mathbf{S}^{\dagger} \cdot \boldsymbol{\epsilon} \tag{6.29}
\end{equation*}
$$

This identification allows us to write the effective $\Lambda^{*} \gamma \Lambda$ and $\Lambda^{*} \gamma \Sigma^{0}$ couplings, $g_{\Lambda^{*} \gamma \Lambda}$ and $g_{\Lambda^{*} \gamma \Sigma^{0}}$, in terms of the couplings $g_{i 1}$ of the $\Lambda^{*}(1520)$ in the transition of the channel $i \rightarrow \Lambda^{*}(1520) \rightarrow \pi \Sigma^{*}$ with its values given in Tab. 6.1, resulting in

$$
\begin{align*}
g_{\Lambda^{*} \gamma \Lambda}^{\left(K \Xi^{*}\right)} & =-\frac{e}{2} \frac{f_{\pi N \Delta}^{*}}{m_{\pi}}\left(G_{2}+\frac{2}{3} \tilde{G}_{2}\right) g_{\Lambda^{*} K \Xi^{*}}, \\
g_{\Lambda^{*} \gamma \Sigma^{0}}^{\left(\pi \Sigma^{*}\right)} & =-\frac{\sqrt{2} e}{3} \frac{f_{\pi N \Delta}^{*}}{m_{\pi}}\left(G_{1}+\frac{2}{3} \tilde{G}_{1}\right) g_{\Lambda^{*} \pi \Sigma^{*}}, \\
g_{\Lambda^{*} \gamma \Sigma^{0}}^{\left(K \Xi^{*}\right)} & =\frac{e}{2 \sqrt{3}} \frac{f_{\pi N \Delta}^{*}}{m_{\pi}}\left(G_{2}+\frac{2}{3} \tilde{G}_{2}\right) g_{\Lambda^{*} K \Xi^{*}}, \\
g_{\Lambda^{*} \gamma \Lambda}^{(\bar{K} N)} & =\frac{e(D+3 F)}{3 \sqrt{2} f_{\pi}} \tilde{G}_{3}^{\prime} g_{\Lambda^{*} \bar{K} N}, \\
g_{\Lambda^{*} \gamma \Sigma^{0}}^{(\pi \Sigma)} & =-\frac{4 e F}{3 f_{\pi}} \tilde{G}_{4}^{\prime} g_{\Lambda^{*} \pi \Sigma}, \\
g_{\Lambda^{*} \gamma \Sigma^{0}}^{(\bar{K} N)} & =\frac{e(F-D)}{\sqrt{6} f_{\pi}} \tilde{G}_{3}^{\prime} g_{\Lambda^{*} \bar{K} N} . \tag{6.30}
\end{align*}
$$

The upper index in brackets indicates which particles are present in the first loop. Adding all processes, we find using

$$
\begin{align*}
g_{\Lambda^{*} \gamma \Lambda} & =g_{\Lambda^{*} \gamma \Lambda}^{\left(K \Xi^{*}\right)}+g_{\Lambda^{*} \gamma \Lambda}^{(\bar{K} N)}, \\
g_{\Lambda^{*} \gamma \Sigma^{0}} & =g_{\Lambda^{*} \gamma \Sigma^{0}}^{\left(\pi \Sigma^{*}\right)}+g_{\Lambda^{*} \gamma \Sigma^{0}}^{\left(K \Xi^{*}\right)}+g_{\Lambda^{*} \gamma \Sigma^{0}}^{(\pi \Sigma)}+g_{\Lambda^{*} \gamma \Sigma^{0}}^{(\bar{K} N)}, \tag{6.31}
\end{align*}
$$

the partial decay width for the processes $\Lambda^{*}(1520) \rightarrow \gamma \Lambda$ and $\Lambda^{*}(1520) \rightarrow$ $\gamma \Sigma^{0}$ is given by

$$
\begin{equation*}
\Gamma=\frac{k}{3 \pi} \frac{M_{Y}}{M_{\Lambda^{*}}}\left|g_{\Lambda^{*} \gamma Y}\right|^{2} \tag{6.32}
\end{equation*}
$$

Table 6.4: Experimental data, quark model results from Ref. [211, 217], and results from this study for the partial decay width of the $\Lambda^{*}$ (1520) into $\gamma \Lambda$ and $\gamma \Sigma^{0}$. The results in brackets come from the use of empirical $\pi Y Y^{*}$ couplings or $S U(3)$ uncertainties.

|  | $\Gamma\left(\Lambda^{*}(1520) \rightarrow \gamma \Lambda\right)[\mathrm{keV}]$ | $\Gamma\left(\Lambda^{*}(1520) \rightarrow \gamma \Sigma^{0}\right)[\mathrm{keV}]$ |
| :--- | :--- | :--- |
| From Ref. [221] | $33 \pm 11$ | $47 \pm 17$ |
| From Ref. [222] | $134 \pm 23$ |  |
| From Ref. [223] | $159 \pm 33 \pm 26$ |  |
| From Ref. [210] | $167 \pm 43_{-12}^{+26}$ | 17 |
| From Ref. [211] | 46 | 157 |
| From Ref. [217] | 258 | $71(60$, with empirical |
| This study | $3(2.5-4)$ | $\Sigma^{*} \rightarrow \pi \Lambda, \pi \Sigma$ couplings $)$ |

where $Y=\Lambda, \Sigma^{0}$ is the final state hyperon and $k=\lambda^{1 / 2}\left(M_{\Lambda^{*}}, 0, M_{Y}^{2}\right) /\left(2 M_{\Lambda^{*}}\right)$ the CM momentum of the decay products.

In Tab. 6.4 the numerical results from this study are compared with experimental data. For the $\gamma \Sigma^{0}$ final state, our result almost matches within errors the value given in Ref. [221], and certainly matches it considering the theoretical uncertainties that we will estimate below. The experimental value from Ref. [221] is the only direct measurement of $\Gamma\left(\Lambda^{*} \rightarrow \gamma \Sigma^{0}\right)$. In the same experiment [221], the $\Gamma\left(\Lambda^{*} \rightarrow \gamma \Lambda\right)$ partial width has also been determined but lies far below more recent measurements, see Tab. 6.4. Note, that the value from Ref. [57] for $\Gamma\left(\Lambda^{*} \rightarrow \gamma \Sigma^{0}\right)$ is around six times larger than the value from Ref. [221]. However, this large value is not a direct measurement (see Ref. [224]) but is extrapolated from $\Gamma\left(\Lambda^{*} \rightarrow \gamma \Lambda\right)$ by using $S U(3)$ arguments in Ref. [222]. Summarizing, the experimental situation is far from being clear. In the present study we compare to the direct measurement of $\Gamma\left(\Lambda^{*} \rightarrow \gamma \Sigma^{0}\right)=$ $47 \pm 17 \mathrm{keV}$ as a reference, but an independent experimental confirmation of


Figure 6.5: Alternative representation of the photonic loop with $\pi \Sigma$ and $\pi \Sigma^{*}$.
this value would be desirable. Efforts in this direction have been announced [225]. It is also worth estimating the theoretical uncertainties. The largest source of uncertainty for us is the implicit use of $S U(3)$ to relate the mesonbaryon octet-baryon decuplet couplings. We have scaled them to the $\pi N \Delta$ coupling. If we use the empirical couplings for $\Sigma^{*} \rightarrow \pi \Lambda, \pi \Sigma$ in agreement with the $\Sigma^{*}$ partial decay widths, the value in brackets in Tab. 6.4 results.

The theoretical value for the $\gamma \Lambda$ final state in Tab. 6.4 is systematically below experiment although there are large discrepancies in the data. This suggests that the decay mechanisms could come from a different source than the coupled hadronic channels. The theoretical value is small because of large cancellations: In the scheme of dynamical generation, the dominant building channel of the $\Lambda^{*}(1520)$ is given by $\pi \Sigma^{*}$ as can be seen in Tab. 6.1. However, in the isospin combination from Eq. (6.9) which is needed in Eq. (6.13), this channel precisely vanishes because of the cancellation of the $\pi^{+} \Sigma^{*-}$ and $\pi^{-} \Sigma^{*+}$ contributions. The same holds for the $\pi \Sigma$ channel in $d$-wave with the cancellation in Eq. (6.16) from the isospin combination in Eq. (6.15). This channel is important as the branching ratio into $\pi \Sigma$ is large. In contrast, the diagrams with $\pi^{+} \Sigma^{*-}$ and $\pi^{-} \Sigma^{*+}$ add in the $I=0$ combination with $\gamma \Sigma^{0}$ in the final state instead of $\gamma \Lambda$, as Eq. (6.13) shows, and the same is true for $\pi \Sigma$ in $d$-wave. As a result, a much larger partial decay width for the $\gamma \Sigma^{0}$ final state is obtained.

The cancellation of the $\pi \Sigma$ and $\pi \Sigma^{*}$ channels can be also understood when we turn the external baryon line around and redraw the decay process as shown in Fig. 6.5. First, we consider the case with the $\Lambda$. The $\pi^{+} \pi^{-}$


Figure 6.6: Cut-off dependency of $\Gamma\left(\Lambda^{*}(1520) \rightarrow \gamma \Lambda\right)[\mathrm{keV}]$. Contributions for different particles in the first loop and coherent sum. Dotted line: $K \Xi^{*}$ in $s$-wave. Dashed line: $\bar{K} N$ in $d$-wave. Thick solid line: Coherent sum.
system is necessarily in $J^{P}=1^{-}$as these are the quantum numbers of the photon. As a consequence, the condition $L+S+I=$ even for the two-pion state where $L=J=1$ and $S=0$ can only be fulfilled if the two-pion state is in $I=1$; this is in contradiction to $I=0$ of the $\bar{\Lambda} \Lambda^{*}$ system. This is independent of the interaction denoted with the gray dashed circle in Fig. 6.5. In contrast, if the baryon on the right side is a $\overline{\Sigma^{0}}$, then the $\overline{\Sigma^{0}} \Lambda^{*}$ system is in an isospin one state, so that a finite contribution is expected. If the $\pi^{+} \pi^{-}$system is replaced with $K^{+} K^{-}$, there is no restriction imposed by $L+S+I=$ even, so this process is possible for both $\Lambda$ or $\overline{\Sigma^{0}}$ on the right side.

The situation is illustrated in Fig. 6.6 and 6.7 where the partial decay widths are plotted as a function of the cut-off in the first loops. Indeed, the large $\pi \Sigma$ and $\pi \Sigma^{*}$ channels that contribute in Fig. 6.7 are missing in Fig. 6.6 and render the width small. Note also that the $d$-wave loops introduce a relatively strong cut-off dependence. Our cut-offs from Secs. 6.3 and 6.3.1 have been uniquely fixed by matching the cut-off scheme to the dimensional regularization scheme of the $M B^{*}$ and $M B$ loop functions that generate dynamically the $\Lambda^{*}(1520)$. The latter have values for the subtraction constants


Figure 6.7: Cut-off dependency of $\Gamma\left(\Lambda^{*}(1520) \rightarrow \gamma \Sigma^{0}\right)[\mathrm{keV}]$. Contributions for different particles in the first loop and coherent sum. Dotted line: $K \Xi^{*}$ in $s$-wave. Dashed line: $\bar{K} N$ in $d$-wave. Dashed dotted line: $\pi \Sigma^{*}$ in $s$-wave. Double dashed dotted line: $\pi \Sigma$ in $d$-wave. Thick solid line: Coherent sum.
which lead to good data description in $\bar{K} N \rightarrow \bar{K} N$ and $\bar{K} N \rightarrow \pi \Sigma[60]$. Therefore, assuming that the strong interaction in these processes fixes the cut-offs, their values should be taken seriously and not changed for the first loop with the photon. On the other hand, the strong cut-off dependence is a large source of theoretical error in the model of the radiative decay such that uncertainties as big as $50 \%$ would not be exaggerated. With this uncertainty the $\Lambda^{*}(1520) \rightarrow \gamma \Sigma^{0}$ is clearly compatible with the only data available. But the $\Lambda^{*}(1520) \rightarrow \gamma \Lambda$ is certainly not. However, the fact that the only measurement for $\Lambda^{*}(1520) \rightarrow \gamma \Sigma^{0}$ is done in an experiment where the $\Lambda^{*}(1520) \rightarrow \gamma \Lambda$ disagrees so strongly with other measurements calls for caution and and further data on this decay rate is most needed.

On the other hand, even with large uncertainties our prediction for $\Lambda^{*}(1520) \rightarrow \gamma \Lambda$ is definitely small. Uncertainties from the implicit $S U(3)$ use in the couplings are estimated of the order of $15 \%$ resulting in the band for the partial decay rate shown in brackets in Tab. 6.4. Hence we have pinned down an observable which is extremely sensitive to extra components of the $\Lambda^{*}(1520)$ resonance beyond the meson-baryon ones provided by the chiral
unitary approach. The sensitivity shows up because of the exact cancellation of the contribution from the most important components provided by the chiral unitary approach.

### 6.5 Estimates of strength of the genuine resonance component

The usual way to include a genuine resonance in the chiral unitary approach is by introducing a CDD pole in the kernel of the interaction in the BSE. The residues of the pole stand for the strength of the coupling of this genuine resonance component to the meson baryon states of our space. These couplings are unknown generally and fits to data are performed to determine them, which in some cases [17] turn out to be compatible with zero, thus giving an indication that the genuine component plays a minor role in the structure of the physical resonance, which then qualifies as a dynamically generated resonance. In the present case we have no much experimental information to determine the strength of the genuine component. The success of the meson baryon components alone in the $d$-wave $\bar{K} N$ scattering indicates a small component of a genuine resonance, which, however, is essential to reproduce the $\Lambda^{*}(1520) \rightarrow \Lambda \gamma$ decay. Even if we could determine this small component of the CDD pole from scattering, which due to its minor role played there would have large uncertainties, this would not help us determining the role played in the radiative decay $\Lambda^{*}(1520) \rightarrow \Lambda \gamma$ since there is a new, independent, and unknown coupling of the CDD component to the photon. One has to find other methods here to make estimates of the strength of the genuine resonance in the physical $\Lambda^{*}(1520)$. Due to this, we shall make use of the results of quark models to try to make some rough estimate.

The first thing one must admit is that, with the large differences found for the decay rates in different quark models (see Tab. 6.4), the uncertainties in the estimates must be large. But even then, the exercise is worth doing and also brings light on how extra experimental data could help in this analysis. In the first step let us take as more significant the most recent results obtained in a relativistic quark model which has proved to have large
predictive power [216,217]. The decay rate obtained with this model is twice as big as experiment. Second, for the extra component we are searching for, there is no need to take a quark wave function which is fitted to data in order to obtain optimal agreement with experiment assuming that this is the only component of the wave function. Hence, we would rather search for a quark component with just a $s$ quark in the $1 p$ level and the $u, d$ quarks in the $1 s$ ground state coupled to isospin 0, i.e. no configuration mixing.

This wave function would help getting larger results for the $\Lambda^{*}(1520) \rightarrow$ $\Lambda \gamma$ while there would be no changes for the $\Lambda^{*}(1520) \rightarrow \Sigma \gamma$ transition, as we have pointed out before in the absence of configuration mixing. It is reasonable to assume that if all the strength of the configuration mixing wave function of [217] is put into this single component, the strength for $\Lambda^{*}(1520) \rightarrow \Lambda \gamma$ decay would increase and the new strength would be roughly of the order of the sum of strengths for $\Lambda^{*}(1520) \rightarrow \Lambda \gamma$ and $\Lambda^{*}(1520) \rightarrow \Sigma \gamma$ transitions obtained with the configuration mixing wave function. Next, we assume that the new contribution interferes constructively with the one from the meson baryon component (although the interference is very small) and then we find that with $20 \%$ of the genuine quark wave function we can reproduce the experimental data for $\Lambda^{*}(1520) \rightarrow \Lambda \gamma$ without spoiling the agreement for $\Lambda^{*}(1520) \rightarrow \Sigma \gamma$ which would simply be reduced by $20 \%$. This latter decay would be exclusively due to the meson baryon component.

The former exercise should be improved in a more realistic approach. Indeed, it is well known from studies of the cloudy bag model [199] that in models where the meson cloud plays a role in the building up of the baryon, the size is mostly due to the meson cloud while the quarks are confined in a very small region. We would invoke this finding to suggest that in a hybrid analysis for the $\Lambda^{*}(1520)$ resonance, the quarks would be confined in a small region, smaller than that assumed in quark models like [217] where all the baryon properties are attributed to the quarks. By recalling that the coupling $\Lambda^{*}(1520) \rightarrow \Lambda \gamma$ (or analogous radiative baryon-baryon transitions) is proportional to the inverse of the radius or the quark core [226], a quark wave function with half the radius of that used in [217] would lead to a twice bigger radiative coupling, and hence a $5 \%$ of the quark wave function would suffice to explain the data.

Rough as the analysis is, it has the virtue of touching the sensitive points of a more thorough analysis which could be performed in the future when more data are available. These new data would certainly be necessary, because with just one piece of data, the rate of $\Lambda^{*}(1520) \rightarrow \Lambda \gamma$, one has ambiguities in the size and strength as we have seen in the exercise. Yet, the rough results indicate that one could indeed live with a small component of a genuine resonance, and a large size of the meson baryon component which would justify the success of the meson cloud component alone in explaining the scattering data.

One could also suggest extra experiments which would help get more information in the future. In the same spirit of the cloudy bag model, we recall that baryon form factors usually have two regions, the one at small momentum transfers which is dictated by the meson cloud and another one at larger momentum transfers which is determined by the size and strength of the quark core. In this sense, future information from $\Lambda^{*}(1520) \rightarrow \Lambda e^{+} e^{-}$ would certainly bring new light into the issue. Increased photon fluxes or kaon fluxes in planned future facilities make this goal attainable. But extra information concerning the $\Lambda^{*}(1520)$ which is now largely studied in several laboratories, could help in the quest of determining the structure of this interesting resonance.

### 6.6 Conclusions

The chiral unitary model for the $\Lambda^{*}(1520)$ has been extended in order to describe the radiative decay of the $\Lambda^{*}$. The study of the two decay modes into $\gamma \Lambda$ and $\gamma \Sigma^{0}$ can help gain insight into the nature of the $\Lambda^{*}$, as to whether it is a genuine three quark state, a dynamically generated resonance, or a mixture of both.

For the $\gamma \Sigma^{0}$ final state we have seen that the model of dynamical generation matches the empirical value, although there are certain theoretical uncertainties from the $d$-wave loops in the model. However, the good reproduction of the empirical value fits in the picture because the dominant channels of our coupled channel model add up for this decay, and in some quark models, the dominant three quark component for this decay is small.

In contrast, we find very little contribution from our model for the $\gamma \Lambda$ final state due to a cancellation of the dominant channels, so that this decay should be dominated by the genuine three-quark component in a more realistic picture of the $\Lambda^{*}(1520)$ as a hybrid with some three constituent quark component and a substantial meson-baryon cloud.

We have made rough estimates using present information from relativistic quark models which point in the direction that a small admixture of a genuine three constituent quarks wave function could explain the data.

More precise experimental information and theoretical tools are needed in order to make more quantitative conclusions about the $\Lambda^{*}(1520)$, but the findings of the present study point in the direction of the $\Lambda^{*}$ being a composite object of a genuine 3-quark state and a dynamical resonance, with the first component dominating the $\Lambda^{*}(1520) \rightarrow \gamma \Lambda$ decay and the second the $\Lambda^{*}(1520) \rightarrow \gamma \Sigma^{0}$ decay. Extra experimental work, measuring other couplings of the $\Lambda^{*}(1520)$, like the one to $\bar{K}^{*}$, as recently shown in Ref. [226], or the $\Lambda^{*}(1520) \rightarrow \Lambda e^{+} e^{-}$reaction, would also bring relevant information on the nature of the $\Lambda^{*}(1520)$.

## Chapter 7

## Radiative decay of the $\Delta^{*}(1700)$

Electromagnetic properties provide information about the structure of strongly interacting systems and allow for independent tests of hadronic models. The radiative decay of the $\Delta^{*}(1700)$ can be studied in a similar way as the radiative decay of the $\Lambda^{*}(1520)$ in chapter 6 . The driving mechanism of the radiative decay of the $\Delta^{*}(1700)$ is, as in case of the $\Lambda^{*}(1520)$, the photon coupling to the last loop of the rescattering series which dynamically generates the $\Delta^{*}(1700)$. In the original coupled channel model of the $\Delta^{*}(1700)$ only the rescattering of the $\left(3 / 2^{+}\right)$decuplet baryons with the $\left(0^{-}\right)$mesons in $s^{-}$ wave is considered. We include in this chapter an additional channel, $\pi N$ in $d$-wave. The second novelty is a careful treatment of questions related to gauge invariance.

### 7.1 Introduction

Several low-lying $\left(3 / 2^{-}\right)$resonances have been generated in the unitary coupled channel approach from Ref. [44]. Poles appear in the complex plane of the scattering amplitude as a consequence of the unitarization. These poles can be identified with resonances from the PDB [57]. Usually, some fine-tuning improves the pole position and brings it closer to its phenomenological value, as for example in case of the $\Lambda^{*}(1520)$ (see chapter 6). There, the introduction of $d$-wave channels [60,209] improves the mass and width and the branching ratios into the $\left(1 / 2^{+}\right)$baryon, $\left(0^{-}\right)$meson channels in
$d$-wave. However, this requires the introduction of additional parameters as, e.g., $d$-wave couplings of the new channels to the $s$-wave channels and also $d$-wave couplings of $d$-wave $\rightarrow d$-wave transitions. Therefore, it is interesting and necessary to test the predictivity of the fine-tuned model in as many different reactions as possible without introducing new parameters. We have seen an example for this in chapter 6 where the radiative decay of the $\Lambda^{*}(1520)$ is studied.

In the same scheme of dynamical generation, a pole in the strangeness zero, isospin-spin three half channel has been identified with the $\Delta^{*}(1700)$ [44]. The coupled channels in this case are $\Delta(1232) \pi, \Sigma^{*}(1385) K$, and $\Delta(1232) \eta$. The $\Delta^{*}(1700)$, together with a series of other production mechanisms, has been included in chapter 4 in the study of the $\gamma p \rightarrow \pi^{0} \eta p$ and $\gamma p \rightarrow$ $\pi^{0} K^{0} \Sigma^{+}$photoproduction reactions, currently measured at ELSA/Bonn. In chapter 4 the $\Delta^{*}(1700)$, together with its strong couplings to $\Delta(1232) \eta$ and $\Sigma^{*}(1385) K$, turned out to provide the dominant contribution. The branching ratios into these two channels are predicted from the scheme of dynamical generation and differ from a simple $S U(3)$ extrapolation of the $\Delta \pi$ channel by up to a factor of 30 as seen in chapter 5 .

The predictions for both reactions are in good agreement with preliminary data [193]. Recently, new measurements at low photon energies have been published [229] which also agree well with the results from chapter 4. This has motivated the study documented in chapter 5 of altogether nine additional pion- and photon-induced reactions. From considerations of quantum numbers and the experimentally established $s$-wave dominance of the $\Sigma^{*}$ production close to threshold, the $\Delta^{*}(1700)$ channel is expected to play a major, in some reactions dominant, role. Indeed, good global agreement has been found for the studied reactions that span nearly two orders of magnitude in their respective cross sections.

Thus, evidence from quite different experiments has been accumulated that the strong $\Delta^{*}(1700) \rightarrow \Sigma^{*} K, \Delta \eta$ couplings, predicted by the coupled channel model, are realistic. This gives support to the scheme of dynamical generation of this resonance. However, in all the photon-induced reactions from the chapters 4 and 5 the initial $\gamma p \rightarrow \Delta^{*}(1700)$ transition has been taken from the experimental [228] helicity amplitudes $A_{1 / 2}$ and $A_{3 / 2}$ [227]:

In Ref. [168] the electromagnetic form factors $G_{1}^{\prime}, G_{2}^{\prime}$, and $G_{3}^{\prime}$, which appear in the scalar and vector part of the $\gamma p \rightarrow \Delta^{*}(1700)$ transition, have been expressed in terms of the experimentally known $A_{1 / 2}$ and $A_{3 / 2}[228]$; this provides the transition on which we rely in all the photoproduction reactions via $\Delta^{*}(1700)$ in the chapters 4 and 5.

Such a semi-phenomenological ansatz is well justified: the photon coupling and the width of the $\Delta^{*}(1700)$ is taken from phenomenology, whereas the strong decays of the $\Delta^{*}(1700)$ into hadronic channels are predictions from the unitary coupled channel model; the strengths of these strong transitions are responsible for the good agreement with experiment found in the chapters 4 and 5. It is, however, straightforward to improve at this point, and this is the aim of this chapter.

Electromagnetic properties provide additional information about the structure of strongly interacting systems and allow for an independent test of hadronic models, in this case the hypothesis that the $\Delta^{*}(1700)$ is dynamically generated. A virtue of this test is that one can make predictions for the radiative decay, or equivalently, the inverse process of photoproduction; the components of the $\Delta^{*}(1700)$ in the $\left(0^{-}, 3 / 2^{+}\right)$meson-baryon base $\left(M B^{*}\right)$ are all what matters, together with the well-known coupling of the photon to these constituents.

In chapter 6, good agreement with experiment has been found for the $\Lambda^{*}(1520) \rightarrow \Sigma^{0} \gamma$ decay, where the dominant channels add up. The work of this chapter is carried out along the lines of chapter 6 but several modifications will be necessary: the $(\pi N)$ channel in $d$-wave plays an important role and is implemented in the coupled channel scheme. Second, a fully gauge invariant phototransition amplitude for the $s$-wave channels is derived that includes also couplings of the photon to the $B^{*}$ baryons.

### 7.2 The model for the radiative $\Delta^{*}(1700)$ decay

In Sec. 7.2.1 the coupled channel model from Ref. [44] is revised and extended to the inclusion of the $\pi N$ channel in $d$-wave, $(\pi N)_{d}$. In Sec. 7.2 .2 the
model for the phototransition amplitude $\Delta^{*}(1700) \rightarrow \gamma N$ is derived: The photon interacts with the dynamically generated resonance via a one-loop intermediate state that is given by all coupled channels which constitute the resonance.

### 7.2.1 The $(\pi N)_{d}$ channel in the unitary coupled channel approach

In the unitarized model from Ref. [44] the $\Delta^{*}(1700)$ resonance appears as a quasi-bound state in the coupled channels $\Delta \pi, \Sigma^{*} K$, and $\Delta \eta$. In this approach, the strong attraction in the latter two channels leads to the formation of a pole in the $D_{33}$ channel which has been identified with the $\Delta^{*}(1700)$ resonance.

In Ref. [44] the chiral interaction from [66] is adapted in a nonrelativistic reduction providing isovector $M B^{*} \rightarrow M B^{*}$ transitions where $M\left(B^{*}\right)$ stands for the octet of $0^{-}$pseudoscalar mesons ( $3 / 2^{+}$decuplet baryons). This interaction is unitarized by the use of the Bethe-Salpeter equation (BSE)

$$
\begin{equation*}
T=(1-V G)^{-1} V \tag{7.1}
\end{equation*}
$$

which is a matrix equation in coupled channels, factorized to an algebraic equation according to the on-shell reduction scheme of [17,44]. The function $G$ is a diagonal matrix with the $M B^{*}$ loop functions $G_{M B^{*}}$ of the channels $i$ which are regularized in dimensional regularization with one subtraction constant $\alpha$. As it will appear in a different context in the phototransition, the function $G_{M B^{*}}$ has been re-derived with the result

$$
\begin{align*}
G_{M B^{*}} & =i \int \frac{d^{4} p}{(2 \pi)^{4}} \frac{2 M}{(p+q)^{2}-M^{2}} \frac{1}{p^{2}-m^{2}} \\
& =-2 M \lim _{d \rightarrow 4}\left[\int \frac{d^{d} \ell_{E}}{(2 \pi)^{d}} \int_{0}^{1} d x \frac{1}{\left(\ell_{E}^{2}+x M^{2}+(x-1)\left(x q^{2}-m^{2}\right)\right)^{2}}\right] \\
& =\frac{2 M}{(4 \pi)^{2}}\left[\alpha+\log \frac{m^{2}}{\mu^{2}}+\frac{M^{2}-m^{2}+s}{2 s} \log \frac{M^{2}}{m^{2}}+\frac{Q(\sqrt{s})}{\sqrt{s}} f_{1}(\sqrt{s})\right] \tag{7.2}
\end{align*}
$$

where $m(\mathrm{M})$ is the meson (decuplet baryon) mass, $q^{2}=s$ is the invariant scattering energy, and

$$
\begin{equation*}
\alpha(\mu)=\gamma-\frac{2}{\epsilon}-\log (4 \pi)-2 \tag{7.3}
\end{equation*}
$$

with the Eucledian integration over $\ell_{E}$ and $\epsilon=4-d$. Values of the regularization scale of $\mu=700 \mathrm{MeV}$ and $\alpha=-2$ are natural, as argued in [44]. The c.m. momentum function $Q$ and $f_{1}$ in Eq. (7.2) are given by

$$
\begin{align*}
Q(\sqrt{s}) & =\frac{\sqrt{\left(s-(M+m)^{2}\right)\left(s-(M-m)^{2}\right)}}{2 \sqrt{s}} \\
f_{1}(\sqrt{s}) & =\log \left(s-\left(M^{2}-m^{2}\right)+2 \sqrt{s} Q(\sqrt{s})\right) \\
& +\log \left(s+\left(M^{2}-m^{2}\right)+2 \sqrt{s} Q(\sqrt{s})\right) \\
& -\log \left(-s+\left(M^{2}-m^{2}\right)+2 \sqrt{s} Q(\sqrt{s})\right) \\
& -\log \left(-s-\left(M^{2}-m^{2}\right)+2 \sqrt{s} Q(\sqrt{s})\right) . \tag{7.4}
\end{align*}
$$

The loop function from Eq. (7.2) has a real part, which is a major difference of the present approach compared to unitarizations with the $K$-matrix. The real parts of the $G_{i}$, together with the attractive kernel $V$ in the isospin $3 / 2$, strangeness 0 channel, provide enough strength for the formation of a pole in the complex plane of the invariant scattering energy $\sqrt{s}$ which is identified with the $\Delta^{*}(1700)$.

However, additional channels will also couple to the dynamically generated resonance, changing in general its position and branching ratios, as these new channels can rescatter as well. In this study, the $(\pi N)_{d}$ channel is included in the analysis, because this is the lightest channel that can couple to the $\Delta^{*}(1700)$ and precise information of the $\pi N \rightarrow \pi N$ transition in the $D_{33}$ channel exists from the partial wave analysis of Ref. [230]. The $(\rho N)_{s}$ channel has been found important [204, 228], but for the radiative decay its influence is expected to be moderate as discussed below.

In order to include the $(\pi N)_{d}$ channel in the coupled channel model, one has to determine the $\left(\pi_{N}\right)_{d} \rightarrow\left(M B^{*}\right)_{s}$ transitions, where $M B^{*}$ stands for the channels $\Delta \pi, \Sigma^{*} K$, and $\Delta \eta$ from [44]. There is no experimental information on these transitions. From the theoretical side, there is no information either due to the large number of low energy constants in the $d$-wave to
$s$-wave transition. Thus, the coupling strengths have to be introduced as free parameters. For the inclusion of $d$-wave potentials, we follow the lines of Ref. $[60,61]$ where it has been shown that the $d$-wave transitions can be factorized on-shell in the same way as the $s$-wave transitions; as a consequence, the meson-baryon $d$-wave loop function is the same as the $s$-wave loop function from Eq. (7.2).

With the channel ordering $i=1 \cdots 4$ for $\Delta \pi, \Sigma^{*} K, \Delta \eta,(\pi N)_{d}$ the interaction kernel is given by

$$
\begin{align*}
& V=  \tag{7.5}\\
& \left(\begin{array}{llll}
-\frac{2}{4 f_{\pi}^{2}}\left(k^{0}+k^{\prime 0}\right) & -\frac{\sqrt{5}}{4 f_{\pi}^{2}}\left(k^{0}+k^{\prime 0}\right) & 0 & Q_{\pi N}^{2} r \beta_{(\pi N)_{d} \rightarrow \Delta \pi} \\
-\frac{-1}{4 f_{\pi}^{2}}\left(k^{0}+k^{\prime 0}\right) & -\frac{3}{\sqrt{2}} \\
4 f_{\pi}^{2} \\
& 0 & \left.k^{0}+k^{\prime 0}\right) & Q_{\pi N}^{2} r \beta_{(\pi N)_{d} \rightarrow \Sigma^{*} K} \\
& & & Q_{\pi N}^{2} r \beta_{(\pi N)_{d} \rightarrow \Delta \eta} \\
& & Q_{\pi N}^{4} r^{2} \beta_{(\pi N)_{d} \rightarrow(\pi N)_{d}}
\end{array}\right)
\end{align*}
$$

where we have multiplied some elements with $r=1 /\left(4 \cdot 93^{2} \cdot 1700\right) \mathrm{MeV}^{-3}=$ $1.7 \cdot 10^{-8} \mathrm{MeV}^{-3}$ in order to obtain dimensionless transition strengths $\beta$ of the order of one. In Eq. (7.5) one can recognize the chiral isovector transitions with $\left(k^{0}+k^{\prime 0}\right)$ from Ref. [44] where $k^{0}=\left(s-M^{2}+m^{2}\right) /(2 \sqrt{s})$ is the meson energy and $f_{\pi}=93 \mathrm{MeV}$. In Eq. (7.5), $Q_{\pi N}$ is the on-shell c.m. momentum of the $\pi N$ system and the $\beta$ are the $s$-wave to $d$-wave transition strengths.

Although a natural value for the subtraction constants is given by $\alpha=$ -2 [44], it is a common procedure [60] to absorb higher order effects in small variations around this value. Thus, as these higher order effects are undoubtly present, we allow for variations of the $\alpha$ of the four channels. Together with the transitions strengths $\beta$, the set of free parameters is fitted to the single-energy-bin solution of the PWA of Ref. [230]. Note that there is a conversion factor according to

$$
\begin{equation*}
\tilde{T}_{i j}(\sqrt{s})=-\sqrt{\frac{M_{i} Q_{i}}{4 \pi \sqrt{s}}} \sqrt{\frac{M_{j} Q_{j}}{4 \pi \sqrt{s}}} T_{i j}(\sqrt{s}) \tag{7.6}
\end{equation*}
$$

in order to express the solution $T$ of the Bethe-Salpeter equation (7.1) in terms of the dimensionless amplitude $\tilde{T}(\sqrt{s})$ plotted in Fig. 7.1. In Eq. (7.6), $M_{i}\left(Q_{i}\right)$ is the baryon mass (c.m. momentum) of channel $i$; in the present case, $i=j=4$.

Table 7.1: Parameter values of the fits to $(\pi N)_{d}$ : the $\alpha_{i}$ are subtraction constants, the $\beta_{i}$ are $(\pi N)_{d} \rightarrow\left(B^{*} M\right)_{s},(\pi N)_{d}$ transition strengths. Note the sign changes of the $\beta_{i}$ between fit 3 and fit $3^{\prime}$.

|  | $\alpha_{\Delta \pi}$ | $\alpha_{\Sigma^{*} K}$ | $\alpha_{\Delta \eta}$ | $\alpha_{(\pi N)_{d}}$ |
| :--- | :--- | :--- | :--- | :--- |
| Fit 1 | -1.96 | -1.28 | -0.87 | -1.00 |
| Fit 2 | -1.25 | -1.29 | -0.66 | -1.96 |
| Fit 3 | -1.21 | -1.21 | -1.21 | -1.00 |
| Fit 3 | -1.22 | -1.22 | -1.22 | -1.00 |
|  | $\beta_{(\pi N)_{d} \rightarrow \Delta \pi}$ | $\beta_{(\pi N)_{d} \rightarrow \Sigma^{*} K}$ | $\beta_{(\pi N)_{d} \rightarrow \Delta \eta}$ | $\beta_{(\pi N)_{d} \rightarrow(\pi N)_{d}}$ |
| Fit 1 | +2.19 | -1.15 | -0.07 | 63.5 |
| Fit 2 | +3.30 | -0.37 | +0.31 | 0 |
| Fit 3 | +3.25 | -0.85 | -0.54 | 0 |
| Fit 3 | -3.19 | +0.92 | +0.42 | 0 |

For energies above 1.7 GeV a theoretical error of 0.08 has been added to the error bars from [230] as additional channels such as $\rho N$ start to open and one can not expect good agreement much beyond the position of the $\Delta^{*}(1700)$. The resulting amplitudes of four different fits are plotted in Fig. 7.1 with the parameter values displayed in Tab. 7.1. In fit 1, all parameters have been left free. The values for the $\beta$ are small: For $\sqrt{s} \sim M_{\Delta^{*}}$, they lead to values in the kernel (7.5) around one order of magnitude smaller than the chiral interactions between channels 1 to 3 ; the size of $\beta_{(\pi N)_{d} \rightarrow(\pi N)_{d}}$ corresponds to a value even two orders of magnitude smaller. These parameter values reflect the fact that the $\Delta^{*}(1700)$ couples only weakly to $\pi N$ and that the $\pi N$ interaction in $D_{33}$ is weak in general.

Thus, the $(\pi N)_{d} \rightarrow(\pi N)_{d}$ transition strength can be set to zero which is done for the fits 2, 3, and 3'. One can even choose all subtraction constants of the $s$-wave channels to be equal, which is done in fit 3 , and still obtain a sufficiently good result as Fig. 7.1 shows. However, for fit three, there is another minimum in $\chi^{2}$, almost as good as the best one found. This fit is called fit 3'. As Tab. 7.1 and Fig. 7.1 show, one obtains an almost identical


Figure 7.1: Fit results: Fit 1: All parameters free. Fit 2: without $\pi N \rightarrow \pi N$ $d$-wave transition kernel. Fit 3: As fit 2, but all subtraction constants for the $s$-wave loops chosen to be equal. Fit 3': As fit 3 but different minimum in $\chi^{2}$. The error bars show the single-energy-bin solution from Ref. [230].

Table 7.2: Position $s_{\text {pole }}^{1 / 2}$ and couplings of the $\Delta^{*}(1700)$. The values in brackets show the original results from [44] without the inclusion of the $(\pi N)_{d^{-}}$ channel. The PDB [228] quotes a value of $s_{\text {pole }}^{1 / 2}=(1620-1680)-i(160-240)$ MeV and couplings corresponding to $\left|g_{(\Delta \pi)_{s}}\right|=1.57 \pm 0.3,\left|g_{(\pi N)_{d}}\right|=$ $0.94 \pm 0.2$. Note the sign change of $g_{(\pi N)_{d}}$ between fit 3 and fit 3 '.

|  | $s_{\text {pole }}^{1 / 2}[\mathrm{MeV}]$ | $g_{\Delta \pi}$ | $\left\|g_{\Delta \pi}\right\|$ | $g_{\Sigma^{*} K}$ | $\left\|g_{\Sigma^{*} K}\right\|$ |
| :--- | :--- | :--- | :--- | :--- | :--- |
|  | $(1827-i 108)$ | $(0.5+i 0.8)$ | $(1.0)$ | $(3.3+i 0.7)$ | $(3.4)$ |
| Fit 1 | $1707-i 160$ | $1.09-i 0.92$ | 1.4 | $3.57+i 1.91$ | 4.0 |
| Fit 2 | $1692-i 166$ | $0.62-i 1.03$ | 1.2 | $3.44+i 2.28$ | 4.1 |
| Fit 3 | $1697-i 214$ | $0.68-i 1.07$ | 1.3 | $3.01+i 1.95$ | 3.6 |
| Fit 3' | $1698-i 216$ | $0.68-i 1.07$ | 1.3 | $3.02+i 1.95$ | 3.6 |
|  | $g_{\Delta \eta}$ | $\left\|g_{\Delta \eta}\right\|$ | $g_{(\pi N)_{d}}$ |  |  |
|  | $(1.7-i 1.4)$ | $(2.2)$ | () |  |  |
| Fit 1 | $-1.98-i 1.68$ | 2.6 | $-0.84-i 0.05$ |  |  |
| Fit 2 | $-1.89-i 1.78$ | 2.6 | $-0.77+i 0.00$ |  |  |
| Fit 3 | $-2.27-i 1.89$ | 3.0 | $-0.89+i 0.15$ |  |  |
| Fit 3' | $-2.27-i 1.89$ | 3.0 | $+0.92-i 0.12$ |  |  |

amplitude with a set of $\beta$ 's with opposite sign as compared to fit 3 (see explanation below).

For the different solutions 1 to $3^{\prime}$, the coupling strengths of the resonance to the different channels can be obtained by expanding the amplitude around the resonance position in a Laurent series (see also Sec. 7.2.5). The residues give the coupling strengths which are uniquely determined up to a global sign which we fix by demanding the real part of the coupling to $\Delta \pi$ to be positive. In Tab. 7.2 the resulting couplings are displayed. The values in brackets quote the values of the original model from Ref. [44] without the inclusion of $(\pi N)_{d}$. Compared to these values, all couplings increase slightly in strength and have different phases. However, the main properties of the resonance are conserved; in particular, the absolute values $|g|$ do not change much. This is important with respect to the previous chapters 4 and 5 where
the model from [44] has been used for the couplings of the $\Delta^{*}(1700)$ to $\Delta \eta$ and $\Sigma^{*} K$.

We give some explicit corrections to the results from chapters 4 and 5 based on the values from fit 1 which is the preferred one as discussed below. The cross sections of the pion induced processes from chapter 5 with the $\Sigma^{*} K$ final state will change by a factor $(|-0.84-i 0.05| / 0.94)^{2}(4.0 / 3.4)^{2}=1.1$, i.e., they stay practically the same. In the pion induced reactions with a $\Delta \eta$ final state the factor is again 1.1. Anticipating the result for fit 1 from Tab. 7.4, that the radiative coupling is close to the experimental one (which is the one used in the chapters 4 and 5), the cross section of the reaction $\gamma p \rightarrow K^{0} \pi^{+} \Lambda$ increases by a factor of $(4.0 / 3.4)^{2}=1.4$ which improves the agreement with data, see Fig. 5.7 and also the recent measurements in Ref. [206]. For the cross section of the $\gamma p \rightarrow \pi^{0} \eta p$ reaction, the transition via the $\Delta^{*}(1700)$ is only one of the reactions, and one has to re-evaluate the coherent sum of all processes of the model of chapter 4, with the new values from Tab. 7.2. The numerical results for this reaction from chapter 4 have been updated in Fig. 5.8. With the values of fit 1 from Tab. 7.2, the cross section stays practically the same as in Fig. 5.8 ( $10 \%$ decrease).

Interestingly, the sign of the coupling to $(\pi N)_{d}$ is reversed in fit $3^{\prime}$ compared to fit 3. This behavior has been noted before for the parameter values in Tab. 3.2. The reason for the difference between fit 3 and 3 ' is that the coupling of the $\pi N$ channel to the $\Delta^{*}(1700)$ is small: Although $(\pi N)_{d}$ is contained to all orders in the rescattering scheme, higher orders are smaller. Then, the (by far) dominant order in the fit to $(\pi N)_{d} \rightarrow(\pi N)_{d}$ is $g_{(\pi N)_{d}}^{2}$ and the relative sign to the other couplings $g_{\Delta \pi}, g_{\Sigma^{*} K}$, and $g_{\Delta \eta}$ is hard to fix. Thus, the important parameters, which have fine-tuned the original model from [44], are the subtraction constants $\alpha_{\Delta \pi}, \alpha_{\Sigma^{*} K}$, and $\alpha_{\Delta \eta}$. Their variation brings the pole down from $s_{\text {pole }}^{1 / 2}=1827-i 108 \mathrm{MeV}[44]$ to the values quoted in Tab. 6.1.

In Sec. 7.3 results for the radiative decay for all four fits are given, in order to obtain an idea of the systematical theoretical uncertainties. The fit 1 is preferred, though, because in the reduction to less free parameters, as it is the case for the fits 2,3 , and 3 ', the remaining free parameters have to absorb effects such as direct $(\pi N)_{d} \rightarrow(\pi N)_{d}$ transitions; results might
become distorted, despite the fact that the fits appear to be good in Fig. 7.1. The larger space of free parameters also helps to fix the ambiguities found in fit 3 and 3': For fit 1 and 2, no alternative solutions with a reversed sign for $g_{(\pi N)_{d}}$ have been found, and, thus, the sign is fixed.

We have also performed a search for poles in the second Riemann sheet. For the different fits, the pole positions are given in Tab. 7.2. The positions are in agreement with the values given by the PDB [228]. Note that one of the virtues of the coupled channel analysis is that a separation of background and resonance part of the amplitude is not necessary; thus, the position of the pole does not suffer from ambiguities from this separation process required by other analyses.

Nevertheless, some theoretical uncertainties are present in the model from the omission of other channels such as $\rho N$ in $s$-wave or even $\rho N, \Delta \pi$ and $K \Sigma$ in $d$-wave all of which have been reported in the PDB [228]. Although the $\rho N$ channel is closed at the position of the $\Delta^{*}(1700)$, it contributes through the real part of the $\rho N$ loop function in the rescattering scheme, and through the finite width of the $\rho$ even to the imaginary part.

However, in the calculation of the radiative decay, which is the aim of this study, no large contributions are expected from these heavy channels. In the study of the radiative decay of the $\Lambda^{*}(1520)$ in chapter 6 we have seen that contributions to the radiative decay width from the heavy channels are systematically suppressed.

### 7.2.2 The phototransition amplitude

The only known radiative decay of the $\Delta^{*}(1700)$, which is also the one of relevance in the chapters 4 and 5 , is into $\gamma N$ and we concentrate on this channel. The coupled channel model for the $\Delta^{*}(1700)$ has the virtue that the radiative decay can be calculated in a parameter-free and well-known way through the coupling of the photon to particles which constitute the resonance. The dominant photon couplings to the coupled channels $\Delta \pi$, $\Sigma^{*} K, \Delta \eta$, and $(\pi N)_{d}$ are displayed in Fig. 7.2 . We have chosen here the $I_{3}=+1 / 2$ or charge $C=+1$ state of the $\Delta^{*}(1700)$. The set of diagrams (1) to (9) is gauge invariant and finite as shown in Sec. 7.2.3.


Figure 7.2: Mechanisms for the $\Delta^{*}(1700)$ decay in $s$ - and $d$-wave loops. The shaded circles represent the $\Delta^{*}(1700)$ in the transition $B^{*} M \rightarrow \Delta \pi$ or $\pi^{+} n \rightarrow \Delta \pi$. The diagrams in the left column are referred to as "meson pole loops". Diagrams (2), (5), and (8) have a $\gamma B M B^{*}$ transition and are referred to as "Kroll-Ruderman loops" (diagrams (3), (6), (9): "baryon pole terms"). Diagram (10) shows the process with a $d$-wave coupling of the $(\pi N)_{d}$ intermediate state to the $\Delta^{*}(1700)$.

There is also a loop diagram where the photon couples to the $\Delta^{+}$in an intermediate $\Delta^{+} \eta$ state. This contribution is doubly suppressed: First, the diagrams with $\gamma B^{*} B^{*}$ couplings are much smaller than the other ones (see Sec. 7.2.3), and, second, the $\Delta \eta$ state is heavy which renders this contribution even smaller.

Apart from the photon coupling to the $\Delta \pi$ and $\Sigma^{*} K$ channels, Fig. 7.2 shows also the coupling to the $\pi N d$-wave channel in diagram (10). Note that there is no Kroll-Ruderman term (which has $\pi N$ in $s$-wave in the $\gamma B M B$ vertex) as the vertex on the right hand side of the $\pi N$ loop is in $d$-wave, and the term vanishes in the integration over the loop momentum. The photon coupling to the $d$-wave loop is evaluated in Sec. 7.2.4.

As for additional photon couplings, as e.g. to the external proton, to vertices of the rescattering, or to intermediate meson-baryon loops, apart from the ones considered, these processes are present [183] in general but negligible as discussed in Sec. 7.2.3.

The $M B B^{*}$ vertices appearing in Fig. 7.2 are provided by the Lagrangian from Ref. [190], given in Eq. (4.56). Explicit Feynman rules for the $M B B^{*}$ vertices and $\gamma M B B^{*}$ vertices can be found in Appendix C.

Let us start with the evaluation of diagram (1) in Fig. 7.2. The amplitude is given by

$$
\begin{align*}
(-i t)_{\pi^{+} \Delta^{0}} & =-\frac{f_{\Delta \pi N}^{*}}{m} \sqrt{\frac{1}{3}} e S_{\mu}^{\dagger} \epsilon_{\nu} t_{\Delta} \\
& \times(-i) \int \frac{d^{4} p}{(2 \pi)^{4}} \frac{2 M}{(q-p)^{2}-M^{2}+i \epsilon} \frac{1}{p^{2}-m^{2}+i \epsilon} \\
& \times \frac{1}{(p-k)^{2}-m^{2}+i \epsilon}(p-k)^{\mu}(2 p-k)^{\nu} \tag{7.7}
\end{align*}
$$

where $m(M)$ is the pion $(\Delta)$ mass, $f_{\Delta \pi N}^{*} \equiv m_{\pi} /\left(\sqrt{2} f_{\pi}\right) \mathcal{C}_{\Delta \rightarrow N \pi}=2.13$ is the $\Delta \pi N$ coupling strength, $e^{2}=4 \pi / 137$ is the electric charge, $S_{\mu}^{\dagger}$ is the spin $1 / 2$ $\rightarrow 3 / 2$ transition operator which we approximate by $S_{\mu}^{\dagger}=\left(0, \mathbf{S}^{\dagger}\right)$, and $\epsilon_{\nu}$ is the polarization of the photon which in Coulomb gauge is given by $\epsilon_{\nu}=(0, \boldsymbol{\epsilon})$. The shaded circle in diagram (1) represents the $T$-matrix element $t_{\Delta}$ of the unitary coupled channel scheme in which the $\Delta^{*}(1700)$ appears dynamically generated. In Sec. 7.2 .5 it will be matched to the results from Sec. 7.2.1. Note the simplified structure of the $\Delta$-propagator in Eq. (7.7). This is the
same simplification as made in the model of dynamical generation [44] of the $\Delta^{*}(1700)$, see Eq. (7.2).

In order to ensure gauge invariance, the loop in Eq. (7.7) can be evaluated using a calculation technique from Refs. [231, 270]. In Sec. 7.2 .3 we will compare this scheme to a straightforward calculation of the loops of Fig. 7.2. The general structure of the loop function, or phototransition amplitude, is given by

$$
\begin{equation*}
T^{\mu \nu}=a g^{\mu \nu}+b q^{\mu} q^{\nu}+c q^{\mu} k^{\nu}+d k^{\mu} q^{\nu}+e k^{\mu} k^{\nu} . \tag{7.8}
\end{equation*}
$$

with the momenta $q$ and $k$ as defined in diagram (1). The terms with $c$ and $e$ do not contribute once contracted with $\epsilon^{\nu}$ according to Eq. (7.7) and using the transversality of the photon, $\epsilon k=0$. The terms with $b$ and $d$ do not contribute as $\epsilon q=0$ in the c.m. frame where $|\mathbf{q}|=0$ and using the fact that $\epsilon^{0}=0$. Thus, the only term that will not vanish in Eq. (7.8) is $\left(a g^{\mu \nu}\right)$.

It can be shown that the sets of diagrams (1) to (6) and (7) to (9) of Fig. 7.2 is gauge invariant. Contracting $T^{\mu \nu}$ with the photon momentum $k^{\nu}$ and using the Ward identity $k_{\nu} T^{\mu \nu} \equiv 0$, leads to the condition $a+d k q=0$. Note that diagram (2) of Fig. 7.2 contributes only to the term with $a$, whereas diagram (1) contributes both to $a$ and $d$. However, evaluating $d$ (and from this, $a$ through the condition $a+d k q=0$ ) has the advantage that the loop integral is finite whereas both diagram (1) and (2) are logarithmic divergent.

Using Feynman parameters and keeping only the terms proportional to $k^{\mu} q^{\nu}$, the second and third line of Eq. (7.7) become

$$
\begin{align*}
& d k^{\mu} q^{\nu}  \tag{7.9}\\
= & -\frac{4 M k^{\mu} q^{\nu}}{\left(4 \pi^{2}\right)} \int_{0}^{1} d x \int_{0}^{1-x} d z \frac{x(z-1)}{x\left[(x-1) q^{2}+z\left(q^{2}-M_{\mathrm{e}}^{2}\right)+M^{2}\right]+(1-x) m^{2}}
\end{align*}
$$

where we have written the product $2 q k=2 q^{0} k^{0}=q^{2}-M_{\mathrm{e}}^{2}$ in the c.m. system where $|\mathbf{q}|=0$ and $M_{\mathrm{e}}$ is the mass of the external baryon, in this case a proton. Note that $k^{0}=|\mathbf{k}|=1 /(2 \sqrt{s})\left(s-M_{\mathrm{e}}^{2}\right)$ where $\sqrt{s} \equiv q^{0}$ which we will use several times in the following.

From Eq. (7.9) we calculate the term $a$ through the condition $a+d k q=0$,

$$
\begin{align*}
G_{\mathrm{g} . \mathrm{i} .}^{I} & \equiv a=-d k q \\
& =\frac{2 M}{(4 \pi)^{2}}\left[\frac{M_{\mathrm{e}}^{2}-M^{2}+m^{2}}{2 M_{\mathrm{e}}^{2}}\right. \\
& +\frac{k \sqrt{s}\left[\left(M^{2}-m^{2}\right)^{2}-2 M_{\mathrm{e}}^{2} M^{2}\right]-m^{2} M_{\mathrm{e}}^{4}}{2 M_{\mathrm{e}}^{4} s} \log \frac{M^{2}}{m^{2}} \\
& +\frac{\left(M_{\mathrm{e}}^{2}-M^{2}+m^{2}\right)\left(2 M_{\mathrm{e}}^{2}-s\right)}{4 M_{\mathrm{e}}^{3} k \sqrt{s}} Q\left(M_{\mathrm{e}}\right) f_{1}\left(M_{\mathrm{e}}\right) \\
& +\frac{M^{2}-M_{\mathrm{e}}^{2}-m^{2}+2 k \sqrt{s}}{4 k s} Q(\sqrt{s}) f_{1}(\sqrt{s}) \\
& +\frac{m^{2}}{2 k \sqrt{s}}\left[\operatorname{Li}_{2}\left(\frac{-M^{2}+M_{\mathrm{e}}^{2}+m^{2}-2 M_{\mathrm{e}} Q\left(M_{\mathrm{e}}\right)}{2 m^{2}}\right)\right. \\
& +\operatorname{Li}_{2}\left(\frac{-M^{2}+M_{\mathrm{e}}^{2}+m^{2}+2 M_{\mathrm{e}} Q\left(M_{\mathrm{e}}\right)}{2 m^{2}}\right) \\
& -\mathrm{Li}_{2}\left(\frac{-M^{2}+s+m^{2}-2 \sqrt{s} Q(\sqrt{s})}{2 m^{2}}\right) \\
& \left.\left.-\mathrm{Li}_{2}\left(\frac{-M^{2}+s+m^{2}+2 \sqrt{s} Q(\sqrt{s})}{2 m^{2}}\right)\right]\right] \tag{7.10}
\end{align*}
$$

which is gauge invariant by construction. The c.m. energy for this and the following expressions of this section have to be taken at the physical sheet, i.e., $\sqrt{s} \rightarrow \sqrt{s}+i \epsilon$. In Eq. (7.10), $\mathrm{Li}_{2}$ is the dilogarithm and $Q$ and $f_{1}$ are given in Eq. (7.4). Having calculated $a$ from $d$, the loop function $G_{\mathrm{g} . \mathrm{i}}^{I}$ corresponds to the meson pole diagram (1) from Fig. 7.2 plus the KrollRuderman term from diagram (2).

Furthermore, the photon can also couple to the baryon as displayed in diagram (3). This diagram also contributes to the term $d k^{\mu} q^{\nu}$ in Eq. (7.8). The contribution to $d$, let it be $d^{I I}$, leads to an extra modification of the
term $a$,

$$
\begin{align*}
G_{\mathrm{g} . \mathrm{i.}}^{I I} & \equiv a^{I I}=-d^{I I} k q \\
& =\frac{2 M}{(4 \pi)^{2}}\left[\frac{M_{\mathrm{e}}^{2}+M^{2}-m^{2}}{2 M_{\mathrm{e}}^{2}}\right. \\
& -\frac{k \sqrt{s}\left(M^{2}-m^{2}\right)^{2}-s M^{2} M_{\mathrm{e}}^{2}}{2 M_{\mathrm{e}}^{4} s} \log \frac{M^{2}}{m^{2}} \\
& -\frac{\left(m^{2}-M^{2}\right)\left(2 M_{\mathrm{e}}^{2}-s\right)-s M_{\mathrm{e}}^{2}}{4 M_{\mathrm{e}}^{3} k \sqrt{s}} Q\left(M_{\mathrm{e}}\right) f_{1}\left(M_{\mathrm{e}}\right) \\
& -\frac{M^{2}-m^{2}+s}{4 k s} Q(\sqrt{s}) f_{1}(\sqrt{s}) \\
& +\frac{M^{2}}{2 k \sqrt{s}}\left[\operatorname{Li}_{2}\left(\frac{-m^{2}+M_{\mathrm{e}}^{2}+M^{2}-2 M_{\mathrm{e}} Q\left(M_{\mathrm{e}}\right)}{2 M^{2}}\right)\right. \\
& +\operatorname{Li}_{2}\left(\frac{-m^{2}+M_{\mathrm{e}}^{2}+M^{2}+2 M_{\mathrm{e}} Q\left(M_{\mathrm{e}}\right)}{2 M^{2}}\right) \\
& -\operatorname{Li}_{2}\left(\frac{-m^{2}+s+M^{2}-2 \sqrt{s} Q(\sqrt{s})}{2 M^{2}}\right) \\
& \left.\left.-\operatorname{Li}_{2}\left(\frac{-m^{2}+s+M^{2}+2 \sqrt{s} Q(\sqrt{s})}{2 M^{2}}\right)\right]\right] . \tag{7.11}
\end{align*}
$$

In order to determine the effective coupling of the photon to the $\Delta^{*}(1700)$ we construct isospin amplitudes from the diagrams given in Fig. 7.2. The isospin states of $\Delta \pi$ and $\Sigma^{*} K$ in $\left(I=3 / 2, I_{3}=1 / 2\right)$ are given by

$$
\begin{align*}
\left|\Delta \pi, I=3 / 2, I_{3}=1 / 2\right\rangle & =\sqrt{\frac{2}{5}}\left|\Delta^{++} \pi^{-}\right\rangle+\sqrt{\frac{1}{15}}\left|\Delta^{+} \pi^{0}\right\rangle+\sqrt{\frac{8}{15}}\left|\Delta^{0} \pi^{+}\right\rangle \\
\left|\Sigma^{*} K, I=3 / 2, I_{3}=1 / 2\right\rangle & =\sqrt{\frac{1}{3}}\left|\Sigma^{*+} K^{0}\right\rangle+\sqrt{\frac{2}{3}}\left|\Sigma^{* 0} K^{+}\right\rangle \tag{7.12}
\end{align*}
$$

with the phase convention $\left|\pi^{+}\right\rangle=-|1,1\rangle$.
With the loop functions from Eqs. (7.10) and (7.11) and standard Feynman rules from Appendix C we can calculate the isospin amplitudes for the sum of all $\Delta \pi$-loops and the $\Sigma^{*} K$-loops according to Eq. (7.12) with the
result

$$
\begin{align*}
(-i \mathbf{t} \cdot \boldsymbol{\epsilon})_{\gamma p \rightarrow \Delta \pi \rightarrow \Delta \pi}^{\left(I=3 / 2, I_{3}=1 / 2\right)} & =\frac{\sqrt{10}}{3} e \mathbf{S}^{\dagger} \cdot \boldsymbol{\epsilon} \frac{f_{\Delta \pi N}^{*}}{m_{\pi}} \\
& \times\left(G_{\text {g.i. }}^{I}+G_{\text {g.i. }}^{I I}\right)_{\mid m=m_{\pi}, M=M_{\Delta}, M_{\mathrm{e}}=M_{N}} T_{\Delta \pi \rightarrow \Delta \pi} \\
(-i \mathbf{t} \cdot \boldsymbol{\epsilon})_{\gamma p \rightarrow \Sigma^{*} K \rightarrow \Delta \pi}^{\left(I=3 / 2, I_{3}=1 / 2\right)} & =\frac{1}{3 \sqrt{2}} e \mathbf{S}^{\dagger} \cdot \boldsymbol{\epsilon} \frac{\mathcal{C}_{\Sigma^{*} \rightarrow N \bar{K}}}{f_{\pi}} \\
& \times\left(G_{\text {g.i. }}^{I}+G_{\text {g.i. }}^{I I}\right)_{\mid m=m_{K}, M=M_{\Sigma^{*}}, M_{\mathrm{e}}=M_{N}} T_{\Sigma^{*} K \rightarrow \Delta \pi} \tag{7.13}
\end{align*}
$$

where we have indicated which masses $m, M, M_{\mathrm{e}}$ have to be used in the loop functions. In Eq. (7.13), $f_{\Delta \pi N}^{*}=2.13$ and

$$
\begin{equation*}
\mathcal{C}_{\Sigma^{*} \rightarrow N \bar{K}}=1.508 \simeq \frac{6(D+F)}{5} \tag{7.14}
\end{equation*}
$$

The strength $\mathcal{C}_{\Sigma^{*} \rightarrow N \bar{K}}$ for the $\Sigma^{*}$ decay into the physically closed channel $N \bar{K}$ has been determined from from a $S U(6)$ quark model [192] in the same way as in the chapters 4 and 5: the $\mathrm{SU}(6)$ spin-flavor symmetry connects the $\pi N N$ coupling strength to the $\pi N \Delta$ strength, and then $S U(3)$ symmetry is used to connect the $\pi N \Delta$ transition with $\bar{K} N \Sigma^{*}$. The use of $S U(6)$ symmetry allows to express $\mathcal{C}_{\Sigma^{*} \rightarrow N \bar{K}}$ in terms of $D$ and $F$.

In the decuplet, the $S U(3)$ symmetry is broken. This can be taken into account phenomenologically by allowing for different $\mathcal{C}$ in the Lagrangian (6.5). For the open channels of the $\Sigma^{*}$ decay modes one obtains $\mathcal{C}_{\Sigma^{*} \rightarrow \Sigma \pi}=$ 1.64 and $\mathcal{C}_{\Sigma^{*} \rightarrow \Lambda \pi}=1.71$ from fitting to the partial decay widths into these channels (see Appendix C). The constant $\mathcal{C}_{\Sigma^{*} \rightarrow N \bar{K}}=1.508$ from Eq. (7.14) is close to these values (compare to $\mathcal{C}_{\Delta \rightarrow N \pi}=f_{\Delta \pi N}^{*}=2.13$ ).

### 7.2.3 Gauge Invariance

The construction of the gauge invariant amplitude in the last section can be compared to a straightforward calculation of the diagrams (1) to (9) in Fig. 7.2. In this section we show that both ways give identical results; at the end of this section we discuss further issues related to gauge invariance.

In Fig. 7.2 there are three types of loops: the Kroll-Ruderman structure, the meson pole term, and the baryon pole term. All loops are logarithmically divergent and we calculate in dimensional regularization for the sake
of conservation of gauge invariance. The Kroll-Rudermann loop function is identical to the common meson-baryon loop function from Eq. (7.2),

$$
\begin{equation*}
G_{\gamma B M B^{*}}=G_{M B^{*}} \tag{7.15}
\end{equation*}
$$

With the momenta assigned as in diagram (1) of Fig. 7.2, we define the meson pole loop function by

$$
\begin{align*}
& (-i) \int \frac{d^{4} p}{(2 \pi)^{4}} \frac{2 M}{(q-p)^{2}-M^{2}+i \epsilon} \frac{1}{p^{2}-m^{2}+i \epsilon} \\
\times & \frac{1}{(p-k)^{2}-m^{2}+i \epsilon}(p-k)^{\mu}(2 p-k)^{\nu} \\
\rightarrow & g^{\mu \nu} G_{\gamma M M} \\
= & \frac{g^{\mu \nu} M\left(2 / \epsilon-\gamma+\log (4 \pi)+\log \mu^{2}\right)}{(4 \pi)^{2}} \\
- & \frac{g^{\mu \nu} 2 M}{(4 \pi)^{2}} \int_{0}^{1} d x \int_{0}^{1-x} d z \log \left(x\left[(x-1) q^{2}+2 z q k+M^{2}\right]+(1-x) m^{2}\right) \\
= & g^{\mu \nu} \frac{2 M}{(4 \pi)^{2}} R(m, M) \tag{7.16}
\end{align*}
$$

with

$$
\begin{align*}
R(m, M) & =\frac{1}{2}\left(1-\alpha-\log \frac{m^{2}}{\mu^{2}}\right)-\frac{\left(M^{2}-m^{2}\right)^{2}+s M_{\mathrm{e}}^{2}}{4 s M_{\mathrm{e}}^{2}} \log \frac{M^{2}}{m^{2}} \\
& +\frac{M_{\mathrm{e}}^{2}-M^{2}+m^{2}}{4 M_{\mathrm{e}} k \sqrt{s}} Q\left(M_{\mathrm{e}}\right) f_{1}\left(M_{\mathrm{e}}\right)-\frac{s-M^{2}+m^{2}}{4 k s} Q(\sqrt{s}) f_{1}(\sqrt{s}) \\
& +\frac{m^{2}}{2 k \sqrt{s}}\left[\operatorname{Li}_{2}\left(\frac{-M^{2}+M_{\mathrm{e}}^{2}+m^{2}-2 M_{\mathrm{e}} Q\left(M_{\mathrm{e}}\right)}{2 m^{2}}\right)\right. \\
& +\operatorname{Li}_{2}\left(\frac{-M^{2}+M_{\mathrm{e}}^{2}+m^{2}+2 M_{\mathrm{e}} Q\left(M_{\mathrm{e}}\right)}{2 m^{2}}\right) \\
& -\operatorname{Li}_{2}\left(\frac{-M^{2}+s+m^{2}-2 \sqrt{s} Q(\sqrt{s})}{2 m^{2}}\right) \\
& \left.-\operatorname{Li}_{2}\left(\frac{-M^{2}+s+m^{2}+2 \sqrt{s} Q(\sqrt{s})}{2 m^{2}}\right)\right] \tag{7.17}
\end{align*}
$$

and $Q, f_{1}$ from Eq. (7.4). The arrow in Eq. (7.16) indicates that we only keep the terms proportional to $g^{\mu \nu}$ because all other possible structures from Eq. (7.8) do not contribute as commented following Eq. (7.8). Similarly,

Table 7.3: Coefficients $A_{i}$ for the diagrams (1) to (9) from Fig. 7.2 with the amplitude given in Eq. (7.19). The lower row shows how the infinities $2 / \epsilon$ $(\epsilon=4-d$, see Eq. (7.3) for $\epsilon \rightarrow 0)$ scale for each diagram. Once multiplied with the corresponding Clebsch-Gordan coefficients (CG) according to Eq. (7.12), the sum over the infinities cancels, $\Sigma_{i}(\mathrm{CG})_{i} r_{i}\left(\frac{2}{\epsilon}\right)=0$.

|  | $(1)$ | $(2)$ | $(3)$ | $(4)$ | $(5)$ | $(6)$ |
| :--- | :--- | :--- | :--- | :--- | :--- | :--- |
| $A_{i}$ | $\sqrt{\frac{1}{3}} G_{\gamma M M}$ | $\sqrt{\frac{1}{3}} G_{\gamma B M B^{*}}$ | $-\sqrt{\frac{2}{3}} G_{\gamma B^{*} B^{*}}$ | $G_{\gamma M M}$ | $G_{\gamma B M B^{*}}$ | $2 G_{\gamma B^{*} B^{*}}$ |
| $r_{i}\left(\frac{2}{\epsilon}\right)$ | $-\frac{1}{2}\left(\frac{2}{\epsilon}\right)$ | $1\left(\frac{2}{\epsilon}\right)$ | $\frac{1}{\sqrt{2}}\left(\frac{2}{\epsilon}\right)$ | $-\frac{\sqrt{3}}{2}\left(\frac{2}{\epsilon}\right)$ | $\sqrt{3}\left(\frac{2}{\epsilon}\right)$ | $-\sqrt{3}\left(\frac{2}{\epsilon}\right)$ |
|  | $(7)$ | $(8)$ | $(9)$ |  |  |  |
|  |  | $\sqrt{\frac{1}{3}} G_{\gamma M M}$ | $\sqrt{\frac{1}{3}} G_{\gamma B M B^{*}}$ | $\sqrt{\frac{2}{3}} G_{\gamma B^{*} B^{*}}$ |  |  |
| $A_{i}$ | $-\frac{1}{2}\left(\frac{2}{\epsilon}\right)$ | $1\left(\frac{2}{\epsilon}\right)$ | $-\frac{1}{\sqrt{2}}\left(\frac{2}{\epsilon}\right)$ |  |  |  |
| $r_{i}\left(\frac{2}{\epsilon}\right)$ |  |  |  |  |  |  |

and with the assignment of momenta as in diagram (3) of Fig. 7.2, the loop function where the photon couples directly to the baryon, is given by

$$
\begin{equation*}
g^{\mu \nu} G_{\gamma B^{*} B^{*}}=g^{\mu \nu} \frac{2 M}{(4 \pi)^{2}} R(M, m) \tag{7.18}
\end{equation*}
$$

Note that the convection part of the $\gamma B^{*} B^{*}$ coupling (the non-magnetic part) is of the same structure and sign as the $\gamma M M$ coupling [168].

The next step is to express the amplitudes of the diagrams in Fig. 7.2 in terms of these three loop functions. Using the Feynman rules from Appendix C, we obtain for the diagrams (1) to (9)

$$
\begin{equation*}
(-i \mathbf{t} \cdot \boldsymbol{\epsilon})_{(i)}=A_{i} e g_{B^{*} M B} \mathbf{S}^{\dagger} \cdot \boldsymbol{\epsilon} T_{B^{*} M \rightarrow \Delta \pi} \tag{7.19}
\end{equation*}
$$

where $g_{B^{*} M B}=f_{\Delta \pi N}^{*} / m_{\pi}$ or $g_{B^{*} M B}=\mathcal{C}_{\Sigma^{*} \rightarrow N \bar{K}} /\left(2 f_{\pi}\right)$ from Eq. (7.14) depending on whether $\left(B^{*} M\right)=(\Delta \pi)$ or $\left(\Sigma^{*} K\right)$. The coefficients $A_{i}$ for the diagrams (i) for $i=1$ to 9 are given in Tab. 7.3.

The sum over all diagrams results in

$$
\begin{align*}
& (-i \mathbf{t} \cdot \boldsymbol{\epsilon})_{\gamma p \rightarrow \Delta \pi \rightarrow \Delta \pi}^{\left(I=3 / 2, I_{3}=1 / 2\right)} \\
= & \frac{\sqrt{10}}{3} e \mathbf{S}^{\dagger} \cdot \boldsymbol{\epsilon} \frac{f_{\Delta \pi N}^{*}}{m_{\pi}} \\
\times & \left(G_{\gamma M M}^{*}+G_{\gamma B M B^{*}}+G_{\gamma B^{*} B^{*}}\right)_{\mid m=m_{\pi}, M=M_{\Delta}, M_{\mathrm{e}}=M_{N}} T_{\Delta \pi \rightarrow \Delta \pi}, \\
& (-i \mathbf{t} \cdot \boldsymbol{\epsilon})_{\gamma p \rightarrow \Sigma^{*} K \rightarrow \Delta \pi}^{\left(I=3 / 2, I_{3}=1 / 2\right)} \\
= & \frac{1}{3 \sqrt{2}} e \mathbf{S}^{\dagger} \cdot \boldsymbol{\epsilon} \frac{\mathcal{C}_{\Sigma^{*} \rightarrow N \bar{K}}}{f_{\pi}} \\
\times & \left(G_{\gamma M M}+G_{\gamma B M B^{*}}+G_{\gamma B^{*} B^{*}}\right)_{\mid m=m_{K}, M=M_{\Sigma^{*}}, M_{\mathrm{e}}=M_{N}} T_{\Sigma^{*} K \rightarrow \Delta \pi} . \tag{7.20}
\end{align*}
$$

The infinities of the nine diagrams, i.e., the terms with $2 / \epsilon$ from the Eqs. $(7.15,7.16,7.18)$ scale as shown in the lower row of Tab. 7.3. In order to construct the isospin states according to Eq. (7.12) each infinity is multiplied with the corresponding Clebsch-Gordan coefficient; as a result, the sums over all infinities cancel for the $\Delta \pi$ loops and also for the $\Sigma^{*} K$ loops. The Ward identity is working. In other words, gauge invariance renders the phototransition amplitude finite and leads to a parameter-free expression. Comparing Eqs. (7.20) and (7.13), obviously

$$
\begin{equation*}
G_{\gamma M M}+G_{\gamma B M B^{*}}+G_{\gamma B^{*} B^{*}}=G_{\mathrm{g} . \mathrm{i} .}^{I}+G_{\mathrm{g} \mathrm{~g} . \mathrm{i} .}^{I I} \tag{7.21}
\end{equation*}
$$

which can also be seen by comparing the explicit expressions given in Eqs. (7.10, 7.11, 7.15, 7.16, 7.18). In other words, the scheme from Eq. (7.8), which allows for the construction of a gauge invariant amplitude through the condition $a+d k q=0$, leads to the same result as a straightforward calculation of the amplitude, in which the infinities cancel systematically. However, this is only the case if all contributions to $d$ are taken into account, in the present case from the meson pole term plus the baryon pole term ( $G_{\mathrm{g} . \mathrm{i} .}^{I}$ and $G_{\mathrm{g} . \mathrm{i} .}^{I I}$.

In the rest of this section, further issues of gauge invariance are discussed such as a comparison to cut-off schemes, the role of magnetic couplings, and gauge invariance in the context of the rescattering scheme.

In chapters 4 and 6 the occurring photon loops have been regularized with a cut-off. As we have now a gauge invariant, parameter-free scheme at hand,


Figure 7.3: Update of the model of the radiative decay of the $\Lambda^{*}(1520)$ from chapter 6 (for the phototransitions via $s$-wave loops). The set of diagrams is gauge invariant. Note the additional contributions with direct coupling of the $\gamma$ to the decuplet baryons.
we would like to compare both methods numerically. As a first test, the scheme has been implemented in the calculation of the radiative decay width of the $\Lambda^{*}(1520)$ from chapter 6 . This means a gauge invariant evaluation of the $s$-wave loops from Fig. 6.3 formed by $\pi \Sigma^{*}$ and $K \Xi^{*}$, plus additional diagrams with $\gamma \Sigma^{*} \Sigma^{*}$ and $\gamma \Xi^{*} \Xi^{*}$ couplings in analogy to the diagrams in Fig. 7.2. The resulting set of gauge invariant diagrams for the radiative decay of the $\Lambda^{*}(1520)$ is displayed in Fig. 7.3 (apart from these, the $d$ wave loops contribute as before, see Fig. 6.3). In practice, the re-calculation only requires the replacement of the terms $\left(G_{i}+\frac{2}{3} \tilde{G}_{i}\right)$ from Eq. (6.30) by $\left(G_{\text {g.i. }}^{I}+G_{\text {g.i. }}^{I I}\right)$ from Eq. (7.21). The final result from chapter 6 for the radiative decay $\Lambda^{*}(1520) \rightarrow \gamma \Sigma^{0}$ changes from $\Gamma=60 \mathrm{keV}$ (Tab. 6.4) to $\Gamma=61 \mathrm{keV}$. Thus, the approximations made in chapter 6 and the violation of gauge invariance are well under control. Note that the additional diagram with a $\gamma \Sigma^{*} \Sigma^{*}$ coupling cancels for the second decay studied in chapter 6, $\Lambda^{*}(1520) \rightarrow \gamma \Lambda$, in the same way as the $\pi \Sigma^{*}$ meson pole and Kroll-Ruderman term. This is a consequence of gauge invariance but can be also seen directly by noting that the $\gamma M M$ interaction and the convection term of the $\gamma B^{*} B^{*}$
interaction have the same structure and sign [168]. Thus, the $\Lambda^{*}(1520) \rightarrow \gamma \Lambda$ radiative width stays as small as already found in Tab. 6.4.

In the cut-off scheme, the meson pole loop is defined as

$$
\begin{align*}
\tilde{G}_{i}^{\text {(cut) }} & =i \int \frac{d^{4} q}{(2 \pi)^{4}} \frac{\mathbf{q}^{2}-(\mathbf{q} \cdot \mathbf{k})^{2} /|\mathbf{k}|^{2}}{(q-k)^{2}-m_{i}^{2}+i \epsilon} \frac{1}{q^{2}-m_{i}^{2}+i \epsilon} \\
& \times \frac{M_{i}}{E_{i}(\mathbf{q})} \frac{1}{P^{0}-q^{0}-E_{i}(\mathbf{q})+i \epsilon}, \\
& =-\int_{0}^{\Lambda} \frac{d q q^{2}}{(2 \pi)^{2}} \int_{-1}^{1} d x \frac{q^{2}\left(1-x^{2}\right)}{2 \omega_{i} \omega_{i}^{\prime}} \frac{1}{k+\omega_{i}+\omega_{i}^{\prime}} \frac{1}{k-\omega_{i}-\omega_{i}^{\prime}+i \epsilon} \\
& \times \frac{M_{i}}{E_{i}(q)} \frac{1}{\sqrt{s}-\omega_{i}-E_{i}(q)+i \epsilon} \frac{1}{\sqrt{s}-k-\omega_{i}^{\prime}-E_{i}(q)+i \epsilon} \\
& \times\left[\left(\omega_{i}+\omega_{i}^{\prime}\right)^{2}+\left(\omega_{i}+\omega_{i}^{\prime}\right)\left(E_{i}(q)-\sqrt{s}\right)+k \omega_{i}^{\prime}\right] . \tag{7.22}
\end{align*}
$$

Here, $x$ is the cosinus of the angle between $\mathbf{q}$ and $\mathbf{k}$ with $\mathbf{k}$ the momentum of the real photon $(|\mathbf{k}| \equiv k) ; m_{i}$ is the meson mass, $P^{0} \equiv \sqrt{s}$, and $\omega_{i}, \omega_{i}^{\prime}$ are the energies of the mesons at momentum $q$ and $q-k$, respectively; $E_{i}$ the energy of the baryon. Note that Eq. (7.22) is slightly different from the corresponding expression in Eq. (6.12) as one kinematical approximation made in Eq. (6.12), the angle average over the term $1-x^{2}$, is not sufficient in the present case, because the momenta of the decay products are higher. One obtains $\tilde{G}$ from Eq. (6.12) by substituting $\left(1-x^{2}\right) \rightarrow 1$ in Eq. (7.22). Note that $\tilde{G}^{(\text {cut })}$ corresponds to $2 / 3$ of $\tilde{G}$.

The meson pole term $\tilde{G}^{(\text {cut })}$ is accompanied by the corresponding KrollRuderman term with cut-off, called $G^{(\text {cut })}$, where explicit expressions can be found, e.g., in Eq. (6.11). The cut-off can be determined by requiring the real part of the Kroll-Ruderman loop function to be equal in both dimensional regularization and cut-off scheme, at the energy of the resonance. In the present case this leads to a cut-off of $\Lambda=881 \mathrm{MeV}$.

In Fig. 7.4 the cut-off loops for $\Delta \pi\left(\tilde{G}^{(\mathrm{cut})}+G^{(\mathrm{cut})}\right)$, are shown as the dotted line. The gauge invariant function $G_{\text {g.i. }}^{I}$ from Eq. (7.10) is plotted with the solid line. The imaginary parts of both results are identical as expected, but, more interestingly, at the energies of the $\Delta^{*}$ of around 1700 MeV , also the real parts coincide closely. The dashed line shows the gauge invariant function $G_{\text {g.i. }}^{I I}$ from Eq. (7.11). This contribution comes from the baryon


Figure 7.4: Real and imaginary parts of $\Delta \pi$ loop functions. Solid line: gauge invariant $G_{\mathrm{g} . \mathrm{i} .}^{I}$ (meson pole term plus Kroll-Ruderman term). Dashed line: gauge invariant $G_{\text {g.i. }}^{I I}$ (baryon pole term). Dotted line: meson pole plus KrollRuderman term in a cut-off scheme $\left(\tilde{G}^{(\text {cut })}+G^{(\text {cut })}\right)$ with $\Lambda=881 \mathrm{MeV}$. The cut-off scheme and $G_{\text {g.i. }}^{I}$ have identical imaginary parts
pole diagrams. As there are two baryon propagators, the diagram should be smaller which is indeed the case as Fig. 7.4 shows. However, results can be affected noticeable and one should include this term in general.

From the comparison in Fig. 7.4 we see that the cut-off scheme as it has been used in [7] (Kroll-Ruderman plus meson pole term) indeed takes into account the dominant contributions. The baryon pole term, which has not been considered in [7], is small. If we would take this term into account in the cut-off scheme, the expression would be finite and in the limit $\Lambda \rightarrow \infty$, both cut-off scheme and gauge invariant scheme would give identical results.

Finally, there are photon couplings to the external baryon of the rescattering scheme, to vertices of the rescattering scheme itself, and to intermediate loops of the rescattering scheme which all have been ignored in the present study. This is, because with these couplings, the first loop has no photon attached any longer and is effectively suppressed because the integration over the momentum vanishes due to the presence of one $s$-wave and one $p$-wave vertex, or one $s$-wave and one $d$-wave vertex. A detailed discussion can be found in Sec. 4.3.2. There is also the magnetic part of the $\gamma B^{*} B^{*}$ vertex which is proportional to $\mathbf{S}_{\Delta} \times \mathbf{k}$ [168] with k the photon momentum. As evaluated in Eq. (4.16), this contribution vanishes for large baryon masses and can be neglected in practice.

### 7.2.4 Photon coupling to the $\pi N$ loop in $d$-wave

Diagram (10) of Fig. 7.2 shows the phototransition amplitude via the $\pi N$ state in $d$-wave. The evaluation follows the same steps as in Sec. 7.2.2. For $\pi N$ in $\left(I, I_{3}\right)=(3 / 2,1 / 2)$, which is the configuration chosen here, $|N \pi\rangle=$ $-\sqrt{1 / 3}\left|n \pi^{+}\right\rangle+\sqrt{2 / 3}\left|p \pi^{0}\right\rangle$. Using standard Feynman rules, we obtain

$$
\begin{equation*}
(-i \mathbf{t} \cdot \boldsymbol{\epsilon})_{\gamma p \rightarrow N \pi \rightarrow \Delta \pi}^{\left(I=3 / 2, I_{3}=1 / 2\right)}=-\frac{\sqrt{2}}{3} e \mathbf{S}^{\dagger} \cdot \boldsymbol{\epsilon} \frac{D+F}{f_{\pi}} \tilde{G}_{N \pi}^{\prime} T_{N \pi \rightarrow \Delta \pi} \tag{7.23}
\end{equation*}
$$

The $d$-wave meson pole loop function $\tilde{G}_{N \pi}^{\prime}$ has been calculated in Eq. (6.17) and [226] and we use the results from there,

$$
\begin{align*}
\tilde{G}_{N \pi}^{\prime} & =i \int \frac{d^{4} q}{(2 \pi)^{4}} \frac{\mathbf{q}^{2}}{(q-k)^{2}-m_{\pi}^{2}+i \epsilon} \frac{1}{q^{2}-m_{\pi}^{2}+i \epsilon} \frac{1}{P^{0}-q^{0}-E_{N}(\mathbf{q})+i \epsilon} \\
& \times \frac{M}{E_{N}(\mathbf{q})}\left(\frac{\mathbf{q}^{2}}{Q_{\pi N}^{2}\left(M_{\Delta^{*}}\right)}\right) \tag{7.24}
\end{align*}
$$

where $Q_{\pi N}\left(M_{\Delta^{*}}\right)$ is the on-shell three-momentum of pion and nucleon for the $\Delta^{*}(1700)$ decay at rest; $M$ is the nucleon mass and the other quantities are defined as in Eq. (7.22).

The development of a gauge invariant scheme as for the $s$-wave loops in Sec. 7.2.2, 7.2.3 is beyond the scope of this thesis. Instead, we follow the lines of chapter 6. In the numerical comparison at the end of Sec. 7.2.3 we have seen that the violation of gauge invariance from the cut-off scheme leads almost to the same results as the gauge invariant calculation, at least sufficiently beyond threshold; thus, the loop function in Eq. (7.24) is regularized with a cut-off.

The cut-off is determined from a comparison between the $d$-wave mesonbaryon loop in dimensional regularization with on-shell factorization of the vertices (i.e., as it appears in the rescattering scheme of Sec. 7.2.1), and the $d$-wave loop with cut-off according to

$$
\begin{equation*}
Q_{\pi N}^{4}(\sqrt{s}) G_{\pi N}(\sqrt{s})=\int_{0}^{\Lambda} \frac{d p p^{2}}{2 \pi^{2}} \frac{p^{4}}{2 \omega} \frac{M}{E(p)} \frac{1}{\sqrt{s}-\omega(p)-E(p)+i \epsilon} \tag{7.25}
\end{equation*}
$$

at the energy $\sqrt{s}$ of the real part of the resonance position given in Tab. 7.2. In Eq. (7.25), $G_{\pi N}$ is the loop in dimensional regularization from Eq. (7.2)


Figure 7.5: Effective resonance representation of the radiative decay.
with the subtraction constants $\alpha_{(\pi N)_{d}}$ from Tab. 7.1; $Q_{\pi N}^{4}$ is the on-shell c.m. momentum from the two $d$-wave vertices in on-shell factorization, $\omega(E)$ is the pion (nucleon) energy, and $M$ the nucleon mass. The cut-off determined in this way is then also used for the regularization of the meson pole term of Eq. (7.24).

### 7.2.5 Effective photon coupling

In the last sections the amplitudes for the process $\gamma p \xrightarrow{\Delta^{*(1700)}} \Delta \pi$ have been determined and are written in terms of the $T^{(i 1)}$, the unitary solution of the BSE (7.1) for meson-baryon scattering with the transitions from channel $i\left(\Delta \pi, \Sigma^{*} K,(\pi N)_{d}\right)$ to the $\Delta \pi$ final state (channel no. 1). In order to determine the partial photon decay width of the $\Delta^{*}(1700)$, the $T^{(i 1)}$ are expanded around the simple pole in the complex plane with the leading term of the Laurent series given by

$$
\begin{equation*}
T^{(i 1)} \simeq \frac{g_{i} g_{\Delta \pi}}{\sqrt{s}-M_{\Delta^{*}(1700)}} \tag{7.26}
\end{equation*}
$$

where $g_{i}$ is the $\Delta^{*}(1700)$ coupling to channel $i$, the product $g_{i} g_{\Delta \pi}$ is provided by the residue and $M_{\Delta^{*}(1700)}$ is the complex pole position given in Tab. 7.2. With this replacement for the $T^{(i 1)}$, the amplitudes from Eqs. (7.13, 7.23) can be matched to the resonant process shown in Fig. 7.5, which is given by

$$
\begin{equation*}
(-i \mathbf{t} \cdot \boldsymbol{\epsilon})=\left(-i g_{\Delta^{*} \Delta \pi}\right) \frac{i}{\sqrt{s}-M_{\Delta^{*}}} g_{\Delta^{*} \gamma \Delta} \mathbf{S}^{\dagger} \cdot \boldsymbol{\epsilon} . \tag{7.27}
\end{equation*}
$$

This identification allows to write the effective $\Delta^{*}(1700) \gamma p$ coupling, $g_{\Delta^{*} \gamma p}$, in terms of the one-loop photoproduction processes discussed in the last
sections:

$$
\begin{align*}
g_{\Delta^{*} \gamma p}^{(\Delta \pi)} & =e g_{\Delta \pi} \frac{\sqrt{10}}{3} \frac{f_{\Delta \pi N}^{*}}{m_{\pi}}\left(G_{\Delta \pi}+\tilde{G}_{\Delta \pi}^{(\mathrm{f})}\right) \\
g_{\Delta^{*} \gamma p}^{\left(\Sigma^{*} K\right)} & =e g_{\Sigma^{*} K} \frac{1}{3 \sqrt{2}} \frac{\mathcal{C}_{\Sigma^{*} \rightarrow N \bar{K}}}{f_{\pi}}\left(G_{\Sigma^{*} K}+\tilde{G}_{\Sigma^{*} K}^{(\mathrm{f})}\right) \tag{7.28}
\end{align*}
$$

with the couplings of the $\Delta^{*}$ to the channel $i, g_{i}$, given in Tab. 7.2. In the same way, the $d$-wave amplitude in Eq. (7.23) is matched, resulting in

$$
\begin{equation*}
g_{\Delta^{*} \gamma p}^{(N \pi)}=-g_{(\pi N)_{d}} e \frac{\sqrt{2}}{3} \frac{D+F}{f_{\pi}} \tilde{G}_{N \pi}^{\prime} \tag{7.29}
\end{equation*}
$$

The effective photon coupling is given by the coherent sum of all processes from Fig. 7.2,

$$
\begin{equation*}
g_{\Delta^{*} \gamma p}=g_{\Delta^{*} \gamma p}^{(\Delta \pi)}+g_{\Delta^{*} \gamma p}^{\left(\Sigma^{*} K\right)}+g_{\Delta * \gamma p}^{(N \pi)} . \tag{7.30}
\end{equation*}
$$

### 7.3 Numerical results

The results for the radiative decay width, given by

$$
\begin{equation*}
\Gamma_{\Delta^{*} \rightarrow \gamma N}=\frac{k}{3 \pi} \frac{M_{p}}{M_{\Delta^{*}}}\left|g_{\Delta^{*} \gamma p}\right|^{2} \tag{7.31}
\end{equation*}
$$

with $g_{\Delta^{*} \gamma p}$ from Eq. (7.30) and $k$ the c.m. momentum of the photon, are summarized in Tab. 7.4. For the experimental value of $\Gamma=570 \pm 254 \mathrm{keV}$, we have summed in quadrature the errors from the $\Delta^{*}(1700)$ width and the branching ratio into $\gamma N$ given in the PDB [228].

From the four different results in Tab. 7.4 we prefer the decay width from fit 1. In the other fits some of the free parameters have been fixed. As argued in Sec. 7.2.1, the remaining free parameters of these extra fits have to absorb effects from this reduction of degrees of freedom and results may become distorted. For fit 3 we have found a very similar solution, called fit 3 ', which has a reversed sign for the coupling of the $(\pi N)_{d}$-channel to the $\Delta^{*}(1700)$. This is a consequence of the weakness of this coupling as explained in Sec. 7.2.1. Under the conditions of fit 1 , we could not find such an alternative minimum in $\chi^{2}$; nevertheless, the two different decay widths of 459 keV and

Table 7.4: Radiative decay width $\Gamma$ of the $\Delta^{*}$ (1700), to be compared with $\Gamma=\mathbf{5 7 0} \pm \mathbf{2 5 4} \mathrm{keV}$ from the PDB [228]. Also, the effective couplings $g_{\Delta^{*} \gamma p}^{(\cdots)}$ of the photon to the $\Delta^{*}(1700)$ via $\Delta \pi, \Sigma^{*} K$, and $(\pi N)_{d}$ loops are displayed, in order to show the interference pattern of the channels (multiplied with $10^{3}$ ).

|  |  |  |  |  |
| :--- | ---: | ---: | ---: | ---: |
|  | $g_{\Delta^{*} \gamma p}^{(\Delta \pi)}$ | $g_{\Delta^{*} \gamma p}^{\left(\Sigma^{*} K\right)}$ | $g_{\Delta^{*} \gamma p}^{(N \pi)}$ | $\Gamma[\mathrm{keV}]$ |
| Fit 1 | $-82-i 71$ | $-30-i 16$ | $-9+i 36$ | $\mathbf{6 0 2}$ |
| Fit 2 | $-85-i 32$ | $-28-i 18$ | $6+i 34$ | $\mathbf{4 0 3}$ |
| Fit 3 | $-89-i 37$ | $-25-i 16$ | $-1+i 40$ | $\mathbf{4 5 9}$ |
| Fit 3' | $-89-i 37$ | $-25-i 16$ | $3-i 41$ | $\mathbf{7 3 0}$ |

730 keV from fit 3 and 3' give an idea of the intrinsic theoretical uncertainties of the present study. Thus, we assign a final value of

$$
\Gamma=602 \pm 140 \mathrm{keV}
$$

to the radiative decay width of the $\Delta^{*}(1700)$.
The contribution from the $\Sigma^{*} K$ channel is smaller than from the $\Delta \pi$ channel as Tab. 7.4 shows. At first sight this seems surprising as $g_{\Sigma^{*} K}$ from Tab. 7.2 is four times larger than $g_{\Delta \pi}$. However, the threshold for this channel is at $\sqrt{s}=1880 \mathrm{MeV}$; at $\sqrt{s}=1.7 \mathrm{GeV}$ it is closed and the $\Sigma^{*} K$ loops from Fig. 7.2 are small. Obviously, channels with a threshold much lower than the $\Delta^{*}(1700)$ mass dominate the decay $(\Delta \pi)$; even $(N \pi)$ contributes despite to the weak coupling to the $\Delta^{*}(1700)$. A similar pattern has been observed in chapter 6 for the radiative decay of the $\Lambda^{*}(1520)$. Thus, the contribution from the $\rho N$ channel, which has not been considered in this study, will only moderately change the results.

In the present scheme, the large contributions from the $\Delta^{++} \pi^{-}$and $\Delta^{0} \pi^{+}$channel add up in the isospin combination from Eq. (7.13), and as a consequence the $\Delta \pi$ channel gives a large contribution to the radiative
decay, in good agreement with experiment. This is in analogy to the decay $\Lambda^{*}(1520) \rightarrow \gamma \Sigma^{0}$ studied in chapter 6 where the dominant $\pi \Sigma$ and $\pi \Sigma^{*}$ channels add up and result in good agreement with data.

In chapter 6, it has also been observed that these channels cancel exactly for the $\Lambda^{*}(1520) \rightarrow \gamma \Lambda$ decay and the discrepancy between experiment and coupled channel model has been attributed to a genuine three quark component in the wave function of the $\Lambda^{*}(1520)$, on top of the meson-baryon component from the coupled channel model. For the $\Delta^{*}(1700)$ there is no such additional channel for which we can test the model. However, there is a long history of calculating radiative decays of excited baryons in quark models, e.g. [232-235]. For the radiative decay of the $\Delta^{*}(1700)$ all these works obtain good agreement with experiment.

Thus, the radiative decay appears well reproduced in both the quark model picture and the present scheme where the degrees of freedom are the mesons and baryons. A calculation of electroproduction within the present framework could help bring further insight into the question of which theoretical framework is more appropriate; e.g., the quark model from [235] has some difficulties in the $D_{33}$ channel, but improving the experimental data situation is certainly desirable.

### 7.4 Conclusions

In the study of the $\Delta^{*}(1700)$ radiative decay, a model has been formulated in which the photon couples to the final loops of the rescattering series that dynamically generates the $\Delta^{*}(1700)$. As a novelty, the $\pi N$ channel in $d-$ wave has been included in the coupled channel scheme. Furthermore, the phototransition via the dominant $s$-wave loops has been treated in a fully gauge invariant way.

Previous studies of numerous pion- and photon-induced reactions have accumulated evidence of a strong coupling of the $\Delta^{*}(1700)$ to $\Delta \eta$ and $\Sigma^{*} K$, which is a prediction of the unitary coupled channel model. The present study provides an extra independent test for the nature of the $\Delta^{*}(1700)$, giving additional support to the hypothesis that this resonance is dynamically generated.

## Chapter 8

## Charge fluctuations and electric mass in a hot meson gas


#### Abstract

In the last chapters we have concentrated on low energy $\pi N$ scattering in the vacuum and medium, on low and intermediate energy phenomenology and the relevance of dynamically generated resonanes in photon- and pioninduced reactions. In the last two chapters of the thesis we will focus on quite a different physical and methodical framework, the study of the Quark-Gluon Plasma. However, in the Introduction in chapter 1.1 we have already pointed out similarities and connections between the two fields. In particular, in this chapter (8) we will find the importance of unitarity on one side and higher order corrections in the interaction and density on the other side. In order to account for both of these relevant points, an addendum is added (chapter 9) where the chiral unitary approch is used in a modification to finite temperature field theory at finite chemical potential and with some additional improvements over other studies concerning the consistent inclusion of particle statistics.


In this chapter, net-Charge fluctuations in a hadron gas are studied using an effective hadronic interaction. The emphasis of this work is to investigate the corrections of hadronic interactions to the charge fluctuations of a noninteracting resonance gas. Several methods, such as loop, density and virial expansions are employed. The calculations are also extended to $S U(3)$ and some resummation schemes are considered. Although the various corrections
are sizable individually, they cancel to a large extent. As a consequence we find that charge fluctuations are rather well described by the free resonance gas.

### 8.1 Introduction

The study of event-by-event fluctuations or more generally fluctuations and correlations in heavy ion collisions has recently received considerable interest. Fluctuations of multiplicities and their ratios [236], transverse momentum [237-240] and net charge fluctuations [241-244] have been measured. Also first direct measurements of two particle correlations have been carried out [245, 246].

Conceptually, fluctuations may reveal evidence of possible phase transitions and, more generally, provide information about the response functions of the system [247]. For example, it is expected that near the QCD critical point long range correlation will reveal themselves in enhanced fluctuations of the transverse momentum $\left(p_{t}\right)$ per particle [248]. Also, it has been shown that the fluctuations of the net charge are sensitive to the fractional charges of the quarks in the Quark Gluon Plasma (QGP) [249, 250].

Most fluctuation measures investigated so far are integrated ones, in the sense that they are related to integrals of many particle distributions [251]. Examples are: Multiplicity, charge and momentum fluctuations which are all related to two-particle distributions. These integrated measures have the advantage that they can be related to well defined quantities in a thermal system. For example, fluctuations of the net charge are directly related to the charge susceptibility. However, in an actual experiment additional, dynamical, i.e non-thermal correlations may be present which make a direct comparison with theory rather difficult. This is particularly the case for fluctuations of the transverse momentum, where the appearance of jet like structures provides nontrivial correlations $[245,252,253]$. These need to be understood and eliminated from the analysis before fluctuation measurements can reveal insight into the matter itself.

In this article we will not be concerned with the comparison with experimental data, and the difficulties associated with it. We rather want to inves-
tigate to which extent interactions affect fluctuations. Specifically, we will study the fluctuations of the net electric charge of the system, the so-called charge fluctuations (CF). CF have been proposed as a signature for the formation of the Quark Gluon Plasma (QGP) in heavy ion collisions [249, 250]. Refs. [249, 250] note that CF per degree of freedom should be smaller in a QGP as compared to a hadron gas because the fractional charges of the quarks enter in square in the CF. Using noninteracting hadrons and quarks, gluons, respectively, it was found that the CF per entropy are about a factor of 3 larger in a hadron gas than in a QGP. The net CF per entropy has in the meantime been measured [241-244]. At RHIC energies the data are consistent with the expectations of a hadron gas, but certainly not with that of a QGP. This might be due to limited acceptance as discussed in [254-256].

The original estimates of the net charge fluctuations per entropy in the hadron gas $[249,250]$ have been based on a system of noninteracting particles and resonances. While this model has been proven very successful in describing the measured single particle yields [257, 258], it is not obvious to which extent residual interactions among the hadronic states affect fluctuation observables. For example, in the QGP phase, lattice QCD calculations for the charge susceptibility and entropy-density differ from the result for a simple weakly interacting QGP. Their ratio, however, agrees rather well with that of a noninteracting classical gas of quarks and gluons [247, 249, 259-261]. As far as the hadronic phase is concerned, lattice results for charge fluctuations are only available for systems with rather large pion masses [259, 261]. In this case, an appropriately rescaled hadron gas model seems to describe the lattice results reasonably well [262]. Lattice calculations with realistic pion masses, however, are not yet available. Thus, one has to rely on hadronic model calculations in order to assess the validity of the noninteracting hadron gas model for the description of CF. In Ref. [263] the electric screening mass $m_{\text {el }}^{2}$ which is closely related to CF has been calculated up to next-to-leading (NLO) order in $\pi \pi$ interaction. However, the fact that thermal loops pick up energies in the resonance region of the $\pi \pi$ amplitude where chiral perturbation theory is no longer valid leads to large theoretical uncertainties.

It is the purpose of this paper to provide a rough estimate of the effect of interactions in the hadronic phase, in particular the effect of the coupling of
the $\rho$-meson to the pions. Since $\rho$-mesons are strong resonances which carry the same quantum numbers as the CF this should provide a good estimate for the size of corrections to be expected from a complete calculation; the latter will most likely come from lattice QCD, once numerically feasible.

As a first step we will consider the case of a heavy $\rho$-meson or, correspondingly, a low temperature approximation. In this case the $\rho$-meson is not dynamical and will not be part of the statistical ensemble. It will only induce an interaction among the pions which closely corresponds to the interaction from the lowest order (LO) chiral Lagrangian.

Although the temperatures in the hadronic phase are well below the $\rho$ mass, it is interesting to estimate the residual $\pi \pi$ correlations introduced when this resonance is treated dynamically. Special attention is paid to charge conservation and unitarity. In addition, we will investigate the importance of quantum statistics. Finally an extension to strange degrees of freedom is provided.

This paper is organized as follows. After a brief review of the charge fluctuations we introduce our model Lagrangian and discuss the heavy rho limit. Next we discuss the treatment of dynamical $\rho$-mesons up to two-loop order and compare with the results obtained in the heavy rho limit. Then, the effect of quantum statistics and unitarity is discussed. Before we show our final results including strange degrees of freedom, we will briefly comment on possible resummation schemes.

### 8.2 Charge fluctuations and Susceptibilities

Before turning to the model interaction employed in this work, let us first introduce some notation and recall the necessary formalism to calculate the CF (for details, see, e.g., Ref. [247]).

In this work we will consider a system in thermal equilibrium. In this case the charge fluctuations $\left\langle\delta Q^{2}\right\rangle$ are given by the second derivative of the appropriate free energy $F$ with respect to the charge chemical potential $\mu$ :

$$
\begin{equation*}
\left\langle\delta Q^{2}\right\rangle=-T \frac{\partial^{2} F}{\partial \mu^{2}}=-V T \chi_{Q} \tag{8.1}
\end{equation*}
$$

Here, $T(V)$ is the temperature (volume) of the system and $\chi_{Q}$ is the charge
susceptibility, which is often the preferred quantity to consider, particularly in the context of lattice QCD calculations. Equivalently, the CF or susceptibility are related to the electromagnetic current-current correlation function [264, 265]

$$
\begin{equation*}
\Pi_{\mu \nu}(\omega, \mathbf{k})=\mathrm{i} \int \mathrm{dtd}^{3} \mathrm{x}^{-\mathrm{i}(\omega \mathrm{t}-\mathbf{k x})}\left\langle\mathrm{J}_{\mu}(\mathrm{x}, \mathrm{t}) \mathrm{J}_{\nu}(0)\right\rangle \tag{8.2}
\end{equation*}
$$

via

$$
\begin{equation*}
\left\langle\delta Q^{2}\right\rangle=V T \Pi_{00}(\omega=0, \mathbf{k} \rightarrow 0)=V T m_{\mathrm{el}}^{2} \tag{8.3}
\end{equation*}
$$

which is illustrated for scalar QED in Appendix D.1. Relation (8.3) also establishes the connection between the CF and the electric screening mass $m_{\mathrm{el}}$.

As noted previously, the observable of interest is the ratio of CF over entropy

$$
\begin{equation*}
D_{S} \equiv \frac{\left\langle\delta Q^{2}\right\rangle}{e^{2} S} \tag{8.4}
\end{equation*}
$$

Given a model Lagrangian, both CF and entropy can be evaluated using standard methods of thermal field theory (see e.g. [265]). CF are often evaluated via the current-current correlator using thermal Feynman rules; evaluating the free energy and using relation (8.1) will lead to the same results as will be demonstrated in Sec. 8.5.

Let us close this section by noting that in an actual experiment a direct measurement of the entropy is rather difficult. However, the number of charged particles $\left\langle N_{\mathrm{ch}}\right\rangle$ in the final state is a reasonable measure of the final state entropy. Therefore, the ratio

$$
\begin{equation*}
D_{c}=4 \frac{\left\langle\delta Q^{2}\right\rangle}{e^{2}\left\langle N_{\mathrm{ch}}\right\rangle} . \tag{8.5}
\end{equation*}
$$

has been proposed as a possible experimental observable for accessing the CF per degree of freedom. For details and corrections to be considered see Ref. [247] and references therein. In this article we will concentrate on the "theoretical" observable $D_{S}$ defined in Eq. (8.4).

### 8.3 Model Lagrangian and $\pi \pi$ interaction in the heavy $\rho$ limit

As already discussed in Sec. 8.1, in this work we want to provide an estimate of the corrections to the CF introduced by interactions among the hadrons in the hadronic phase. Since it is impossible to account for all hadrons and their interactions, we will concentrate on a system of pions and $\rho$-mesons only, with some extensions to $S U(3)$ in later sections. A suitable effective Lagrangian for this investigation is the "hidden gauge" approach of Refs. [266, 267]. In this model the $\rho$-meson is introduced as a massive gauge field. The $\pi \rho$ interaction results from the covariant derivative $D_{\mu} \Phi=\partial_{\mu} \Phi-\frac{i g}{2} \quad\left[\rho_{\mu}, \Phi\right]$ acting on the pion field $U(x)=\exp \left[i \Phi(x) / f_{\pi}\right]$ in the LO chiral Lagrangian

$$
\begin{equation*}
\mathcal{L}_{\pi \pi}^{(2)}=\frac{f_{\pi}^{2}}{4} \operatorname{Tr}\left[\partial_{\mu} U^{\dagger} \partial^{\mu} U+\mathcal{M}\left(U+U^{\dagger}\right)\right] \tag{8.6}
\end{equation*}
$$

by the replacement $\partial_{\mu} \rightarrow D_{\mu}$. Here,

$$
\Phi=\left(\begin{array}{cc}
\pi^{0} & \sqrt{2} \pi^{+}  \tag{8.7}\\
\sqrt{2} \pi^{-} & -\pi^{0}
\end{array}\right), \quad \rho_{\mu}=\left(\begin{array}{cc}
\rho_{\mu}^{0} & \sqrt{2} \rho_{\mu}^{+} \\
\sqrt{2} \rho_{\mu}^{-} & -\rho_{\mu}^{0}
\end{array}\right)
$$

and $f_{\pi}=93 \mathrm{MeV}$ is the pion decay constant. An extension of the heavy gauge model to $S U(3)$ has been applied for vacuum and in-medium processes (see, e.g., Refs. [268, 269]) and is straightforward [270]. This extension is considered in Sec. 8.7.1.

The resulting $\pi \rho$ interaction terms are

$$
\begin{equation*}
\mathcal{L}_{\rho \pi \pi}=\frac{i g}{4} \operatorname{Tr}\left(\rho_{\mu}\left[\partial^{\mu} \Phi, \Phi\right]\right) \tag{8.8}
\end{equation*}
$$

and

$$
\begin{equation*}
\mathcal{L}_{\rho \rho \pi \pi}=-\frac{g^{2}}{16} \operatorname{Tr}\left(\left[\rho_{\mu}, \Phi\right]^{2}\right) . \tag{8.9}
\end{equation*}
$$

Chiral corrections to the interaction in Eq. (8.8) are of $\mathcal{O}\left(p^{5}\right)$ or higher as pointed out in Ref. [271]. The interaction of Eq. (8.9) does not depend on the pion momentum, thus violating the low energy theorem of chiral symmetry [271]. Nevertheless, this term is required by the gauge invariance
of the $\rho$-meson [157] and in fact cancels contributions in the pole term and crossed pole term of $\pi \rho$ scattering via Eq. (8.8).

To leading order in the pion field we, thus, have the following model Lagrangian:

$$
\begin{equation*}
\mathcal{L}=\mathcal{L}_{\Phi}+\mathcal{L}_{\rho}+\mathcal{L}_{\rho \pi \pi}+\mathcal{L}_{\rho \rho \pi \pi} \tag{8.10}
\end{equation*}
$$

with the free field terms

$$
\begin{align*}
\mathcal{L}_{\Phi} & =\frac{1}{4} \operatorname{Tr}\left(\partial_{\mu} \Phi \partial^{\mu} \Phi\right)-\frac{1}{4} \operatorname{Tr}\left(m_{\pi}^{2} \Phi^{2}\right) \\
\mathcal{L}_{\rho} & =-\frac{1}{8} \operatorname{Tr}\left(G_{\mu \nu} G^{\mu \nu}\right)+\frac{1}{4} \operatorname{Tr}\left(m_{\rho}^{2} \rho_{\mu} \rho^{\mu}\right) \tag{8.11}
\end{align*}
$$

and the interaction terms $\mathcal{L}_{\rho \pi \pi}$ and $\mathcal{L}_{\rho \rho \pi \pi}$ as given in Eq. (8.8) and (8.9), respectively. For the kinetic tensor of the $\rho, G_{\mu \nu}=\partial_{\mu} \rho_{\nu}-\partial_{\nu} \rho_{\mu}$, we restrict ourselves to the Abelian part; a non-Abelian $\rho$ would lead to additional $3 \rho$ and $4 \rho$ couplings. In the thermal loop expansion this would result in closed $\rho$ loops which are kinematically suppressed. The coupling constant $g$ is fixed from the $\rho \rightarrow \pi \pi$ decay to be $g=g_{\rho \pi \pi}=6$ and and we use $m_{\pi}=138 \mathrm{MeV}$ and $m_{\rho}=770 \mathrm{MeV}$ throughout this paper.

As a first approximation, we start with the low temperature limit of the $\pi \pi$ interaction in which the $\rho$-meson mediates the interaction of the pions but does not enter the heatbath as an explicit degree of freedom. To this end we construct an effective interaction based on $s$-, $t$-, and $u$-channel $\rho$-meson exchange as given by second order perturbation theory of the interaction $\mathcal{L}_{\rho \pi \pi}$. Furthermore, we assume that the momentum transfer $k^{2}$ of two pions interacting via a $\rho$ is much smaller than the mass of the $\rho$-meson, $m_{\rho}^{2} \gg k^{2}$, i.e. we replace the propagator of the exchanged $\rho$-meson by $-1 / m_{\rho}^{2}$. Thus, we arrive at the following effective interaction

$$
\begin{equation*}
\mathcal{L}_{\pi \pi}^{\mathrm{eeff}}=\frac{g^{2}}{2 m_{\rho}^{2}}\left(\left(\pi^{-} \overleftrightarrow{\partial_{\mu}} \pi^{+}\right)^{2}-2\left(\pi^{0} \overleftrightarrow{\partial_{\mu}} \pi^{+}\right)\left(\pi^{0} \overleftrightarrow{\partial^{\mu}} \pi^{-}\right)\right) \tag{8.12}
\end{equation*}
$$

Note that in this limit, subsequently referred to as the "heavy $\rho$ limit" the $\rho \rho \pi \pi$ term from Eq. (8.9) does not contribute at order $g^{2}$.

The effective Lagrangian of Eq. (8.12) shows the identical isospin and momentum structure as the kinetic term of Eq. (8.6) at $1 / f_{\pi}^{2}$. However,
comparing the overall coefficient one arrives at

$$
\begin{equation*}
m_{\rho}^{2}=3 f_{\pi}^{2} g^{2} \tag{8.13}
\end{equation*}
$$

which differs by a factor of $3 / 2$ from the well known KSFR relation [272] $m_{\rho}^{2}=2 f_{\pi}^{2} g^{2}$. We should point out the same factor has been observed in the context of the anomalous $\gamma \pi \pi \pi$ interaction [273]. As discussed in more detail in Appendix D.2.1 the KSFR relation is recovered if one restricts the model to the $s$-channel diagrams for the isovector $p$-wave ( $T_{11}$ ) amplitude. Once also $t$ - and $u$-channels are taken into account the factor $3 / 2$ appears. For this study, we prefer the interaction (8.12) over $\mathcal{L}_{\pi \pi}^{(2)}$ from Eq. (8.6) as it delivers a better data description at low energies in the $\rho$-channel (see Appendix D.2.1). The simplification from the "heavy $\rho$ " limit of the $\pi \pi$ interaction will later be relaxed in favor of dynamical $\rho$-exchange. However, the interaction in the heavy $\rho$ limit will still serve as a benchmark for the more complex calculations.

Since we are interested in the electromagnetic polarization tensor, the interaction of Eq. (8.12), together with the kinetic term of the pion, is gauged with the photon field by minimal substitution, leading to

$$
\begin{align*}
\mathcal{L}_{\pi \gamma} & =-\frac{1}{4}\left(F^{\mu \nu}\right)^{2}-m_{\pi}^{2}\left(\pi^{+} \pi^{-}+\frac{1}{2}\left(\pi^{0}\right)^{2}\right)+\left(D_{\mu}^{*} \pi^{-}\right)\left(D^{\mu} \pi^{+}\right)+\frac{1}{2}\left(\partial_{\mu} \pi^{0}\right)^{2} \\
& +\frac{g^{2}}{2 m_{\rho}^{2}}\left(\pi^{-} D_{\mu} \pi^{+}-\pi^{+} D_{\mu}^{*} \pi^{-}\right)^{2} \\
& -\frac{g^{2}}{m_{\rho}^{2}}\left(\pi^{0} D^{\mu} \pi^{+}-\pi^{+} \partial^{\mu} \pi^{0}\right)\left(\pi^{0} D_{\mu}^{*} \pi^{-}-\pi^{-} \partial_{\mu} \pi^{0}\right) \tag{8.14}
\end{align*}
$$

with the covariant derivative of the photon field $D_{\mu}=\partial_{\mu}+i e A_{\mu}, e>0$, and the photon field tensor $F^{\mu \nu}$, leading to the $\gamma \pi \pi$ and $\gamma \gamma \pi \pi$ interactions of scalar QED, plus $\gamma \pi \pi \pi \pi$ and $\gamma \gamma \pi \pi \pi \pi$ vertices. Vector meson dominance leads to $\gamma \rho^{0}$ mixing as pointed out, e.g., in Ref. [268], additionally to the vertices of Eq. (8.14). However, since the correlator of Eq. (8.3) is evaluated at the photon point, the form factor is unity and the process $\gamma \rightarrow \rho^{0} \rightarrow \pi \pi$ which emerges in the systematic approach of Ref. [274] does not contribute to the $\gamma \pi \pi$ coupling. Thus, no modification of Eq. (8.14) is required. Note also that the anomalous interaction providing $\gamma \pi \rho$ vertices [268] does not
contribute in the long-wavelength limit studied here. This follows a general rule noted in Ref. [260].

In the following chapters, the $\rho$ will be also treated dynamically. The interaction with the photon is then given by the scalar QED vertices from above, plus a $\gamma \rho \pi \pi$ vertex which is obtained from Eq. (8.8) by minimal substitution. With the same procedure the direct $\gamma \rho$ interaction is constructed from Eq. (8.11), leading to the vertices

$$
\begin{align*}
& \xlongequal[\rho^{-(\sigma, k) \rho^{-}\left(\nu, k^{\prime}\right)}]{\sum_{\hat{\gamma}}^{\gamma(\mu, q)}} \hat{=} e\left(k^{\nu} g_{\mu \sigma}+k^{\prime \sigma} g_{\mu \nu}-\left(k+k^{\prime}\right)^{\mu} g_{\sigma \nu}\right), \\
& \frac{\rho^{ \pm}(\alpha) \rho^{ \pm}(\beta)}{=} 2 e^{2}\left(g_{\mu \beta} g_{\alpha \nu}-g_{\mu \nu} g_{\alpha \beta}\right) \tag{8.15}
\end{align*}
$$

in the imaginary time formalism.

### 8.4 Charge fluctuations at low temperatures

Having introduced the effective interaction in the heavy $\rho$ limit, we can evaluate the correction to the CF due to this interaction. Before discussing the results let us first remind the reader about the basic relations for CF in a noninteracting gas of pions and $\rho$-mesons.

### 8.4.1 Charge fluctuations for free pions and $\rho$-mesons

In order to illustrate the relations of Sec. 8.2 and to establish a baseline it is instructive to calculate $D_{S}$ from Eq. (8.4) for the free pion gas in two ways: once via Eq. (8.3) and also directly from statistical mechanics. The interaction from Eq. (8.14) reduces to scalar QED in the zeroth order in $g$. To order $e^{2}$ the selfenergy is given by the set of gauge invariant diagrams in Fig. 8.1 and reads

$$
\begin{equation*}
\Pi^{00}\left(k^{0}=0, \mathbf{k} \rightarrow \mathbf{0}\right)=e^{2}(C+D), \quad\left\langle\delta Q^{2}\right\rangle=e^{2} T V(C+D) \tag{8.16}
\end{equation*}
$$



Figure 8.1: Photon selfenergy at $e^{2}$ for the free pion gas $(C, D)$ and the free $\rho$ gas $\left(C^{\rho}, D^{\rho}\right)$.
according to Eq. (8.3) with

$$
\begin{equation*}
C=\frac{1}{\pi^{2}} \int_{0}^{\infty} d p \omega n[\omega], \quad D=\frac{1}{\pi^{2}} \int_{0}^{\infty} d p p^{2} \frac{n[\omega]}{\omega} \tag{8.17}
\end{equation*}
$$

where $\omega=\sqrt{p^{2}+m_{\pi}^{2}}$ the pion energy, $n[\omega]=1 /(\exp (\beta \omega)-1)$ the BoseEinstein factor, and $\beta=1 / T$. The CF from Eq. (8.16) can also be derived from statistical mechanics,

$$
\begin{gather*}
\left\langle\delta Q^{2}\right\rangle=\left.e^{2} T^{2} \frac{\partial^{2}}{\partial \mu^{2}}\right|_{\mu=0} \log Z  \tag{8.18}\\
\log Z_{0}(\mu)=-V \int \frac{d^{3} p}{(2 \pi)^{3}} \sum_{\mu_{i}= \pm \mu, 0} \log \left(1-e^{-\beta\left(\omega+\mu_{i}\right)}\right) . \tag{8.19}
\end{gather*}
$$

Both the photon selfenergy and Eq. (8.18) lead to the same CF also at the perturbative level as will be seen in Sec. 8.5. The value for the chemical potential of $\mu_{i}= \pm \mu$ in Eq. (8.19) corresponds to charged pions and $\mu_{i}=0$ is assigned to neutral pions which do not contribute to the CF but to the entropy $S=\partial(T \log Z) / \partial T$ of the free gas,

$$
\begin{equation*}
S_{0}=\frac{1}{2 \pi^{2}} \frac{V}{T} \int_{0}^{\infty} d p p^{2} n[\omega]\left(3 \omega+\frac{p^{2}}{\omega}\right) . \tag{8.20}
\end{equation*}
$$

In the high temperature limit, or for massless pions, the relevant thermodynamical quantities are given by

$$
\begin{equation*}
\left\langle\delta Q^{2}\right\rangle=\frac{e^{2} V}{3} T^{3}, \quad S=\frac{2 \pi^{2} V}{15} T^{3}, \quad\left\langle N_{\mathrm{ch}}\right\rangle=\frac{2 \zeta(3) V}{\pi^{2}} T^{3} \tag{8.21}
\end{equation*}
$$

where $\left\langle N_{\text {ch }}\right\rangle$ is defined as in Ref. [249]. For the quantity $D_{S}$ from Eq. (8.4) we obtain $D_{S}=0.185$ for massive free pions at $T=170 \mathrm{MeV}$ and $D_{S}=0.253$ for massless pions. For $D_{c}$ from Eq. (8.5), the values are 4.52 and 5.47, respectively.

The classical (Boltzmann) limit is obtained by replacing the Bose-Einstein distribution $n$ in Eqs. (8.17) and (8.20) by the Boltzmann distribution $n_{\mathrm{B}}=\exp (-\beta \omega)$. In this case at $T=170 \mathrm{MeV}$ we obtain $D_{S}=0.156$ and $D_{S}=1 / 6$ for massive and massless pions, respectively. For all masses and temperatures, $D_{c}=4$ the classical limit. For a QGP made out of massless quarks and gluons, $D_{S}=0.034$, following the same arguments as in [249]. This is about a factor of five smaller than a pion gas.

The CF for the free $\rho$ gas are given by the diagrams with the double lines in Fig. 8.1. With the $\rho$ propagator

$$
\begin{equation*}
D^{\mu \nu}=\frac{1}{k^{2}-m_{\rho}^{2}+i \epsilon}\left(g^{\mu \nu}-\frac{k^{\mu} k^{\nu}}{m_{\rho}^{2}}\right) \tag{8.22}
\end{equation*}
$$

and the interaction from Eq. (8.15) the photon selfenergy turns out to be

$$
\begin{equation*}
\Pi_{\rho}^{00}\left(k^{0}=0, \mathbf{k} \rightarrow \mathbf{0}\right)=3 e^{2}\left[C^{\rho}+D^{\rho}\right] \tag{8.23}
\end{equation*}
$$

where the upper index means that the pion mass is substituted by the $\rho$-mass in $C$ and $D$ from Eq. (8.17). The factor of three corresponds to the sum over the physical polarizations of the $\rho$. The same factor also appears in $\log Z_{0}$ of Eq. (8.19) for the $\rho$.

### 8.4.2 $\pi \pi$ interaction in the heavy $\rho$ limit to order $e^{2} g^{2}$

At order $e^{2} g^{2}$ the Feynman rules derived from the heavy $\rho$ limit Eq. (8.14) lead to the set of five diagrams (eff1) to (eff5) depicted in Fig. 8.2. They are gauge invariant as shown in Appendix D.3.4. The summation over Matsubara frequencies has been performed by a transformation into contour integrals following Ref. [265]. The limit $\left(k^{0}=0, \mathbf{k} \rightarrow \mathbf{0}\right)$ for the external photon has to be taken before summation and integration, as discussed in Appendix D.1. The loop momenta factorize so that the diagrams of Fig. 8.2 can be expressed in terms of the quantities $C$ and $D$ from Eq. (8.17) as shown in Tab. 8.1. The sum of the diagrams is cast in a surprisingly simple form,

(eff1)

(eff4)

(eff2)

(eff5)

(eff3)

(eneff)

Figure 8.2: Selfenergy for $\pi \pi$ interaction in the heavy $\rho$ limit at order $e^{2} g^{2}$ (diagrams (eff1) to (eff5)). Expansion of $\log Z$ at $g^{2}$ for the calculation of the entropy (diagram (eneff)).

Table 8.1: Static selfenergy $\Pi^{00}\left(k^{0}=0, \mathbf{k} \rightarrow \mathbf{0}\right)$ from Fig. 8.2 with $C$ and $D$ from Eq. (8.17).

| Diagram | Contribution |
| :--- | :--- |
| (eff1) | $-\frac{3}{2} \frac{e^{2} g^{2}}{m^{2}} C^{2}$ |
| (eff2) | $-\frac{e^{2} g^{2}}{m_{\rho}^{2}} D\left(D-3 C-\beta \frac{\partial}{\partial \beta}(C-D)\right)$ |
| (eff3) | $+\frac{e^{2} g^{2}}{m_{\rho}^{2}} D(2 D-C)$ |
| (eff4) | $-5 \frac{e^{2} g^{2}}{m_{\rho}^{2}} C D$ |
| (eff5) | $-\frac{5}{2} \frac{e^{2} g^{2}}{m_{\rho}^{2}} D^{2}$ |

$$
\begin{equation*}
\sum_{i=1}^{5} \Pi_{i}\left(k^{0}=0, \mathbf{k} \rightarrow \mathbf{0}\right)=-\frac{e^{2} g^{2}}{m_{\rho}^{2}}\left[\frac{3}{2}(C+D)^{2}-\beta D \frac{\partial}{\partial \beta}(C-D)\right] . \tag{8.24}
\end{equation*}
$$

The entropy correction at $g^{2}$ is calculated from $\log Z$ given by diagram (eneff) in Fig. 8.2,

$$
\begin{equation*}
S_{1}=-\frac{3 g^{2} V}{2 T}\left(\frac{m_{\pi}}{m_{\rho}}\right)^{2} D(C+D) \tag{8.25}
\end{equation*}
$$

Note that using the LO chiral Lagrangian from Eq. (8.6) instead of the $\pi \pi$ interaction in the heavy $\rho$ limit, results would simply change by a factor of $(2 / 3)^{2}$, up to tiny corrections, which are due to higher order contributions involving the chiral symmetry breaking term $\sim \mathcal{M}$ from Eq. (8.6). Numerical results can be found in Sec. 8.6.2, which supersede our findings from Ref. [9].

### 8.5 The $\rho$-meson in the heatbath

In this section we will relax the assumption of a heavy non-dynamical $\rho$ meson. This will allow for an estimate of the CF from the residual interactions of the $\rho$ when this particle is treated as an explicit degree of freedom. It will also avoid some problems induced in the calculation from vertices of higher order in momenta such as encountered in the $\mathcal{L}_{\pi \pi}^{(4)}$ calculation in Ref. [263] (see discussion in Sec. 8.6.1, 8.6.2). The $\rho \pi \pi$ interaction from Eq. (8.8) involves vertices only linear in momentum and a smoother temperature dependence is expected.

We start with the calculation of the diagrams in the first two columns of Fig. 8.3 because this subset corresponds to the heavy $\rho$ limit from Sec. 8.3; by increasing the $\rho$-mass from its physical value to infinity in these diagrams, the previous results from Tab. 8.1 are recovered as illustrated in Appendix D.3.1. Note that there is no need to include $\gamma \rho^{0}$ mixing or anomalous vertices as we have already seen in Sec. 8.3.

Here and in the following sections, the $\rho$ is treated as a stable particle (propagator from Eq. (8.22)) and we ignore imaginary parts at the cost of unitarity violations as will be discussed in Sec. 8.6.1. A $\rho$ with finite width would induce problems concerning gauge invariance: one would have to couple the photon to all intermediate $\rho$ selfenergy diagrams that build up


(4a)


(5c1)


(eff5)



Figure 8.3: Overview of the relevant two-loop diagrams at $e^{2} g^{2}$ for the photon selfenergy and at $g^{2}$ for the entropy. Diagram (1a2) is calculated in detail in Appendix D.3, where also the results for all other diagrams and a proof of gauge invariance are found. The diagrams on the right hand side (eff1-eff5) correspond to the heavy $\rho$ limit of the ones given on the left. This limit, indicated with arrows, is numerically shown in Appendix D.3.1.
the $\rho$ width in the Dyson-Schwinger summation. In principle, this is possible - see the last part of Sec. 8.8 - but goes beyond the scope of this work. The results for the diagrams with dynamical $\rho$ from Fig. 8.3 are found in Eq. (D.16,D.18) and Fig. D. 2 of Appendix D.3.1, together with a detailed calculation of one of the diagrams and a discussion of the infrared divergences. In Appendix D.3.4 the gauge invariance of the diagrams is shown.

At order $e^{2} g^{2}$ there are additional diagrams with direct $\gamma \rho \rho$ and $\gamma \gamma \rho \rho$ couplings from Eq. (8.15) and also with the $\rho \rho \pi \pi$ coupling from Eq. (8.9) which is required by the gauge invariance of the $\rho$-meson. The resulting diagrams are displayed in Fig. 8.4. Some of these diagrams contain more than one $\rho$-propagator. They are sub-dominant because every $\rho$ propagator counts as $1 / m_{\rho}^{2}$. Furthermore, Fig. 8.4 shows diagrams which have a closed pion loop with only one vertex of the $\rho \pi \pi$ type (see, e.g., diagram (2c)). The latter diagrams vanish due to the odd integrand in the loop integration. The set of diagrams from Figs. 8.3 and 8.4 is complete at order $e^{2} g^{2}$.

The non-vanishing diagrams from Fig. 8.4 are best calculated by evaluating the corresponding partition function, $\log Z$, at finite chemical potential $\mu$ and differentiating with respect to $\mu[260,263]$ (see also Eq. (8.18)). For a calculation at finite $\mu$ we first convince ourselves that for the simple interaction from Eq. (8.12) the use of Eq. (8.18) leads to the same results as in Sec. 8.4.2. The calculation at finite $\mu$ implies a shift in the zero-momenta of the propagators and derivative vertices, $p^{0} \rightarrow p^{0} \pm \mu[263,265]$, depending on the charge states of the particles. The correction to $\log Z(\mu)$ from diagram (a) in Fig. 8.5 with the interaction from Eq. (8.12) is given by
$\log Z_{(\mathrm{a})}(\mu)=\frac{-g^{2}}{16 m_{\rho}^{2}} \beta V\left[3\left(V_{+}-V_{-}\right)^{2}+m_{\pi}^{2}\left(U_{+}+U_{-}\right)\left(4 D+U_{+}+U_{-}\right)\right]$
with

$$
\begin{equation*}
U_{ \pm}=\frac{1}{\pi^{2}} \int_{0}^{\infty} d k \frac{k^{2}}{\omega} n[w \pm \mu], \quad V_{ \pm}=\frac{1}{\pi^{2}} \int_{0}^{\infty} d k k^{2} n[w \pm \mu] \tag{8.27}
\end{equation*}
$$

and $D$ from Eq. (8.17). Applying Eq. (8.18) to $\log Z_{(a)}(\mu)$ reproduces the result for the photon selfenergy in the heavy $\rho$ limit from Eq. (8.24) which is shown to be gauge invariant in Appendix D.3.4.
(1b)


(2b)

(3b)
(2c)
(2a)


(3a)

(8a)

(3c)


Figure 8.4: Additional, sub-leading, diagrams at $e^{2} g^{2}$ with direct $(\gamma) \gamma \rho \rho$ couplings and with $\rho \rho \pi \pi$ interaction. Also, the diagrams which vanish are shown [(1c2), (2c), (3c), (4c), (5c2), (7c), (8c)].

(a)

(b)

(c)

(d)

Figure 8.5: Correction to $\log Z(\mu)$. Diagram (a) shows the $\pi \pi$ interaction in the heavy $\rho$ limit, diagrams (b)-(d) the interaction via explicit vector meson from Eqs. (8.8) and (8.9).


Figure 8.6: Corrections to $m_{\text {el }}$ or CF. Dashed-dotted line: result from the gauge invariant subset of diagrams from the first two columns of Fig. 8.3. Dashed line: Result from diagrams (b)+(c) from Fig. 8.5. Dotted line: heavy $\rho$ limit from Sec. 8.4.2. Solid lines: free $\pi$ gas, free $\rho$ gas, and the $\rho \rho \pi \pi$ interaction from Fig. 8.5 (d).

Thus having established that equivalence of photon selfenergy and charge fluctuations (Eq. (8.3)) holds on the perturbative level, we are encouraged to evaluate the diagrams of Fig. 8.4 by differentiating the appropriate terms in $\log Z$ with respect to the chemical potential. The diagrams for $\log Z$ corresponding to the photon self energies given in Figs. 8.3 and 8.4 are displayed in Fig. 8.5(b,c,d). (Details can be found in Appendix D.3.3).

In Fig. 8.6 corrections to the electric mass of a free pion gas due to different sets of diagrams are shown. As a reference, we also plot the results for gases of noninteracting pions and noninteracting $\rho$-mesons (" free $\pi$ " and "free $\rho$ "). The electric mass from the diagrams of Fig. 8.2 with the $\pi \pi$
interaction in the heavy $\rho$ limit is plotted as the dotted line. The electric mass from the diagrams in the first two columns of Fig. 8.3 with dynamical $\rho$ is plotted as the dashed-dotted line. At low temperatures, both results coincide (in detail this is also plotted in Fig. D.2). However, at higher temperatures we observe significant differences which shows, thus, that the $\rho$ obtains importance as an explicit degree of freedom.

The diagrams (b) and (c) from Fig. 8.5 correspond to the first two columns of Fig. 8.3. Additionally, they provide photon selfenergies with $\gamma \rho \rho$ and $\gamma \gamma \rho \rho$ vertices from Fig. 8.4, diagrams (2a), (3a), (7a), and (8a). As shown in Fig. 8.6 (dashed line), these additional $\gamma \rho$ couplings obtain some minor influence above $T \sim 150 \mathrm{MeV}$.

Additionally, in Fig. 8.4 there are diagrams with $\rho \rho \pi \pi$ couplings from Eq. (8.9). The diagrams (1b), (2b), (3b), (4b), and (7b) correspond to diagram (d) in Fig. 8.5. In the heavy $\rho$ limit these diagrams do not contribute. However, for dynamical $\rho$-mesons these diagrams contribute significantly due to the sum over the spin of the $\rho$. In Fig. 8.6 the resulting electric mass is displayed as the solid line (" $\rho \rho \pi \pi$ ").

### 8.6 Relativistic virial expansion

In Ref. [263] the electric mass has been determined using chiral $\pi \pi$ interaction and thermal loops leading to results that show large discrepancies to a virial calculation of $m_{\text {el }}^{2}$. Before we discuss these differences in Sec. 8.6.1, 8.6.2 let us review the theoretical framework first. The virial expansion is an expansion of thermodynamic quantities in powers of the classical (Boltzmann) densities, while the interaction enters as experimentally measured phase-shifts. Consequently, all orders of the interaction are taken into account. Thermal loops, on the other hand, respect quantum statistics (Bose-Einstein in our case) and, thus, contain an infinite subclass of the virial expansion. However, the interaction only enters up to a given order. Thus, the loop and virial expansion represent quite different approximations and it will depend on the problem at hand which is the more appropriate one. The effect on quantum statistics can be considerable. For example at $T=170 \mathrm{MeV}$ the values for the electric mass $m_{\mathrm{el}}^{2}$ of the free $\pi$ gas or the two-loop diagrams, Eq. (8.24),
change by $20 \%$ and $38 \%$ (!) respectively, if we take the Boltzmann limit. Therefore, it is desirable to have a density expansion that respects particle statistics as well as sums all orders of the interaction. While this might be very difficult if not impossible to do in general, it can be done up to second order in the (Bose-Einstein) density.

The partition function can separated into a free and an interacting part,

$$
\begin{equation*}
\log Z=\log Z_{0}+\sum_{i_{1}, i_{2}} z_{1}^{i_{1}} z_{2}^{i_{2}} b\left(i_{1}, i_{2}\right) \tag{8.28}
\end{equation*}
$$

in an expansion in terms of the chemical potential $\mu$ with $z_{j}=\exp \left(\beta \mu_{j}\right)$ for $j=1,2$ the fugacities. In the $S$-matrix formulation of statistical mechanics from Ref. [275] the second virial coefficient $b\left(i_{1}, i_{2}\right)$ can be separated into a statistical part and a kinematic part containing the vacuum $S$-matrix according to

$$
\begin{equation*}
b\left(i_{1}, i_{2}\right)=\frac{V}{4 \pi i} \int \frac{d^{3} k}{(2 \pi)^{3}} \int d E e^{-\beta \sqrt{k^{2}+E^{2}}} \operatorname{Tr}_{i_{1}, i_{2}}\left[A S^{-1}(E) \frac{\overleftrightarrow{\partial}}{\partial E} S(E)\right]_{c} \tag{8.29}
\end{equation*}
$$

where $A$ is the (anti)symmetrization operator for interacting (fermions) bosons and the trace is over the sum of connected diagrams (index "c"). In Eq. (8.29), $V$ is the Volume, $k$ is the momentum of the center of mass in the gas rest frame and $E=s^{1 / 2}$ stands for the total c.m. energy. The labels $i_{1}, i_{2}$ indicate a channel of the $S$-matrix with $i_{1}+i_{2}$ particles in the initial state. For the second virial coefficient, $i_{1}=i_{2}=1$.

For $\pi \pi$ scattering, Eq. (8.29) can be integrated over $k$ and the $S$-matrix can be expressed via phase shifts, weighted with their degeneracy [263]. With $B_{2}=b\left(i_{1}, i_{2}\right) / V$ in the limit $V \rightarrow \infty$ one obtains

$$
\begin{aligned}
B_{2}^{(\pi \pi), \text { Boltz }}(\mu=0) & =\frac{1}{2 \pi^{3} \beta} \int_{2 m_{\pi}}^{\infty} d E E^{2} K_{2}(\beta E) \sum_{\ell, I}(2 I+1)(2 \ell+1) \frac{\partial \delta_{\ell}^{I}(E)}{\partial E} \\
& =\frac{1}{2 \pi^{3}} \int_{2 m_{\pi}}^{\infty} d E E^{2} K_{1}(\beta E) \sum_{\ell, I}(2 I+1)(2 \ell+1) \delta_{\ell}^{I}(8.30)
\end{aligned}
$$

where the second line has been obtained after integration by parts (assuming $\delta_{\ell}^{I} \rightarrow 0$ as $E \rightarrow 2 m_{\pi}$ ). The sum over phase shifts $\delta_{\ell}^{I}$ (isospin $I$, angular
momentum $\ell)$ is restricted to $\ell+I=$ even and $K_{i}$ are the modified Bessel functions of the second kind. The virial expansion in this or similar form has been applied in numerous studies of the thermal properties of interacting hadrons as, e.g., [276,277], among them the electric mass [263]. However, the original reference [275] gives prescriptions of how to at least partially include particle statistics for the asymptotic states of the interaction. This means the summation of the so-called exchange diagrams. We retake this idea and also include a finite chemical potential. This is achieved by projecting the binary collisions of pions in different charge states to the isospin channels [263]. Additionally, the interaction $T$ matrix is boosted from the gas rest frame to the c.m. frame and the $T$-matrix is parametrized via phase shifts with the final result

$$
\begin{align*}
B_{2}^{(\pi \pi), \text { Bose }}(\mu)= & \frac{\beta}{4 \pi^{3}} \int_{2 m_{\pi}}^{\infty} d E \int_{-1}^{1} d x \int_{0}^{\infty} d k \frac{E k^{2}}{\sqrt{E^{2}+k^{2}}}[ \\
& \delta_{0}^{2}(E)\left(n\left[\omega_{1}+\mu\right] n\left[\omega_{2}+\mu\right]+n\left[\omega_{1}-\mu\right] n\left[\omega_{2}-\mu\right]\right) \\
+ & \delta_{0}^{2}(E)\left(n\left[\omega_{1}+\mu\right] n\left[\omega_{2}\right]+n\left[\omega_{1}-\mu\right] n\left[\omega_{2}\right]\right) \\
+ & 3 \delta_{1}^{1}(E)\left(n\left[\omega_{1}+\mu\right] n\left[\omega_{2}\right]+n\left[\omega_{1}-\mu\right] n\left[\omega_{2}\right]\right) \\
+ & \delta_{0}^{2}(E)\left(\frac{1}{3} n\left[\omega_{1}+\mu\right] n\left[\omega_{2}-\mu\right]+\frac{2}{3} n\left[\omega_{1}\right] n\left[\omega_{2}\right]\right) \\
+ & 3 \delta_{1}^{1}(E) n\left[\omega_{1}+\mu\right] n\left[\omega_{2}-\mu\right] \\
+ & \left.\delta_{0}^{0}(E)\left(\frac{2}{3} n\left[\omega_{1}+\mu\right] n\left[\omega_{2}-\mu\right]+\frac{1}{3} n\left[\omega_{1}\right] n\left[\omega_{2}\right]\right)\right] . \tag{8.31}
\end{align*}
$$

The second line of Eq. (8.31) corresponds to $\pi \pi$ scattering with a net charge of the $\pi \pi$ pair of $|C|=2$, the third and fourth line to $|C|=1$ and the lines 4 to 6 to $C=0$. The boosted Bose-Einstein factors which arise after summations over exchange diagrams are

$$
\begin{gather*}
n\left[\omega_{1,2} \pm \mu\right]=\frac{1}{e^{\beta\left(\omega_{1,2} \pm \mu\right)}-1}, \quad \omega_{1}=\gamma_{f}\left(\frac{1}{2} E+\frac{k Q x}{\sqrt{E^{2}+k^{2}}}\right) \\
\omega_{2}=\gamma_{f}\left(\frac{1}{2} E-\frac{k Q x}{\sqrt{E^{2}+k^{2}}}\right) \\
\gamma_{f}=\left(1-\frac{k^{2}}{E^{2}+k^{2}}\right)^{-\frac{1}{2}}, \quad Q=\frac{1}{2} \sqrt{E^{2}-4 m_{\pi}^{2}} \tag{8.32}
\end{gather*}
$$

with the pion c.m. momentum $Q \equiv Q_{\text {c.m. }}$.
Obviously, the chemical potential can not be factorized in Eq. (8.31) so that the expansion is rather in powers of Bose-Einstein factors $n$ than in powers of $e^{\beta \mu}$ as in a conventional virial expansion. Eq. (8.31) contributes also to higher virial coefficients. The situation resembles the case of a free Bose-Einstein gas that contributes to all virial coefficients which can be seen by expanding the Bose-Einstein factor in powers of $e^{\beta \mu}$. Therefore, in the following we will refer to the expansion (8.31) as "(low) density expansion". The term "virial expansion" will be reserved for the well known expansion in terms of classical (Boltzmann) distributions. We note, that in the Boltzmann limit the standard expression for the virial coefficient, e.g. Eq. (9) of Ref. [263],

$$
\begin{align*}
B_{2}^{(\pi \pi), \text { Boltz }}(\mu) & =\frac{1}{2 \pi^{3}} \int_{2 m_{\pi}}^{\infty} d E E^{2} K_{1}(\beta E) \\
& \times\left[2 \cosh (2 \mu \beta) \delta_{0}^{2}+2 \cosh (\mu \beta)\left(\delta_{0}^{2}+3 \delta_{1}^{1}\right)+\delta_{0}^{2}+3 \delta_{1}^{1}+\delta_{0}^{0}\right] \tag{8.33}
\end{align*}
$$

is recovered which at $\mu=0$ reduces to the expression in Eq. (8.30).
The connection of $B_{2}(\mu)$ to physics is given by

$$
\begin{equation*}
\log Z(\mu)=V B_{2}(\mu), \quad P(\mu)=\frac{B_{2}(\mu)}{\beta}, \quad m_{\mathrm{el}}^{2}=e^{2}\left(\frac{\partial^{2} P}{\partial \mu^{2}}\right)_{\mu=0} \tag{8.34}
\end{equation*}
$$

where $P$ is the correction to the pressure. Note that for the electric mass the contribution $\sim \delta_{0}^{0}$ vanishes in the Boltzmann limit (and is small anyways). The form of Eq. (8.31) makes it as easy to use as the common virial expansion, inserting the $\pi \pi$ phase shifts $\delta_{0}^{0}, \delta_{1}^{1}$, and $\delta_{0}^{2}$ which we adopt from Ref. [277]. The inelasticities of the $\pi \pi$ amplitude are small in the relevant energy region and we have not taken them into account in Eq. (8.31).

### 8.6.1 Density expansion versus thermal loops

It is instructive to see to which extent the thermal loop expansion and the extension of the virial expansion from Eq. (8.31) agree. To this end we need to match both approaches by extracting the scattering amplitude from our model Lagrangian and insert it into Eq. (8.31). For simplicity, we first study
the $\pi \pi$ interaction in the heavy $\rho$ limit at $g^{2}$ and evaluate Eq. (8.31). As this interaction is not unitary, one has to go back to the original $S$-matrix formulation and express it in terms of the (on-shell) $T$-matrix [275] which can then be calculated from theory. Given the normalization of the $T$-matrix used in this paper, $S=1-\frac{i Q}{8 \pi \sqrt{s}} T$, the right hand side of Eq. (8.29) can be written as

$$
\begin{equation*}
\left(S^{-1} \frac{\partial S}{\partial E}-\frac{\partial S^{-1}}{\partial E} S\right)=-\frac{i}{8 \pi} \frac{\partial}{\partial E}\left[\frac{Q}{E}\left(T+T^{\dagger}\right)\right]+\frac{1}{64 \pi^{2}}\left(\frac{Q}{E} T^{\dagger}\right) \frac{\overleftrightarrow{\partial}}{\partial E}\left(\frac{Q}{E} T\right) \tag{8.35}
\end{equation*}
$$

Using the relation between $S$-matrix and phase shifts, $S=e^{2 i \delta}$, we find

$$
\begin{equation*}
\frac{\partial}{\partial E} \delta_{\ell}^{I} \hat{=}-\frac{\partial}{\partial E}\left(\frac{2 Q}{E} \operatorname{Re} T_{\ell}^{I}\right)+\frac{8 Q^{2}}{E^{2}}\left(\operatorname{Re} T_{\ell}^{I} \frac{\overleftrightarrow{\partial}}{\partial E} \operatorname{Im} T_{\ell}^{I}\right) \tag{8.36}
\end{equation*}
$$

where the connection between isospin amplitudes $T^{I}$ and their projection into partial waves $T_{\ell}^{I}$ is given in Eq. (D.10). Inserting this expression into Eq. (8.31) leads to the density expansion based on a given model amplitude. We note that the second term in Eq. (8.36) is quadratic in the amplitude and vanishes for real amplitudes. Therefore, close to threshold, where the amplitudes are small and real, the quadratic term can be neglected. However, with increasing energy unitarity requires that the imaginary part of the amplitude will become sizable so that the second term cannot any longer be neglected. This is especially the case if the amplitude is resonant. Consequently, the use of point-like interactions at tree level which are always real and not unitary might lead to rather unreliable predictions for thermodynamic quantities.

Before we discuss the importance of unitarity, let us first establish that the density expansion of Eq. (8.31) and the loop expansion lead to the same results if both methods are based on the same point-like interaction. The partial amplitudes $T_{\ell}^{I}$ for the $\pi \pi$ interaction in the heavy $\rho$ limit are obtained from Eq. (D.9) by neglecting $s, t, u$, and $\Gamma$ in the denominators and $s \equiv E^{2}$. Inserting the result in Eq. (8.31) and calculating the pressure from Eq. (8.34) we obtain exactly the same result as for the thermal loops from Eq. (8.26) at $\mu=0$. We have also verified that this agreement holds in a simple $\phi^{4}$ theory of uncharged interacting bosons. Calculating the electric mass in both approaches for the $\pi \pi$ interaction in the heavy $\rho$ limit (Eqs. $(8.24,8.3)$ and (8.31,8.34)), we again find perfect agreement.


Figure 8.7: Electric mass from dynamical $\rho$ exchange. Solid line: from the diagrams in the first two columns of Fig. 8.3. Dotted line: Bose-Einstein density expansion from dynamical $\rho$ exchange (no imaginary parts, $\Gamma_{\rho} \rightarrow 0$ ). Dashed line: Same, but $\Gamma_{\rho}=150 \mathrm{MeV}$.

Consequently, and not so surprisingly, both thermal loop and density expansion lead to the same result, if the interaction in the density expansion is truncated at the appropriate (unitarity violating) level. This is also true in the classical (Boltzmann) limit. In this limit, a similar equivalence has been found in [263] using an effective range expansion for the amplitude; see also [278] for a related equivalence for propagators.

While it is comforting to see that both approaches agree in the same order of density and interaction, this agreement highlights a possible problem for the loop expansion. If the order of the interaction considered violates unitarity the second term of Eq. (8.36) is ignored and the loop expansion may lead to unreliable results for the pressure etc. This is of particular importance if the amplitudes are resonant, as it is the case for the $\rho$-exchange.

In order to see these effects we concentrate on the gauge invariant set of diagrams given in the first two columns of Fig. 8.3. The result for these diagrams is given in Eqs. (D.16, D.18) and plotted in Fig. 8.7 as the solid line. In the calculation of these thermal loops we have made the following approximations, see Appendix D.3.1: (I) The poles of the $\rho$ have been neglected in
the contour integration (see the explanation following Eq. (D.22)). (II) The $\rho$ has no width, i.e. the $\rho$ propagator is given by $D^{\mu \nu}$ from Eq. (8.22). (III) Only the real parts of the thermal loops have been considered.

In the following we test these approximations by comparing the thermal loop result with a suitable "toy model" low density expansion. For the interaction driving the low density expansion we take the partial waves from Eq. (D.9) and project out the $T_{\ell}^{I}$ by the use of Eq. (D.10). Furthermore, we set $\Gamma_{\rho}=0$ in Eq. (D.9) in this interaction. Third, we consider only the term linear in $T$ in Eq. (8.36) for the density expansion. This means that imaginary parts are neglected. The low density expansion, constructed in this way, exhibits the same approximations (II) and (III) as the calculation of the thermal loops above, i.e. the zero width and the reduction to the real part only. The result of this "toy model" low density expansion is plotted in Fig. 8.7 as the dotted line.

Both the results from thermal loops (solid line) and the density expansion (dotted line) agree closely. The small deviation of both curves is due to the additional approximation (I) which we have made in the calculation of the thermal loops, i.e. neglecting the poles in the contour integration. Note also that other partial waves than $T_{0}^{0}, T_{1}^{1}$, and $T_{0}^{2}$ are present in the results from the thermal loops because the $\rho$ exchange contains all partial waves. However, from the agreement found here, we may conclude that these higher partial waves give negligible contributions (at least in the present $\rho$-exchange model).

In our "toy model" low density expansion, we can allow for a finite width in the $\rho$-propagator. This implies that the $\rho$-propagator is given by $D_{\rho}=\left[p^{2}-m_{\rho}^{2}+i m_{\rho} \Gamma(\sqrt{s})\right]^{-1}$, where $\Gamma(\sqrt{s})=\Gamma\left(m_{\rho}\right)\left(m_{\rho}^{2} / E^{2}\right)\left(E^{2}-\right.$ $\left.4 m_{\pi}^{2}\right)^{3 / 2} /\left(m_{\rho}^{2}-4 m_{\pi}^{2}\right)^{3 / 2}$. With this modification, we evaluate again the electric mass. However, as we still use only the term linear in $T$ in Eq. (8.36), any imaginary parts of the amplitude arising from the finite width are still ignored. The result is shown as the dashed line in Fig. 8.7; the electric mass hardly changes.

Let us now discuss the effect of the imaginary parts of the amplitude. To simplify the discussion let us restrict ourselves to the vector-isovector $(I, J)=(1,1)$ channel, which is dominated by the $\rho$-resonance. We will also
work in the Boltzmann limit as effects due to unitarity are independent of the statistical ensemble. The model amplitude is simply the $s$-channel $\rho$ exchange diagram with a $\rho$-propagator as given above. This amplitude is unitary by construction and describes the scattering data in the $(I, J)=$ (1,1)-channel well (see Fig. D.1). With the (complex) $T$-matrix $T_{1}^{1}$ the electric mass $m_{\mathrm{el}}$ is given by

$$
\begin{align*}
m_{\mathrm{el}}^{2}(\mu=0, I=\ell=1) & =-\frac{6 e^{2} \beta}{\pi^{3}} \int_{2 m_{\pi}}^{\infty} d E Q E K_{1}(\beta E) \operatorname{Re} T_{1}^{1} \\
& +\frac{24 e^{2}}{\pi^{3}} \int_{2 m_{\pi}}^{\infty} d E Q^{2} K_{2}(\beta E)\left(\operatorname{Re} T_{1}^{1} \frac{\stackrel{\partial}{\partial}}{\partial E} \operatorname{Im} T_{1}^{1}\right) \tag{8.37}
\end{align*}
$$

where the first term is linear and the second quadratic in the amplitude. It is the second, quadratic, term where the imaginary part of the amplitude enters. In Fig. 8.8 the different contributions to the electric mass according to the decomposition Eq. (8.37) are plotted. As a reference we also show the result using experimentally measured phase shift $\delta_{1}^{1}$ (solid black line). Obviously the contribution from the quadratic term (" $T_{1}^{1}$, quad.") is dominant, and adding the linear (" $T_{1}^{1}$, lin.") and quadratic terms we obtain good agreement with the result from the $\delta_{1}^{1}$ experimental phase shift. This is to be expected as $T_{1}^{1}$ fits vacuum data well. Note that the linear term alone vastly underpredicts the electric mass. Thus the imaginary part of the amplitude is essential for the proper description of the fluctuations.

Furthermore, the electric mass from the free $\rho$-gas (Boltzmann statistics) (gray line, Fig. 8.8) agrees well with the results from the experimental phase shift and unitary $\rho$-model. Indeed, it can be shown that the $\pi \pi$-interaction via unitary $s$-channel $\rho$-exchange in the limit of vanishing width leads to a contribution to $\log Z$ equal to that of a free $\rho$-gas $[277,279]$. In this limit $\delta_{1}^{1}(E)=\pi \Theta\left(E-m_{\rho}\right)$ allowing for an explicit evaluation of Eq. (8.31) in the Boltzmann approximation. For Bose-Einstein statistics the situation is more complicated. In Ref. [279] it has been shown for meson-baryon interaction that also in this case the interaction of two particles via a narrow resonance $N^{*}$ leads to the same grand canonical potential as from a free $N^{*}$-gas with


Figure 8.8: Main plot: Contribution to $m_{\mathrm{el}}^{2}$ in the $(I, \ell)=(1,1)$-channel of $\pi \pi$ scattering. Solid line: From experimental phase shift. Dotted lines: Terms linear and quadratic in $T$ from Eq. (8.37) with $T$ from a unitary $\rho$ exchange model. Dashed line: Sum of linear and quadratic term. Gray line: free $\rho$-gas (Boltzmann).
Insert: With $T$ from the LO chiral Lagrangian. Dashed line: Tree level. Solid line: Unitarization with $K$-matrix.
the corresponding Fermi-statistics for $N^{*}$; however, the proof requires a selfconsistent medium modification of the $N^{*}$ width and a consideration of larger classes of diagrams.

While in our toy model we could simply restore unitarity by introducing a $\rho$ width, in a more complete calculation this is considerably more difficult. For example, using a $\rho$-propagator with a width in the diagrams of Figs. 8.3 and 8.4 leads to additional photon couplings to the intermediate pion loops, which generate the $\rho$-width. This is simply a consequence of gauge invariance (see Appendix D.3.4). Therefore, introducing unitary amplitudes while maintaining gauge invariance is a non-trivial task.

An alternative approach to assess the role of unitarity is to unitarize a given amplitude using the $K$-matrix approach (see, e.g. [187]). This approach does not add any additional dynamics, and therefore provides a good estimator on the importance of unitarity alone. Using the $K$-matrix approach we can in principle take any of the interactions discussed in this paper. Here we choose the interaction in the (1,1)-channel from the LO chiral Lagrangian given in Eq. (8.6). Details of the calculation can be found in Appendix D.2.2. Maintaining gauge invariance in a $K$-matrix unitarization scheme requires special care and is beyond the scope of this paper. Ignoring this issue, we can compare the electric mass from the unitarized version using Eqs. (D.14,8.31,8.34) with the tree level amplitude using $T_{1}^{1}$ from Eq. (D.13) and then Eqs. $(8.36,8.31,8.34)$. The results are plotted in the insert of Fig. 8.8 and show only a small correction due to unitarization.

Consequently, unitarity by itself is not as crucial as the dynamics which generates the resonance. In other words as long as the phase shift is slowly varying with energy unitarity corrections are small. A resonant amplitude on the other hand corresponds to a very rapidly varying phase-shift. Since it is the derivative of the phase-shift which enters the density expansion, resonant amplitudes are expected to dominate. Consequently, a resonance gas should provide a good leading order description of the thermodynamics of a strongly interacting system.

Note that the unitarized amplitude $T_{(u), 1}^{1}$ from Eq. (D.13) corresponds to a unitarization via the Bethe-Salpeter equation in the limit where the real parts of the intermediate $\pi \pi$-loops are neglected; the freedom in the choice


Figure 8.9: CF or electric mass for the interacting $\pi \pi$ system. The $\mathcal{L}_{\pi \pi}^{(2)}$, $\mathcal{L}_{\pi \pi}^{(4)}$ calculations are from Ref. [263] and the conventional virial expansion (Boltzmann statistics) reproduces results from Ref. [263].
of the real part (loop regularization) can be used to fit to experimental phase shifts, which in turn introduces the missing dynamics (see, e.g., Refs. $[1,16]$ ) This should lead to more reliable predictions [280].

To conclude this analysis of the density expansion, it appears that the low density expansion of Eq. (8.31), using experimental phase shifts, will give the most reliable results, while a simple hadron gas calculation should provide a reasonable first estimate for the fluctuations of a system. Finally, there are certain features of the $\rho$ model from Sec. 8.5 which can not be taken into account in the low density expansion: The $\rho \rho \pi \pi$ and $(\gamma) \gamma \rho \rho$ interactions discussed in Sec. 8.5 (Fig. 8.4) are a consequence of the $\rho$ being treated as a heavy gauge particle; these features will be missed in the low density or virial expansion in which the $\rho$ is not more than a resonant structure in the $\pi \pi$ amplitude. These considerations will be taken into account in the final numerical result from Sec. 8.8.

### 8.6.2 Numerical results for the interacting pion gas

In Fig. 8.9 the results so far obtained are compared to Ref. [263] (gray dashed lines). The electric mass for pions interacting in the heavy $\rho$ limit from Sec. 8.4.2 is indicated with the dotted line. Taking into account that the interaction from Eq. (8.12) is around $3 / 2$ times stronger than the one from the LO chiral Lagrangian, the calculation is consistent with the $\mathcal{L}_{\pi \pi}^{(2)}$ calculation from Ref. [263] which we have also checked analytically. The result for dynamical $\rho$ exchange (black dashed line) contains the contributions from free pion and $\rho$ gas and the diagrams from Fig. 8.5 (b), (c), and (d). The difference to the heavy $\rho$ limit shows the importance of the $\rho$ as an explicit degree of freedom in the heatbath.

Up to $T \sim 130 \mathrm{MeV}$ the dynamical $\rho$ exchange contributes with the same sign as the virial expansion from Ref. [263] although they differ largely in size due to the lack of imaginary part in the loop calculation, especially in the $(I, \ell)=(1,1)$-channel as discussed above. Also the $\rho$-model does not describe the $(I, \ell)=(2,0)$ amplitude very well.

For the low density expansion from Eq. (8.31) and the virial expansion from Eq. (8.30) we use the phase shifts from Ref. [277]. Note that there is a partial cancellation from the $\delta_{1}^{1}$ and $\delta_{0}^{2}$ partial waves [263].

The $\mathcal{L}_{\pi \pi}^{(4)}$ calculation from Ref. [263], which of course contains also the $\mathcal{L}_{\pi \pi}^{(2)}$ contribution, shows a very distinct result. The reason is twofold: on one hand, unitarity is not preserved (see discussion in Sec. 8.6.1). On the other hand, the thermal loops in the $\mathcal{L}_{\pi \pi}^{(4)}$ calculation pick up high c.m. momenta where the theory is no longer valid and the dependence of the NLO interaction on high powers of momenta introduces artifacts. Note that the size of the correction from $\mathcal{L}_{\pi \pi}^{(4)}$ alone is larger than the one from $\mathcal{L}_{\pi \pi}^{(2)}$ for $T>80 \mathrm{Mev}$.

The results for the observable $D_{S}$ from Eq. (8.4) are displayed in Fig. 8.10. Corrections to the entropy are included: from Eq. (8.25) for the heavy $\rho$ limit, from Eq. (D.16) for the case with dynamical $\rho$ and from Eq. (8.34) for the low density expansions. For the thermal loops, indicated by "models with $\rho$ exchange", $D_{S}$ is suppressed. This is due to the large negative correction to $\left\langle\delta Q^{2}\right\rangle$ as has been seen in Fig. 8.9. The virial expansion and the density expansion coincide better with each other than in Fig. 8.9 and can be roughly


Figure 8.10: $D_{S}$ from Eq. (8.4) for the interacting $\pi \pi$ system. The heavy $\rho$ limit from Eqs. $(8.24,8.25)$ includes the contribution from the free $\pi$ gas. The contributions from free $\pi$ and free $\rho$ gases are included for the result from Eqs. (D.23,D.25,D.26), the latter indicated as "dynamical $\rho$ ". The virial and density expansions are indicated according to their underlying particle statistics with "Boltzmann" and "Bose-Einstein", respectively.
approximated by a gas of noninteracting pions and rhos.
Having contrasted virial expansions and dynamic $\rho$ model in Sec. 8.6.1, the most realistic results for CF and $D_{S}$ for the interacting $\pi \pi$ system are given by the Bose-Einstein density expansion from Eq. (8.31). While this result is somewhat below the estimate of a gas of free pions and $\rho$-meson, it is nowhere near the value of $D_{S}^{Q G P} \simeq 0.034$ for the quark gluon plasma.

### 8.7 Higher order corrections

Both the density expansions and $\rho$ models from the last sections are quadratic in density, i.e., the statistical factor $n$. However, at the temperatures of the hadronic phase higher effects in density play an important role. Virial expansions become complicated beyond the second virial coefficient and no experimental information exists on three body correlations. Performing resummations is, therefore, of interest. This will include the density and strong coupling $g$ to all orders. Of course, this can not be done in a systematic way;


Figure 8.11: Resummation schemes: necklace (n) and ring (r). Below, the tadpole medium correction of the $\rho$ propagator is displayed $(\mathrm{t})$.
resummations only contain certain classes of diagrams at a given order in perturbation theory. In all resummations, $\log Z$ is calculated at finite $\mu$ and then Eq. (8.18) is applied in order to obtain the electric mass. We have convinced ourselves in Sec. 8.5 that this is a charge conserving procedure.

We start with two natural extensions of the basic interaction diagram (a) in Fig. 8.5, displayed in Fig. 8.11 (n) and (r) using for both of them the effective interaction of the heavy $\rho$ limit from Eq. (8.12). Alternatively, one can use the LO chiral Lagrangian from Eq. (8.6). As found in Sec. 8.3 , results for the dominant part of this interaction are obtained by simply multiplying $g$ in the following by a factor of $2 / 3$. However, one should keep in mind the unitarity problems of these simplified point-like interactions which have been addressed in Sec. 8.6.1.

For the calculation of diagram ( n ) we utilize an equation of the Faddeev type. The Faddeev equations, usually used in three-body scattering processes as in Ref. [1] in a different context, are an easy way to sum processes whose elementary building blocks are of different types, as in this case loops of



Figure 8.12: Left panel, main plot: Resummations (n) and (r) from Fig. 8.11 and their expansions up to $g^{2} / m_{\rho}^{2}$ (dashed line) and up to $g^{4} / m_{\rho}^{4}$ (dotted line) which are the same for ( n ) and (r). Insert: resummation ( t ), for order $g^{4} / m_{\rho}^{4}$ and higher. Right panel: CF over entropy, $D_{S}$. Result for heavy $\rho$ limit and dynamical $\rho$ as in Fig. 8.10. The result including the resummations (see text) is shown as the dashed dotted line $\left(\Sigma^{\prime}\right)$.
neutral pions with chemical potential $\mu=0$ and charged loops:

$$
\begin{align*}
\log Z_{(\mathrm{n})}(\mu) & =\frac{1}{2} \beta V\left(\frac{1}{2} a_{0} b_{ \pm}+a_{ \pm}\left(b_{ \pm}+b_{0}\right)\right) \\
b_{ \pm} & =g^{\prime} c_{ \pm}+g^{\prime} l_{ \pm}\left(b_{ \pm}+b_{0}\right) \\
b_{0} & =\frac{1}{2} g^{\prime} c_{0}+\frac{1}{2} g^{\prime} l_{0} b_{ \pm} \tag{8.38}
\end{align*}
$$

with $g^{\prime}=-g^{2} / m_{\rho}^{2}$. The first loop in the chain is labeled $a$, the last one $c$, and $l$ means an intermediate loop. The indices " $\pm$ " and " 0 " label charged and uncharged loops, respectively. It is instructive to expand Eq. (8.38) loop by loop which shows that the structure indeed reproduces all sequences of charged and uncharged loops, of all lengths. There is a symmetry factor of $1 / 2$ for every loop of neutral pions and a global factor of $1 / 2$ for every pion chain. The solution of Eq. (8.38) is found in Appendix D.4. The result of resummation (n) is plotted in Fig. 8.12 together with its expansion up to $g^{2} / m_{\rho}^{2}$ (dashed line) and up to $g^{4} / m_{\rho}^{4}$ (dotted line).

The summation (r) of Fig. 8.11 with the interaction from Eq. (8.12) exhibits a symmetry factor of $1 / N$ for a ring with $N$ "small" loops (see Fig. 8.11) which after summing over $N$ leads to the occurrence of a logarithmic cut in the zero-component $p^{0}$ of the momentum of the "big" loop. Due to this obstacle for the contour integration method [265], usually only the static
mode $p^{0}=0$ is calculated, although new studies overcome this problem [281]. In the present approach, we can calculate the ring with $N$ "small" loops explicitly before summing over $N$. This avoids, thus, the problem of the logarithm at the cost of having to cut the series at some $N_{\max }$. On the positive side, all modes are included, and not only the $p^{0}=0$ static contribution. The result up to eight "small" loops has already converged up to $T \sim 200 \mathrm{MeV}$ and is displayed in Fig. 8.12 as (r). The explicit solution can be found in Appendix D.4.

Note that in the resummation schemes we do not consider the vacuum parts of the loops, i.e. we do not renormalize the vacuum amplitude. This excludes potential double counting issues in the final numerical results in Sec. 8.8 where resummations and density expansion are added: Renormalizations of the vacuum amplitude are supposed to be be included in the phase shifts that are used in the density expansion.

There is an additional resummation scheme that sums up the $\rho \rho \pi \pi$ interaction required by the gauge invariance of the $\rho$ (see Eq. (8.9)): One can consider diagram (b) and (c) of Fig. 8.5, dress the $\rho$ propagator as indicated in ( t$)$ of Fig. 8.11, and finally take the heavy $\rho$ limit as in Sec. 8.3. This leads to the same result as a renormalization of the static $\rho$ propagator $-1 / m_{\rho}^{2}$ of diagram (a) in Fig. 8.5 for the $\pi \pi$ interaction in the heavy $\rho$ limit: The resummed pion tadpoles can be incorporated by a mass shift,

$$
\begin{equation*}
m_{\rho^{ \pm}}^{2} \rightarrow m_{\rho}^{2}+\frac{g^{2}}{4}\left(U_{+}+U_{-}+2 D\right), \quad m_{\rho^{0}}^{2} \rightarrow m_{\rho}^{2}+\frac{g^{2}}{2}\left(U_{+}+U_{-}\right) \tag{8.39}
\end{equation*}
$$

for charged and neutral $\rho$. The contribution to $m_{\mathrm{el}}$ from this modification is shown in the insert of Fig. 8.12 as ( t ). The thermal $\rho_{0}$ mass from Eq. (8.39) at $\mu=0$ is $m_{\rho_{0}}=824 \mathrm{MeV}$ at $T=170 \mathrm{MeV}$ which is slightly more than in other studies [282]. This is certainly due to the omission of the $\rho \rightarrow \pi \pi \rightarrow \rho$ selfenergy which also contributes and is required by the gauge invariance of the $\rho$-meson. In the counting of the present study, the $\rho \rightarrow \pi \pi \rightarrow \rho$ selfenergy is statically included in the resummation (n) of Fig. 8.11.

To the right in Fig. 8.12 the normalized CF over entropy, $D_{S}$ from Eq. (8.4), are plotted. For comparison, the result at $g^{2}$ from the dynamical $\rho$ exchange (see Fig. 8.10) is shown with the dashed line. We include now the resummation (n) but only with three or more loops, or in other words, at
$g^{4}$ and higher in the interaction in order to avoid double counting with the $g^{2}$ contribution. We have already seen in Fig. 8.12, left panel, that both resummations ( n ) and ( r ) contain the same diagram at order $g^{4}$ (linear chain of three loops). Thus, again in order to avoid double counting, we include the resummation (r) requiring at least three of the "small" loops, see Fig. 8.11; this means that only contributions of order $g^{6}$ and higher are included. Finally, we add the resummation ( t$)$ including the orders $g^{4}$ and higher, which again avoids double counting of the $g^{2}$-contribution. Summing in this way the resummations to the $g^{2}$-result (dashed line) for both $\left\langle\delta Q^{2}\right\rangle$ and $S$, the resulting $D_{S}=\left\langle\delta Q^{2}\right\rangle / S$ is indicated as $\Sigma^{\prime}$ with the dashed-dotted line in Fig. 8.12.

The resummations have a large effect on $\left\langle\delta Q^{2}\right\rangle$ (see Fig. 8.12, left) whereas their effect on the entropy is much smaller; the entropy is efficiently suppressed for higher orders in the coupling. This explains, why the result $\Sigma^{\prime}$ shows such a large difference compared to the results at order $g^{2}$ (dashed line).

For the resummations (n) and (r), we have ensured that we recover the results from Eqs. (8.24) and (8.26) at the same order of the interaction. We have also verified that the results from Ref. [283] at external momentum $p$ of the $\rho$ being zero $\left(p^{0}=0, \mathbf{p} \rightarrow 0\right)$ match the $\rho$ self energies at $\mu=0$ that are implicitly or explicitly contained in the resummations (n) and ( t ).

A possible extension of the diagrams discussed here is given by resummations of super-daisy type: the pion propagator is dressed by a series of pion tadpoles; the propagator of the tadpole loop itself is again dressed which constitutes a self consistency condition. E.g., this leads to a thermal mass of the pion $m_{\pi} \sim 170 \mathrm{MeV}$ at $T \sim 170 \mathrm{Mev}$. However, one should realize that the lower orders in the coupling $g$ of a super-daisy expansion are already covered by the resummations considered before: it is easy to see that the super-daisy resummation introduces additional diagrams only at order $g^{8}$ and higher $\left(g^{6}\right.$ and higher for resummation (t)) and, thus, can be neglected.

### 8.7.1 Extension to $S U(3)$

In order to obtain a more realistic model for the grand canonical partition function, the leading contributions from the interaction of the full $S U(3)$ meson and vector meson octets is considered. Obviously, the leading contribution to the CF from strange degrees of freedom is simply the free kaon gas. Here we want to discuss corrections due to interactions of kaons with pions. The most important of those is the resonant $p$-wave interaction involving an intermediate $K^{*}(892)$ meson. This is quite analogous to the $\rho$ meson in the $\pi \pi$ case, discussed previously. The $\Phi$ meson, on the other hand, only enters if interactions between kaons are considered. These are sub-leading as pions are more abundant and thus $\pi K$ interactions are more important.

As in the previous sections we describe the meson-meson interaction by dynamical vector meson exchange, second, by an effective interaction, and, third, by realistic phase shifts via a relativistic Bose-Einstein density expansion. For processes which contain at least one pion, the dynamical vector meson exchange is mediated by the $K^{*}(892)$. The effective contact interaction is taken from the LO chiral meson-meson Lagrangian in its $S U(3)$ version, $\mathcal{L}_{\pi K}^{(2)}$ for the $\pi K$-interaction. The density expansion of $\pi K$ scattering is obtained following the same steps as in Sec. 8.6. Details of the calculations are summarized in Appendix D.5.

In Fig. 8.13 the CF from the different models are shown. For the virial and density expansion the phase shifts have been taken from the parametrization of Ref. [284] for the attractive channels $\delta_{0}^{1 / 2}, \delta_{1}^{1 / 2}, \delta_{2}^{1 / 2}$, corrected for the parameters of the $K_{0}^{*}(1350)$ resonance (nowadays, $K_{0}^{*}$ (1430) in the PDG [57]) as reported in Ref. [276]. The repulsive $\delta_{0}^{3 / 2}$ phase shift is from Ref. [285]. The phase shifts plotted in Fig. 4 of [276] up to $s^{1 / 2}=1 \mathrm{GeV}$ have been reproduced.

The situation resembles the case of $\pi \pi$ scattering from Fig. 8.9: Thermal loops with dynamical vector exchange or with effective interaction via $\mathcal{L}_{\pi K}^{(2)}$ show large discrepancies to the virial and density expansions, this time even more than in the $\pi \pi$ case. The reasons are similar as those found in Sec. 8.6.1: The repulsive $(I, \ell)=(3 / 2,0)$ partial wave is not well described by $\pi K$ scattering via $K^{*}(892)$ and unitarity problems of the thermal loops show


Figure 8.13: Corrections to the electric mass or $\mathrm{CF},\left\langle\delta Q^{2}\right\rangle /\left(e^{2} V T^{3}\right)$ for $\pi K$ interaction. The density and virial expansions are from Eqs. (D.37) and (D.39), respectively. The loop expansions " $\pi K$ dynamical" and " $\pi K$ contact" are from Eqs. (D.34,D.35) and (D.36), respectively. The solid line shows the electric mass of a gas of free $\kappa(800), K^{*}(892), K_{0}^{*}(1430)$, and $K_{2}^{*}(1430)$ mesons.
up. The contributions from both the virial expansion and the low density expansion are large compared to the virial corrections in the $\pi \pi$ sector (see Fig. 8.9). This seems surprising as in the $\pi K$ system the kaon has a large mass which should suppress contributions kinematically. However, in the considered channels of $\pi K$ scattering, four resonances are present, $\kappa(800)$, $K^{*}(892), K_{0}^{*}(1430)$, and $K_{2}^{*}(1430)$ and we know from Sec. 8.6.1 that resonances give a large positive contribution to $m_{\mathrm{el}}{ }^{1}$. The electric masses from these resonances, treated as free gases (Boltzmann), is plotted in Fig. 8.13 with the solid line. We find the same pattern as in the discussion of Fig. 8.8 for the free $\rho$ : the virial corrections from resonant phase shifts are well described by a free gas of the same resonances. Furthermore, the repulsive $\delta_{0}^{3 / 2}$ phase shift is very small.

As the outcome for the density expansion in Fig. 8.13 shows, the inclusion of Bose-Einstein statistics is important (compare to the virial expansion which uses Boltzmann statistics). We consider the density expansion to provide the most reliable prediction.

### 8.8 Numerical results

In the discussions in Secs. 8.6.1 and 8.7.1 good reasons have been found that at quadratic order in density $n$ the Bose-Einstein density expansion gives the most realistic results. For the final numerical results we include therefore the $\pi \pi$ and the $\pi K$ density expansion from Eqs. (8.31) and (D.37). The dashed-dotted lines in Fig. 8.14 show electric mass and normalized charge fluctuations $D_{S}$ from Eq. (8.4) for the sum of the two density expansions. At order $n^{2}$, there are additional photon selfenergy diagrams with $(\gamma) \gamma \rho \rho$ and $\rho \rho \pi \pi$ vertices from Fig. 8.4. As discussed at the end of Sec. 8.6.1 these contributions are not included in the density expansion but a consequence of the $\rho$ being introduced as a heavy gauge field. The same applies to the $K^{*} K^{*} \pi \pi$ diagram (d) from Eq. (D.35). Thus, we include these additional contributions for $\left\langle\delta Q^{2}\right\rangle$ and $S$.

[^6]

Figure 8.14: Final results for charge fluctuations (electric mass) and $D_{S}$. Results of the Bose-Einstein density expansions with the dashed-dotted lines. Adding $\rho \rho \pi \pi$ and $K^{*} K^{*} \pi \pi$ contributions, the resummations, and free mesons up to 1.6 GeV , the results are indicated with the dashed lines. For comparison, $m_{\mathrm{el}}^{2}$ and $D_{S}$ from free mesons alone, without any interactions, are also plotted (dotted lines).

At higher orders in density one has to rely on resummation schemes. Including the resummations in the final results does not to lead to double counting: Resummations at order $g^{4}$ and upwards in the strong coupling correspond to diagrams with three and more loops and, thus, to contributions higher than quadratic in density. We include (with $g^{4}$ and higher) the summations (n), (r), and ( t ) from Sec. 8.7. Note that for $m_{\mathrm{el}}$ there is a partial cancellation of sizable contributions from the resummations and the $(\gamma) \gamma \rho \rho$, $\rho \rho \pi \pi, K^{*} K^{*} \pi \pi$ diagrams.

In order to obtain a more realistic picture, we also include as free gases all mesons from the PDG [57] which have not been considered so far, up to a mass of 1.6 GeV . Note that we do not add free mesons that have the same quantum numbers as the density expansions namely $\sigma(600), \rho(770)$, $\kappa(800), K^{*}(892), K_{0}^{*}(1430)$, and $K_{2}^{*}(1430)$. We have seen in Sec. 8.6.1 that their contribution via phase shifts in the density expansions is roughly of the size as if they had been included as free particles. Adding all contributions mentioned, the results are indicated with the dashed lines in Fig. 8.14.

Compared to the density expansions the final results do not change much. The influence of heavier mesons than those considered in this study is, thus, well controlled. Many of the heavier resonances that have been added here as free gases are axials which decay into three particles. To include them
in a density expansion would require the consistent treatment of three body correlations.

Concluding, we can assign $D_{S} \simeq 0.09$ for temperatures $120<T<200$ MeV which coincides (incidentally) quite well with the result if one simply considers free, noninteracting, mesons up to masses of 1.6 GeV . The latter case is indicated with the dotted lines in Fig. 8.14.

Theoretical uncertainties in the present study arise from the omission of diagrams such as the (small) eye shaped diagram mentioned in Ref. [263] already at $g^{4}$. Furthermore, both resummations and density expansions are incomplete as they only partly include the in-medium renormalization of the resonances which drive the meson-meson scattering, such as the $\sigma(600)$, $f_{0}(980)$, or the $\rho(770)$ itself $[279,283]$. In this context one can think of a more complete microscopical model: We have found in Sec. 8.5 and 8.6.1 that unitarity and a good description of the vacuum data up to high energies and in all partial waves are important. Such models exist, e.g., the chiral unitary approach from Ref. [16]. The medium implementation of such a model has been done in a different context, see $[286,287]$ and references therein. A generalization of the virial expansion from Ref. [288] to finite chemical potential and including Bose-Einstein statistics, as carried out here, would be feasible in principle. Such an ansatz [280] would allow to take simultaneously into account the medium renormalization of the (dynamically generated) resonances and the calculation of the grand canonical partition function at finite $\mu$ as needed for a calculation of $m_{\mathrm{el}}^{2}$.

### 8.9 Summary and Conclusions

For an estimate of charge fluctuations (CF) in the hadronic phase of heavy ion collisions, we have calculated the effect of particle interactions. For the perturbative expansion up to two thermal loops, the $\pi \pi$ interaction has been described by vector meson exchange. The correlations induced by a dynamical $\rho$ have been found significant by comparing to an effective theory where the $\rho$ is frozen out.

The photon self energies are charge conserving and shown to be equivalent to the loop expansion of the grand canonical partition function at finite
chemical potential. We have pointed out that the inclusion of imaginary parts is essential for a proper description of the thermodynamics, especially if resonant amplitudes are involved. To second order in the density, it has been possible to include Bose-Einstein statistics in the conventional virial expansion. This "density expansion" can change the conventional results significantly. Moreover, for real amplitudes, we could show the equivalence of the loop expansion and the density expansion at all temperatures. However, the inclusion of unitary (complex) amplitudes is more straightforward in the density (virial) expansion. To the extend that two-particle correlations are dominant, the density expansion with Bose-Einstein statistics is, thus, the method of choice as it provides the same statistics as the thermal loop expansion and unitarity is automatically implemented by the use of realistic phase shifts.

For an estimate of three- and higher particle correlations, a variety of summation schemes has been presented, all of which tend to soften the large first order correction of the thermal loop expansion. For the CF, higher order corrections have a large influence whereas higher orders for the entropy are small.

For the CF over entropy, $D_{S}$, it has been shown that the influence of heavy particles beyond the interactions considered are well under control; a final value of $D_{S} \simeq 0.09$ has been found for temperatures $120<T<200$ MeV . This result agrees quite well with the outcome from the free resonance gas, supporting the notion that resonant amplitudes dominate the thermodynamics. As lattice gauge calculations with realistic quark masses become available it would be interesting to see at which point these start to significantly deviate from a hadron gas.

## Chapter 9

## Charge susceptibility in a hot pion gas with unitarized chiral interaction

Charge susceptibility, or charge fluctuations (CF), have been proposed as a possible observable sensitive to the quark gluon plasma (QGP) in heavy ion collisions. For a calculation of this observable from the hadronic phase, we use chiral perturbation theory in a unitarized version which has delivered a good vacuum data description of $\pi \pi$ scattering. Here, we concentrate on the spin zero, charge two channel. A density expansion, introduced in Sec. 8.6 , is used that goes beyond the conventional virial expansion and delivers quite different results. The rescattering loops of the unitary interaction allow also for inclusion of corrections beyond second order in density. As it is of interest in the framework of lattice calculations, we also study the scaling of the susceptibility with the pion mass.

### 9.1 Introduction

Charge fluctuations (CF) may serve as a possible signal of a deconfined state of matter in heavy ion collisions as proposed some time ago [249, 250]. We refer to chapter 8 and the Introduction in Sec. 1.1 for an overview. The charge fluctuations $\left\langle\delta Q^{2}\right\rangle$ are closely related to the electric mass $m_{\mathrm{el}}^{2}$ and the
charge susceptibility $\chi_{Q}$ by

$$
\begin{equation*}
\left\langle\delta Q^{2}\right\rangle=-V T \chi_{Q}=V T m_{\mathrm{el}}^{2} \tag{9.1}
\end{equation*}
$$

where $V$ and $T$ are the volume and temperature, respectively. In the following we will use all three terms synonymously depending on the context. In lattice calculations, the susceptibility is often the preferred quantity.

In Ref. [263] corrections to the electric mass in the hadronic phase have been calculated. In chapter 8 (see Ref. [8]) the effort has been retaken and further corrections and refinements have been implemented using a $\rho$ model, a density expansion, and resummations schemes.

Unitarity in the meson-meson interaction is important (see chapter 8, Ref. [8]). For the temperatures of interest of $T=100-200 \mathrm{MeV}$, the thermal loops pick up momenta in the $\pi \pi$ amplitude which are far beyond the applicability of chiral perturbation theory. At these momenta, the point like interaction of the lowest (or higher) order chiral Lagrangian leads to unitarity violations which distort the numerical values of the electric mass easily by a factor of three or four. A possible way out is the inclusion of explicit resonances which preserves unitarity of the interaction and delivers realistic amplitudes even at high momenta. In chapter 8 the $\rho$ exchange is the driving interaction for the isospin $I$, spin $\ell,(I, \ell)=(1,1)$ channel. With this, two loop photon self energies $\Pi_{00}\left(\omega_{\mathbf{k}}=0, \mathbf{k} \rightarrow 0\right)$ are calculated, or, alternatively, the grand canonical partition function $\log Z(\mu)$ at finite chemical potential $\mu$. Both ways are known to be equivalent for free gases [260,263],

$$
\begin{equation*}
m_{\mathrm{el}}^{2}=\Pi_{00}\left(\omega_{\mathbf{k}}=0, \mathbf{k} \rightarrow 0\right)=\frac{e^{2}}{\beta V}\left(\frac{\partial^{2} \log Z(\mu)}{\partial \mu^{2}}\right) \tag{9.2}
\end{equation*}
$$

but also hold on the perturbative level as shown (to our knowledge, for the first time) in Sec. 8.5 [8].

The virial expansion offers an alternative formalism to evaluate thermodynamical observables such as CF. An extension of the virial expansion has been introduced in Sec. 8.6, which is called Bose-Einstein density expansion or simply density expansion in chapter 8 and also here. This comprises the inclusion of Bose-Einstein statistics in the asymptotic states of the mesonmeson interaction which is taken into account by a summation over exchange
diagrams [275]. With this extension, it has been shown in Sec. 8.6.1 [8] for the interactions at hand that the two loop results and the novel density expansion are equivalent,

$$
\begin{equation*}
m_{\mathrm{el}}^{2}=\frac{e^{2}}{\beta}\left(\frac{\partial^{2} B_{2}^{\mathrm{Bose}}(\mu)}{\partial \mu^{2}}\right) . \tag{9.3}
\end{equation*}
$$

Here, $B_{2}^{\text {Bose }}$ is the coefficient of the density expansion that corresponds to the lowest virial coefficient, $B_{2}^{\text {Boltz }}$.

Unitarity is many times easier to incorporate in the density expansion, as many microscopical theories are formulated in terms of phase shifts in partial waves. In thermal loops it can be difficult to disentangle partial waves; e.g., for the $\rho$ exchange from Sec. 8.5, the $\rho$ in the two-loop diagrams is not only in the $s$ channel of $\pi \pi$ interaction but also in $t$ and $u$ (see Figs. 8.3, 8.4) so that the $\rho$ even contributes to other partial waves than $(I, \ell)=(1,1)$. Therefore, it is clearer and more economic to prefer the density expansion over the thermal loop expansion for the calculation of the charge susceptibility and other thermodynamical observables; thanks to Eq. (9.3) it is ensured that the density expansion includes the statistical features of thermal loops.

In chapter 8 it was found that large discrepancies between a thermal loop expansion and a virial expansion, first noticed in Ref. [263], are due to the fact that the loop expansion violates unitarity and ignores the $\rho$ whereas the virial expansion, using phase shifts, respects unitarity and includes automatically the phenomenology through the use of realistic phase shifts. However, the thermal loop expansion has also an advantage: In resummation schemes, higher order effects in density and interaction can be taken into account; these contributions are considerable as we have seen in Sec. 8.7 and both density and virial expansion are quadratic in density. An interesting way to include the respective advantages of the two schemes is offered by $U \chi P T$ : the intermediate states of the rescattering series, that unitarizes the amplitude, allow for an in-medium dressing by using thermal loops, and, thus, higher order effects in density are taken into account. In other words, the microscopical rescattering model has the major advantage that chiral symmetry is implemented perturbatively and the effect of the hot mesonic matter and finite chemical potential can be straightforward included in the rescattering
loops by using standard methods of thermal field theory. For the inverse amplitude method the temperature part has been included in a different physical context in Refs. [286-288,297] (and references therein).

We will implement the medium dressings not only for finite temperature, as it has been done in previous studies [288], but also for finite chemical potential $\mu$ which is a necessary ingredient in the calculation of the charge susceptibility even if the actual chemical potential is zero. In a meson gas with $\mu \neq 0$, isospin is evidently broken; e.g., the energy difference for a $\pi^{+}$ compared to a $\pi^{-}$, both at rest, is 100 MeV for $\mu=50 \mathrm{MeV}$.

We use a model for $\pi \pi$ interaction which unitarizes the $\pi \pi$ amplitude by the use of the Bethe-Salpeter equation [39]. Originally introduced for the isoscalar $(0,0)$ sector, the coupled channel chiral unitary approach successfully generates the $\sigma(600)$ and $f_{0}(980)$ by multiple rescattering of the pions. The amplitude depends very weakly on the only free parameter which is a cutoff of the intermediate meson-meson loops. By using the lowest order chiral Lagrangian as the driving interaction, the scalar-isoscalar sector is very well described up to $s^{1 / 2}=1.2 \mathrm{GeV}$ which means a major breakthrough in the understanding of this sector (see also [39,42]). The model of [39] delivers also a sufficient description of phase shifts in the $(I, \ell)=(2,0)$ channel which is the one we are currently interested in.

There are other channels besides the $\rho$ channel in $\pi \pi$ scattering. The isoscalar $(I, \ell)=(0,0)$ channel exhibits two resonant structures, the $\sigma(600)$ and the $f_{0}(980)$. Although this channel is interesting for other reasons which concern, e.g., the nature of the $\sigma$, its contribution to the charge susceptibility is sub-leading: In a conventional virial expansion it gives zero contribution. Going beyond the Boltzmann limit by using the density expansion from Eq. (8.31), there is a small contribution which, however, becomes even smaller considering that the spin and isopin degeneracies of this channel are both one.

The third relevant channel is in isospin, spin $(I, \ell)=(2,0)$. This repulsive channel, due to its multiplicity of five, gives a large contribution to the charge susceptibility that is of the same order of magnitude as the $(1,1)$ channel and with opposite sign. Therefore, we concentrate here on the $(2,0)$ channel. The isospin breaking makes it necessary to work in charge channels rather than
isospin channels. We will concentrate on $\pi \pi$ scattering in the net charge $|C|=2$ channels which in the isospin symmetric case is pure isospin two. Other isospin two components in $|C|=1$ and $|C|=0$ are left to a more systematical study.

For lattice calculations the scaling of the amplitudes with the pion mass is of interest. Therefore, we will give some results for typical lattice pion masses of $m_{\pi} \sim m_{\rho} / 2$ which can be compared to lattice results in the net charge $|C|=2$ channel.

### 9.2 Unitarized chiral perturbation theory at finite temperature

In this section we shortly revise the necessary ingredients from unitarized chiral perturbation theory ( $U \chi P T$ ) and derive the medium dressing of the loops. There are various theoretical schemes for the unitarization of amplitudes [39, 42]. Besides the inverse amplitude method and the $N / D$ method [17], the Bethe-Salpeter equation (BSE) offers a transparent way of obtaining a unitarized amplitude which fulfills

$$
\begin{equation*}
\operatorname{Im} T^{-1}(\sqrt{s})=-\sigma(\sqrt{s}) \tag{9.4}
\end{equation*}
$$

with the phase space $\sigma=-q_{\text {c.m. }} /(16 \pi \sqrt{s})$ in the present normalization for $s$-wave scattering. With $V$ being the interaction kernel, the unitarized amplitude $T$ reads

$$
\begin{equation*}
T(\sqrt{s})=\left(1-V G_{\mathrm{vac}}\right)^{-1} V \tag{9.5}
\end{equation*}
$$

where $G_{\text {vac }}$ is the meson-meson loop function [39]

$$
\begin{equation*}
G_{\mathrm{vac}}=\int_{0}^{\Lambda} \frac{q^{2} d q}{(2 \pi)^{2}} \frac{\omega_{1}+\omega_{2}}{\omega_{1} \omega_{2}\left(s-\left(\omega_{1}+\omega_{2}\right)^{2}+i \epsilon\right)} \tag{9.6}
\end{equation*}
$$

and $\omega_{1}^{2}=q^{2}+m_{1}^{2}, \omega_{2}^{2}=q^{2}+m_{2}^{2}$ are the meson energies with mass $m_{1}$ and $m_{2}$. For a consideration of the $(0,0)$ sector coupled channel dynamics of the full $0^{-}$pseudoscalar meson octet is necessary; e.g., the dominant decay channel of the $f_{0}(980)$ is in $K \bar{K}$. In the present situation, we are interested


Figure 9.1: Unitarization of the $\pi \pi$ amplitude using the Bethe-Salpeter equation. The loop function guarantees analyticity of the amplitude and in the resummation it provides the right-hand cut required by unitarity.
in $(I, \ell)=(2,0)$ with a net charge of $|C|=2$, and the two only channels, $\pi^{+} \pi^{+}$and $\pi^{-} \pi^{-}$, are decoupled. Theoretically, $K \bar{K}$ can couple in $|C|=2$ but is assumed to be thermally suppressed. In Fig. 9.1 a diagrammatic representation of the Bethe-Salpeter equation for meson-meson scattering is shown.

For a more comprehensive introduction of $U \chi P T$, see Ref. [17] and the remarks in Sec. 1.2. In the present study, we use a cutoff for the regularization of the intermediate $\pi \pi$ two-particle propagator $G$ from Eq. (9.5). Alternatively, dimensional regularization can be employed which for the $(2,0)$ channel delivers slightly better results for the phase shift compared to data analyses; however, the dependence of the charge susceptibility on the regularization will be very weak.

In Ref. [39] the kernel is taken from the lowest order chiral Lagrangian which is enough for the excellent data description of the $(I, \ell)=(0,0)$ channel up to $\sqrt{s}=1.2 \mathrm{GeV}$ which is also the procedure followed here. The kernel can be refined by explicitly taking into account the NLO chiral meson-meson Lagrangian $\mathcal{L}_{4}^{(\pi \pi)}$ [42] as it has been very successfully carried out using the inverse amplitude method [293]. However, using $\mathcal{L}_{4}^{(\pi \pi)}$ in finite temperature calculations bears some risks as has been noticed in a $\mathcal{O}\left(p^{4}\right)$ calculation of $m_{\mathrm{el}}^{2}$ in Ref. [263]; see also a discussion in Sec. 8.6.2. The uncertainties in such a calculation come from the high momenta of the $\pi \pi$ scattering amplitude that are picked up by the thermal loops where chiral perturbation theory in its original form is no longer valid and the $\mathcal{O}\left(p^{4}\right)$ amplitude diverges.

The amplitude is very weakly dependent on the value of $\Lambda$ as has been seen in Ref. [39]. There, $\Lambda=1.03 \mathrm{GeV}$ has been found as an optimal value for a description of the $(I, \ell)=(0,0)$ sector. For the $(I, \ell)=(2,0)$ partial wave, in which we are interested here, this value has to be fine tuned and
we find $\Lambda=1.4-1.5 \mathrm{GeV}$ a value that gives a good data description. With $\Lambda=1.03$, there is a $20 \%-30 \%$ deviation from data at high energies.

Note that when including $\mathcal{L}_{\pi \pi}^{(4)}$ in the calculation as in Ref. [42], one obtains a good data description in both partial waves with a unique set of low energy constants. However, we have found good reasons above and in Sec. 8.6.2 why the use of $\mathcal{L}_{\pi \pi}^{(4)}$ bears risks in a calculation of $m_{\text {el }}^{2}$ at finite temperature. Therefore, in this study we regard the effect of higher order chiral Lagrangians as being absorbed in the value of the cutoff which is a standard way of proceeding [39]. From this consideration it is clear that two different values of $\Lambda$ in different partial waves are necessary.

Another improvement of the interaction kernel in the vacuum model can be achieved by including contributions from the unphysical, left hand cut. In [17] a test was done of the contribution of the left-hand cut in mesonmeson scattering with the conclusion that the contribution is small. But more important: It is weakly energy-dependent in the region of physical energies. This is the key to the success of the method explored here, since any constant contribution in a certain range of energies can be accommodated in terms of the cutoff $\Lambda$.

### 9.2.1 Extension to finite $T$ and $\mu$

As mentioned in Sec. 9.1, it is necessary to work in charge channels rather than in isospin channels because isospin is manifestly broken at finite chemical potential. The isospin 2 amplitude, which is the one of interest here, is the only relevant one contributing to $\pi \pi$ scattering with a net charge of two. Additionally, channels with a net charge of one and zero also carry components from isospin two. In here, we will restrict to $\pi^{+} \pi^{+} \rightarrow \pi^{+} \pi^{+}$and $\pi^{-} \pi^{-} \rightarrow \pi^{-} \pi^{-}$scattering as these two channels give the strongest contribution to the fluctuations in isospin two. For an estimate, we use the density expansion from Eq. (8.31) and phase shifts from Ref. [277]. With this, the electric mass from $|C|=1$ and $|C|=0$ roughly behaves like $1: 4$ and $1: 50$ compared to $m_{\text {el }}^{2}$ from $|C|=2$, respectively. In the present study, we are rather interested in the general consequences of $U \chi P T$ for the electric mass. In a more systematical study, also other charge channels should be included.

The on-shell $T$-matrix at tree level for both $\pi^{+} \pi^{+} \rightarrow \pi^{+} \pi^{+}$and $\pi^{-} \pi^{-} \rightarrow$ $\pi^{-} \pi^{-}$from the lowest order chiral Lagrangian is given by

$$
\begin{equation*}
T_{\text {tree }}=\frac{s-2 m_{\pi}^{2}}{f_{\pi}^{2}} \tag{9.7}
\end{equation*}
$$

When summing up the rescattering series in the BSE, Eq. (9.5), one has to take into account a symmetry factor of $1 / 2$ for every loop of identical particles so that the resummed $T$-matrix is given by

$$
\begin{equation*}
T(\sqrt{s})=\left(1-\frac{T_{\text {tree }}}{2} G\right)^{-1} T_{\text {tree }} \tag{9.8}
\end{equation*}
$$

The loop function $G_{\text {vac }}$ is modified in the thermal heat bath and replaced by

$$
\begin{align*}
G=-\int_{0}^{\infty} \frac{q^{2} d q}{8 \pi^{2} \omega_{1} \omega_{2}} & {\left[\frac{1+n\left[\omega_{1}-\mu_{1}\right]+n\left[\omega_{2}-\mu_{2}\right]}{\omega_{1}+\omega_{2}-\mu_{1}-\mu_{2}-p^{0}}\right.} \\
& +\frac{1+n\left[\omega_{1}+\mu_{1}\right]+n\left[\omega_{2}+\mu_{2}\right]}{\omega_{1}+\omega_{2}+\mu_{1}+\mu_{2}+p^{0}} \\
& +\frac{-n\left[\omega_{1}-\mu_{1}\right]+n\left[\omega_{2}+\mu_{2}\right]}{\omega_{1}-\omega_{2}-\mu_{1}-\mu_{2}-p^{0}} \\
& \left.+\frac{-n\left[\omega_{1}+\mu_{1}\right]+n\left[\omega_{2}-\mu_{2}\right]}{\omega_{1}-\omega_{2}+\mu_{1}+\mu_{2}+p^{0}}\right] \tag{9.9}
\end{align*}
$$

which has been derived using standard techniques of thermal field theory in the imaginary time formalism. The labels 1,2 refer to the meson $i$ with chemical potential $\mu_{i}$. In Eq. (9.9), $p^{0}=2 \pi i n T$ is the zero component of the external momentum which is continued to the real axis by means of the prescription $p^{0} \rightarrow \sqrt{s}+i \epsilon$.

The loop function in Eq. (9.9) contains both matter and vacuum part. The latter is given by the first and second term and by removing the BoseEinstein factors $n$ from the numerators; with the analytic continuation to real values of $\sqrt{s}$ and for $\mu_{1}=\mu_{2}=T=0$, Eq. (9.9) is evidently identical to the scalar loop function in Eq. (9.6). For the temperature part we note that in the limit $\mu_{1}, \mu_{2} \rightarrow 0$ the ordinary s-channel thermal loop function is recovered which has been calculated e.g. in Refs. [286, 297].

The analytic structure of $G$ from Eq. (9.9) for $m_{1}=m_{2}$ which is the case considered here is particularly simple. The first and second term bear
a right-hand cut for vacuum and matter part, starting at $\pi \pi$ threshold, and the third and fourth term have no discontinuity above threshold; in fact, their sum is zero at $\mu=0$ (still, they contribute to the charge susceptibility). Below threshold, there are additional cuts which play no role in the present case; for a thorough discussion on the issue, see Ref. [294].

The finite ( $T, \mu$ ) amplitude for $\pi^{+} \pi^{+} \rightarrow \pi^{+} \pi^{+}$plus $\pi^{-} \pi^{-} \rightarrow \pi^{-} \pi^{-}$scattering is now given by Eq. (9.8) with $G$ from Eq. (9.9) and the tree level amplitude from Eq. (9.7). One might ask why the kernel in Eq. (9.7) is still independent of the chemical potential, as all zero components of momenta are shifted by $\pm \mu$, in propagators and also in vertices [263,265]; this has to do with the implementation of the amplitude in the density expansion as will be discussed at the end of Sec. 9.2.2.

Unitarity at finite $\mu$ and $T$ can now be re-written as

$$
\begin{equation*}
\operatorname{Im} T^{-1}=-\sigma_{T} \tag{9.10}
\end{equation*}
$$

with $T$ from Eq. (9.8) and the thermal phase space

$$
\begin{align*}
\sigma_{T}=-\frac{q_{(\mu)}}{16 \pi\left(\sqrt{s}+\mu_{1}+\mu_{2}\right)} & \left(1+\frac{1}{\exp \left[\beta\left(\sqrt{m_{1}^{2}+q_{(\mu)}^{2}}-\mu_{1}\right)\right]-1}\right. \\
& \left.+\frac{1}{\exp \left[\beta\left(\sqrt{m_{2}^{2}+q_{(\mu)}^{2}}-\mu_{2}\right)\right]-1}\right) \tag{9.11}
\end{align*}
$$

where

$$
\begin{equation*}
q_{(\mu)}=\frac{\lambda^{1 / 2}\left(\left(\sqrt{s}+\mu_{1}+\mu_{2}\right)^{2}, m_{1}^{2}, m_{2}^{2}\right)}{2\left(\sqrt{s}+\mu_{1}+\mu_{2}\right)} \tag{9.12}
\end{equation*}
$$

is the c.m. momentum shifted by the presence of the chemical potential. From the above formalism the isospin breaking at finite $\mu$ becomes obvious; we can not combine the $\pi^{+} \pi^{+} \rightarrow \pi^{+} \pi^{+}$and $\pi^{-} \pi^{-} \rightarrow \pi^{-} \pi^{-}$to isospin 2 but have to treat them separately.

The discussion from Ref. [294] on discontinuities, or cuts, on the real axis can be directly applied. We can interpret the appearance of the statistical factors as matter induced stimulated emission and absorption by rewriting $1+n_{1}+n_{2}$ above, where $n_{1}$ and $n_{2}$ are the second and third term in the


Figure 9.2: Left: Real and imaginary part of vacuum propagator (dasheddotted line, dotted line, respectively) and medium $\pi \pi$ propagator (solid line, dashed line) at $\mu=0 \mathrm{MeV}, T=190 \mathrm{MeV}$. Right: The $\pi^{+} \pi^{+}$medium propagator at $\mu=80 \mathrm{MeV}, T=190 \mathrm{MeV}$.
bracket in Eq. (9.11): It is $1+n_{1}+n_{2}=\left(1+n_{1}\right)\left(1+n_{2}\right)-n_{1} n_{2}$. The first term can be interpreted as the probability for stimulated emission into $\pi \pi$ at one vertex of the loop and $-n_{1} n_{2}$ as the stimulated re-absorption on the other vertex.

In Fig. 9.2 the vacuum and matter propagators are shown at $\mu=0 \mathrm{MeV}$ and $\mu=80 \mathrm{MeV}$. The imaginary part of the medium propagator is thermally suppressed at larger $\sqrt{s}$ which simply reflects the fact that the stimulated emission and absorption discussed before is reduced because due to the thermal distribution there are less high energy pions. A finite $\mu$ manifests itself mainly in a shift of the threshold cusp of the propagator as the right hand side of Fig. 9.2 shows.

Phase shifts, assuming inelasticity zero, are obtained in a standard way from Eq. (9.8). For the reaction $\pi^{+} \pi^{+} \rightarrow \pi^{+} \pi^{+}$we obtain

$$
\begin{equation*}
\delta(\sqrt{s}, T, \mu)_{\pi^{+} \pi^{+} \rightarrow \pi^{+} \pi^{+}}=\frac{1}{2} \arctan \left(\frac{-\operatorname{Re} T}{\frac{8 \pi \sqrt{s}}{q_{c . m .}}+\operatorname{Im} T}\right)-\pi \theta(-\operatorname{Re} T) \tag{9.13}
\end{equation*}
$$

with the step function $\theta$ and the $\pi \pi \mathrm{c} . \mathrm{m}$. momentum $q_{c . m}$. For the reaction $\pi^{-} \pi^{-} \rightarrow \pi^{-} \pi^{-}$, the phase shift is $\delta(\sqrt{s}, T, \mu)_{\pi^{-} \pi^{-} \rightarrow \pi^{-} \pi^{-}}=\delta(\sqrt{s}, T,-\mu)_{\pi^{+} \pi^{+} \rightarrow \pi^{+} \pi^{+}}$. Although we use the same symbol $T$ for $T$-matrix and temperature in Eq. (9.13) and in the following, confusions should be excluded. We have explicitly


Figure 9.3: Vacuum and medium phase shifts at different $T$ and $\mu$ for the reaction $\pi^{+} \pi^{+} \rightarrow \pi^{+} \pi^{+}$. The phase shift analyses are from Ref. [295] (dots) and [296] (triangle up for fit A, crosses for fit B).
checked that the $T$-matrix fulfills the thermal unitarity condition from Eq. (9.11) so that Eq. (9.13) is well defined.

In Fig. 9.3 the vacuum phase shift is plotted with the solid line together with some results of phase shift analyses. For the temperature range of interest, $T=100-200 \mathrm{MeV}$, the relevant part in energy is up to $\sqrt{s}$ around $800-1000 \mathrm{MeV}$. There is some discrepancy for higher momenta which, however, is of no relevance because the final results will be almost independent of the actual value of the cutoff $\Lambda$ that is used as fit parameter to the data. Using dimensional regularization will help at this point and is a possible improvement.

Switching on the temperature, the phase shift is lowered (dashed line). As also shown with the dotted line, a cutoff of $\Lambda=1.3 \mathrm{GeV}$ only moderately changes the result. At finite $\mu$ the phase shift is significantly different as shown for $\mu=80 \mathrm{MeV}$ with the dashed-dotted line.

### 9.2.2 Density expansion

In Sec. 8.6 a density expansion has been proposed that takes into account the Bose-Einstein statistics of the asymptotic states of two-body scattering. This is an extension of the conventional viral expansion and takes into account the so-called exchange-diagrams from Ref. [275] where the original $S$-matrix expansion in statistical mechanics has been developed. The analog quantity to the second virial coefficient $B_{2}^{\text {Boltz }}(\mu)$ reads, in the notation from Sec. 8.6 (Ref. [8]),

$$
\begin{align*}
B_{2}^{(\pi \pi), \text { Bose }}(\mu)= & \frac{\beta}{4 \pi^{3}} \int_{2 m_{\pi}}^{\infty} d E \int_{-1}^{1} d x \int_{0}^{\infty} d P \frac{E P^{2}}{\sqrt{E^{2}+P^{2}}} \\
\times & {\left[\delta(\sqrt{s}, T, \mu)_{\pi^{+} \pi^{+} \rightarrow \pi^{+} \pi^{+}} n\left[\omega_{1}+\mu\right] n\left[\omega_{2}+\mu\right]\right.} \\
& \left.+\delta(\sqrt{s}, T, \mu)_{\pi^{-} \pi^{-} \rightarrow \pi^{-} \pi^{-}} n\left[\omega_{1}-\mu\right] n\left[\omega_{2}-\mu\right]\right] . \tag{9.14}
\end{align*}
$$

In here, we have reduced the expression from Eq. (8.31) to account for the only two channels of this study, $\pi \pi$ with charge $|C|=2$, with the phase shifts given in the last section. In contrast to Eq. (8.31) we have now two independent phase shifts in $|C|=2$ due to the explicit isospin breaking. The boosted Bose-Einstein factors in Eq. (9.14) which arise after summations over exchange diagrams are given by Eq. (8.32); see Sec. 8.6 for a thorough discussion of the density expansion.

One point left for discussion from Sec. 9.2.1 is the question why in the kernel Eq. (9.7) the chemical potential is not present; it is well-known [263, 265] that at finite $\mu$ the zero momentum not only of the propagator but also of the vertices has to be shifted according to $p^{0} \rightarrow p^{0} \pm \mu$ depending on the charge state of the particles. The shift in the vertices is analogous to a photon insertion in the vertex as can be seen, e.g., in Sec. $8.5[8]$ and, thus, important for the preservation of gauge invariance.

However, in Eq. (9.7) the chemical potential is simply not present because it has been re-absorbed in the Bose-Einstein factors from Eq. (8.32). This can be shown in an explicit calculation, but an analogy might clarify it faster: Using the tree level amplitude from the LO chiral Lagrangian in the density


Figure 9.4: Charge fluctuations for different $\mu$ and cutoffs $\Lambda$.
expansion, the result is identical to the two loop, eight-shaped thermal loop (see, e.g., diagram (a) of Fig. 8.5) with the same interaction as has been shown in Sec. 8.6.1. In a summation over the Matsubara frequencies of that diagram, the residues of the poles in the $p^{0}$ plane, which are shifted by $\pm \mu$, determine the value that the $p^{0}$-components of the vertices take. This cancels precisely the $\mu$-dependence of the vertices. The $\mu$ dependence appears only in the statistical factors $n$; see Eq. (D.29) for an explicit example of the contour integration method. This is why the density expansion in Eq. (9.14) requires interactions with zero-components in the vertices that are not shifted by $\pm \mu$. Of course, for intermediate loops, the situation is different, and the loop function $G$ from Eq. (9.9) indeed carries an explicit $\mu$ dependence.

### 9.2.3 The charge susceptibility

In Fig. 9.4 the electric mass is plotted. For zero chemical potential, we have plotted $m_{\mathrm{el}}^{2}$ for two quite different cutoffs $\Lambda$ and the result stays practically the same (solid and dashed black lines). First, this is due to the weak cutoff dependence of the chiral unitary method which we have seen already in Fig. 9.3. Second, this is because the $\mu$ dependence of $m_{\mathrm{el}}^{2}$ on the vacuum loop is
much weaker than the $\mu$ dependence on the thermal part of the loop which does not depend on $\Lambda$. The electric mass which is given by the second derivative with respect to $\mu$ is then practically independent of the cutoff.

One can consider the limit of vanishing real part of the $\pi \pi$ loop function and also set the matter part to zero (see Eq. (9.9)). It is easy to see that the $S$-matrix with this simplification reads

$$
\begin{equation*}
S_{K}(\sqrt{s})=\frac{1+i Q K}{1-i Q K}, \quad K=-\frac{T_{\text {tree }}}{16 \pi \sqrt{s}} \tag{9.15}
\end{equation*}
$$

with the tree level amplitude $T_{\text {tree }}$ from Eq. (9.7) and the c.m. momentum $Q$ from Eq. (8.32). Thus, the Bethe-Salpeter equation in the limit considered is equivalent to a $K$-matrix approach as expected. Numerically one obtains the curve labeled as "K-matrix result" in Fig. 9.4. The result is close to the solid curve and confirms the findings from above that the electric mass depends only weakly on the real part of the $\pi \pi$ loop.

One of the novel ingredients in the present study is the introduction of the electric chemical potential $\mu$ in the loops. If this dependence is removed, the only $\mu$ dependence is given by the Bose-Einstein factors from Eq. (8.32). The resulting electric mass at $\mu=0 \mathrm{MeV}$ is plotted with the dashed dotted line and shows a $15 \%$ deviation from the full result. The electric mass at $\mu=80 \mathrm{MeV}$, which is also shown in Fig. 9.4, is quite different from the $\mu=0 \mathrm{MeV}$ result.

For comparison, we also plot the case when, instead of the unitarized amplitude, only the tree level term at $1 / f_{\pi}^{2}$ is used (gray line). This is the kernel Eq. (9.7) of the Bethe-Salpeter resummation and corresponds to the first diagram on the right hand side of Fig. 9.1. It violates unitarity and for an implementation we have to modify Eq. (9.14) slightly. Comparing the present normalization of the $T$-matrix, $S=1-\frac{i Q}{8 \pi \sqrt{s}} T$, with the $S$-matrix parametrization via phase shifts, $S=e^{2 i \delta}$, the replacement in Eq. (9.14) is, up to linear order in the $T$-matrix and exact for $T$ real,

$$
\begin{equation*}
\delta_{\ell}^{I} \hat{=}-\frac{Q}{E}\left(T_{\ell}^{I}+T_{\ell}^{I, \dagger}\right) \tag{9.16}
\end{equation*}
$$

where the connection between $T^{2}$ and $T_{0}^{2}$ is given by an additional factor of $32 \pi$. See also Appendix D.2.1 for the normalizations of amplitudes. With


Figure 9.5: Electric mass or charge fluctuations at $T=190 \mathrm{MeV}$ as a function of $\mu$.
this replacement, one inserts the tree level amplitude from Eq. (9.7) in Eq. (9.14). As Fig. 9.4 shows the result can be twice as large as from the unitary amplitude which again shows the relevance of unitarity for temperatures $T>100 \mathrm{MeV}$.

In Fig. 9.5 the electric mass as a function of $\mu$ is plotted at a fixed temperature of $T=190 \mathrm{MeV}$. Again, we also show the outcome when the $\mu$ dependence of the thermal loops is removed in the way described above. In the latter case, the result can change up to a factor of two as the figure shows (dashed vs. solid black line). Moreover, at $\mu=m_{\pi}$, when BoseEinstein condensation sets in, the susceptibility becomes singular. As in Fig. 9.4 we also show the electric mass using only the (unitarity violating) tree level $\pi \pi$ amplitude (gray line). Again, the result is quite different from the unitary interaction.

In Fig. 9.6 some simplifications of the current model are plotted in order to obtain an overview of the relevance of the different ingredients of the model. If one replaces the density expansion (Bose-Einstein) by a conven-


Figure 9.6: Electric mass in various limits of the present model. For comparison, also the results of the full model, identical to the curves from Fig. 9.4, are plotted (Solid black and solid gray lines).
tional virial expansion, i.e., using Eq. (8.33) instead of Eq. (9.14), the result changes by up to a factor of two (dashed black line vs. solid black line). If one removes additionally the temperature dependence of the thermal loops, i.e., using only the vacuum part of the propagator from Eq. (9.9), the result gets also changed, but more moderately (dotted black line vs. dashed black line).

The more moderate change in the latter case can be understood in terms of the loop resummation which is included from the BSE (9.8): Replacing the density expansion by the conventional virial expansion affects the tree level result at $1 / f_{\pi}^{2}$ of $\pi \pi$ scattering, which is included in the rescattering series as shown in Fig. 9.1. However, removing the thermal part of the loop function occurs first at $1 / f_{\pi}^{4}$ and the change is more moderate. Note, however, that this is no longer true when one would get in the vicinity of a dynamically generated resonance as it is the case in the $(I, \ell)=(0,0)$ channel. There, the thermal loops can be much larger than tree level. It is in fact the Bethe-Salpeter equation which renders the amplitude finite in the resonance region.


Figure 9.7: The scaling of the phase shift with the pion mass $m_{\pi}=m_{\rho} / 2$ (left) and for the charge susceptibility (right).

In Fig. 9.6, also some results for $\mu=80 \mathrm{MeV}$ are plotted. In this case, the difference between conventional virial expansion and density expansion is even more dramatic (dashed gray line vs. solid gray line). The inclusion of the correct Bose-Einstein statistics in the way it is provided by the density expansion is necessary; otherwise the result can be different by a factor of 3 and more. Again, removing the thermal part of the loop function modifies the result (dotted gray line vs. dashed gray line) which results in a $30 \%$ change of the electric mass.

### 9.2.4 Scaling of the amplitude with the pion mass

In the chiral non-perturbative scheme which we have used the amplitude and thermodynamical observables can be expressed in terms of the pion masses (and the pion masses in the terms of the quark masses). For lattice calculations it is of help to know the scaling of the susceptibility with the pion mass because the pion masses on the lattice are larger than the physical ones. In Fig. 9.7 the temperature phase shifts and charge susceptibilities are plotted for the physical pion mass and a typical lattice value of $m_{\pi}=m_{\rho} / 2$. The susceptibility decreases as the pion is heavier. In Fig. 9.8 we display the susceptibility as a function of the pion mass at different temperatures.


Figure 9.8: Electric mass in the charge $|C|=2$ channel as a function of the pion mass at different temperatures ( $T$ in MeV indicated in the figure).

### 9.2.5 Discussion of the results and outlook

For a systematic comparison with lattice results such as Ref. [261,262] we have to include the charge one and zero channels in which the $\rho$-meson plays an important role, but also components from isospin two (as said before, contributions from isospin zero are negligible). We have provided in this study the framework of how to implement chiral unitary methods in the calculation of the electric mass; the missing channels will be included straightforward [280].

Another issue is the off-shell treatment of the interaction vertices. While one of the virtues and strength of $U \chi P T$ is the on-shell factorization of the vertices as discussed in Sec. 1.2, this is not necessarily the case for the finitetemperature part of the loop function of Eq. (9.9). We have already seen that for the nuclear matter environment the off-shell parts indeed induce additional corrections (see Sec. 3.4.1). In here, we expect a similar situation; these refinements will not change the thermal unitary condition from Eq. (9.11) but induce higher order vertex corrections, or, in practical terms, thermal tadpoles similar to that displayed in Fig. 3.8 (1), with the nucleon
lines replaced by pion lines.
Finally, one should also consider the propagation of the intermediate pions. This is in analogy to the pion polarization in nuclear matter considered in Sec. 3.2.2; the most consistent approach will be to include the interaction of the propagating pion with the hot medium via an additional density expansion, in a similar way as it has been carried out in a virial expansion in Ref. [298]. Work along these lines is being carried out [280].

### 9.3 Summary

A unitary chiral method has been applied to the calculation of the charge susceptibility in a hot pion gas. Unitarized chiral methods have been very successful in the description of vacuum $\pi \pi$ scattering and allow for a straightforward medium implementation for finite temperatures and chemical potential. For the charge susceptibility, the most relevant partial wave besides the $\rho$ channel is the isospin two, spin zero amplitude which has been object of this study. Results for the susceptibility in the net charge $|C|=2$ channel have been predicted.

At zero chemical potential the electric mass is changed by 10-15 \% when including $\mu$ in the rescattering loops. At finite $\mu$, the influence of the matter propagator can be larger. The most drastic changes have been observed when replacing the traditional virial expansion with a density expansion which points out the necessity of including Bose-Einstein statistics in the asymptotic states of the two-body scattering.

The novelties of this chapter compared to previous studies are the consistent treatment of particle statistics and the inclusion of finite chemical potential in the rescattering loops generated by the Bethe-Salpeter equation. With this, we could include higher order corrections in density and interaction in a controlled way. Although the current state of the study is still qualitative, the inclusion of the missing channels is straightforward.

## Appendix A

## The $d$-wave in the deuteron

In the sections 2.3.2 and 2.3.3, the influence of the $d$-wave and the interference of $d$-wave and $s$-wave have been discussed for a number of diagrams, namely the absorption, and the 5 th diagram of Fig. 2.6. We display some explicit formulas for the coupling of spin and angular momentum involved in these calculations.

In momentum representation, the deuteron wave functions in Eqs. (2.23), (2.29), and (2.32) are given by

$$
\begin{array}{r}
F_{d}=F_{d}^{(s)}+F_{d}^{(d)}, \quad F_{d}^{(i)}\left(p, \theta_{p}, \phi_{p}\right)=(2 \pi)^{3 / 2} f_{\nu}^{(i)}(\hat{\mathbf{p}}) \Psi^{(i)}(p), \\
\Psi^{(i)}(p)=\sqrt{\frac{2}{\pi}} \sum_{j=1}^{n} \frac{\left[C_{j} \text { for } i=s, D_{j} \text { for } i=d\right]}{p^{2}+m_{j}^{2}} . \tag{A.1}
\end{array}
$$

The index $i$ stands for $s$ or $d$-wave, and the parameterizations of the radial part $\Psi(p)$, by means of the $C_{j}, D_{j}$, and $m_{j}$, are taken from Ref. [101] $(n=11)$ for the CD-Bonn and Ref. [102] $(n=13)$ for the Paris wave function. The normalization is $\int d p p^{2}\left(\Psi^{(s)}(p)^{2}+\Psi^{(d)}(p)^{2}\right)=1$ in both cases.

The angular structure of the $d$-wave is given by the angular momentum $l=2, l_{z}=\mu$ and the spin wave function $\chi$

$$
\begin{align*}
& f_{\nu}^{(d)}(\hat{\mathbf{p}})=\sum_{\mu} C(2,1,1 ; \quad \mu, \nu-\mu, \nu) Y_{2, \mu}\left(\theta_{p}, \phi_{p}\right) \chi_{S_{z}=\nu-\mu}^{S=1}, \\
& f_{\nu}^{(s)}=Y_{00} \chi_{S_{z}=\nu}^{S=1} . \tag{A.2}
\end{align*}
$$

The angular structure $f_{\nu}^{(i)}$, also normalized to one, is preserved under Fourier transforms from coordinate space. The index $\nu=-1,0,1$ indicates at which

3rd component $J_{z}$ the total angular momentum is fixed. The $C$ 's in Eq. (A.2) are the Clebsch-Gordan coefficients that couple the nucleon spins to the angular momentum $l=2$ of the $d$-state, in order to give a total angular momentum $J$ of one. The nucleons are necessarily in a spin triplet with $S=1$ for $s$ and $d$-wave.

Fixing in all calculations the third component of $J$ at $J_{z}=\nu=0$, we obtain for the $d-s$ interference of the diagrams of absorption in Fig. 2.3

$$
\begin{align*}
& \left(\vec{\sigma}_{\text {nucleon } 1,2} \cdot \mathbf{q}\right)\left(\vec{\sigma}_{\text {nucleon } 1,2} \cdot \mathbf{q}^{\prime}\right) f_{0}^{(d)}(\widehat{\mathbf{q}+\mathbf{l}}) f_{0}^{(s)}\left(\widehat{\mathbf{q}^{\prime}+\mathbf{1}}\right) \\
= & \frac{1}{2 \sqrt{2} \pi}\left[1-3 \cos ^{2} \theta_{\widehat{\mathbf{q}+1}}\right] \mathbf{q} \cdot \mathbf{q}^{\prime} \\
+ & \frac{3}{2 \sqrt{2} \pi} \cos \theta \widehat{\mathbf{q + 1}} \sin \theta_{\widehat{\mathbf{q + 1}}}\left[\sin \phi \widehat{\mathbf{q + 1}}\left(\mathbf{q}^{\prime} \times \mathbf{q}\right)_{x}-\cos \phi \widehat{\mathbf{q + 1}}\left(\mathbf{q}^{\prime} \times \mathbf{q}\right)_{y}\right] \tag{A.3}
\end{align*}
$$

where symmetries in the amplitude $A$ in Eq. (2.29) have been used. The $\sigma$ matrices act on the same nucleon. The second line corresponds to the first term in the decomposition $\vec{\sigma} \mathbf{q}^{\prime} \vec{\sigma} \mathbf{q}=\mathbf{q}^{\prime} \cdot \mathbf{q}+i\left(\mathbf{q}^{\prime} \times \mathbf{q}\right) \cdot \vec{\sigma}$, and the third line to the term with crossed momenta.

Another spin structure is given by the diagrams in Fig. 2.4 and the 5th diagram of Fig. 2.6. Here, the $\pi N N$ vertices are attached at different nucleons, and we obtain for the $d-s$ interference for Fig. 2.4

$$
\begin{align*}
& \left(\vec{\sigma}_{\text {nucleon } 1,2} \cdot \mathbf{q}\right)\left(\vec{\sigma}_{\text {nucleon } 2,1} \cdot \mathbf{q}^{\prime}\right) f_{0}^{(d)}(\widehat{\mathbf{q}+\mathbf{1}}) f_{0}^{(s)}\left(\widehat{\mathbf{q}^{\prime}+\mathbf{1}}\right) \\
= & \frac{1}{2 \sqrt{2} \pi}\left[1-3 \cos ^{2} \theta_{\widehat{\mathbf{q}+1}}\right]\left(q_{x} q_{x}^{\prime}+q_{y} q_{y}^{\prime}-q_{z} q_{z}^{\prime}\right) \\
+ & \frac{3}{2 \sqrt{2} \pi} \cos \theta \widehat{\mathbf{q + 1}} \sin \theta \widehat{\mathbf{q + 1}}\left[\sin \phi \widehat{\mathbf{q + 1}}\left(q_{z}^{\prime} q_{y}+q_{z} q_{y}^{\prime}\right)+\cos \phi \widehat{\mathbf{q + 1}}\left(q_{z}^{\prime} q_{x}+q_{z} q_{x}^{\prime}\right)\right] . \tag{A.4}
\end{align*}
$$

The term $\left(q_{x} q_{x}^{\prime}+q_{y} q_{y}^{\prime}-q_{z} q_{z}^{\prime}\right)$ in the second line of Eq. (A.4) is also present in the $s$-wave $\rightarrow s$-wave transition and shows explicitly the $1 / 3$-contribution of the diagrams in Fig. 2.4 to the ones of Fig. 2.3, as has been derived in a different way in Eqs. (2.25), (2.26), and (2.27).

For the 5th diagram of Fig. 2.6, one obtains the angular structure by replacing the momentum components $q_{i} \rightarrow q_{i}^{\prime}$ in Eq. (A.4), where $\mathbf{q}^{\prime}$ is the
momentum of the disconnected pion (the $q$ in the angles $\theta_{\widehat{\mathbf{q}+1}}$ and $\phi \widehat{\mathbf{q}+1}$ of Eq. (A.4) is not changed).

The angular structure of $d$-wave $\rightarrow d$-wave transition for the diagrams is calculated in analogy to Eqs. (A.3) and (A.4), but the resulting expressions are lengthier due to the occurring double sums from Eq. (A.2).

In the calculations for absorption, the results have been numerically tested for choices of $\nu$ equal to $\pm 1$ instead of 0 , and they stay the same, as it is required.

## Appendix B

## Evaluation of the the Pauli blocked $\pi N$ loop with pion polarization

Several integrations of the $\pi N$ loop from Eq. (3.6) can be solved analytically. Furthermore, it is helpful to separate real and imaginary part. Integrating over $q^{0}$ and $x=\cos \angle(\mathbf{P}, \mathbf{q})$ one obtains, including the boosted cut-off and the Pauli blocking for particle and hole part

$$
\begin{align*}
G_{\pi N}\left(P^{0}, \mathbf{P}, \rho\right)= & a_{\pi N}+\frac{M_{N}}{(2 \pi)^{2} P} \int_{0}^{\infty} d q q \int_{0}^{\infty} d \omega S(\omega, q ; \rho)[ \\
& \theta\left(x_{2}^{p}-x_{1}^{p}\right)\left(\log \left|\frac{\omega-P^{0}+\sqrt{M_{N}^{2}+P^{2}+q^{2}-2 P q x_{2}^{p}}}{\omega-P^{0}+\sqrt{M_{N}^{2}+P^{2}+q^{2}-2 P q x_{1}^{p}}}\right|\right. \\
& \left.\left.-i \pi g_{p}\left(x_{1}^{p}, x_{2}^{p}\right)\right)\right) \\
+ & \theta\left(x_{2}^{h}-x_{1}^{h}\right)\left(\log \left|\frac{-\omega-P^{0}+\sqrt{M_{N}^{2}+P^{2}+q^{2}-2 P q x_{2}^{h}}}{-\omega-P^{0}+\sqrt{M_{N}^{2}+P^{2}+q^{2}-2 P q x_{1}^{h}}}\right|\right. \\
& \left.\left.\left.+i \pi g_{h}\left(x_{1}^{h}, x_{2}^{h}\right)\right)\right)\right] \tag{B.1}
\end{align*}
$$

where we have suppressed some indices, see Eq. (3.6) and $P \equiv|\mathbf{P}|$. The values of $x_{1,2}^{p, h}$ include the theta-functions from the cut-off in Eq. (3.7) and
the Pauli blocking. With

$$
l_{\mathrm{cut}}=\frac{1}{P q} \begin{cases}\left(\omega P^{0}-\sqrt{s} \sqrt{\omega^{2}+q_{\max }^{2}-q^{2}}\right) & \text { for } \omega^{2}+q_{\max }^{2}-q^{2}>0  \tag{B.2}\\ \left(\omega P^{0}+\sqrt{s} \sqrt{-\left(\omega^{2}+q_{\max }^{2}-q^{2}\right)}\right) & \text { for } \omega^{2}+q_{\max }^{2}-q^{2} \leq 0\end{cases}
$$

and $l_{\text {Pauli }}=\left(P^{2}+q^{2}-k_{F}^{2}\right) /(2 P q)$ we obtain
$x_{2}^{p}=\min \left(1, l_{\text {Pauli }}\right), x_{1}^{p}=\max \left(-1, l_{\text {cut }}\right), x_{2}^{h}=1, x_{1}^{h}=\max \left(-1, l_{\text {cut }}, l_{\text {Pauli }}\right)$
For the imaginary part in Eq. (B.1) the functions $g$ determine whether the position of the pole is between $x_{1}$ and $x_{2}$ according to $g\left(x_{1}, x_{2}\right)=1$ if $x_{1}<x_{\text {pole }}<x_{2}$ and zero otherwise with

$$
\begin{equation*}
x_{\text {pole }}=\frac{1}{2 P q}\left(M_{N}^{2}+P^{2}-\left(P^{0} \pm \omega\right)^{2}+q^{2}\right) \tag{B.4}
\end{equation*}
$$

where the "-" refers to the particle part with index $p$ and the " + " to the hole part with index $h$ for the $g$ 's of Eq. (B.1). Note that the log-term as well as the imaginary part are well defined in the limit $P \rightarrow 0$. The imaginary part can be even further evaluated. E.g., without the pion polarization, i.e., $S(\eta, q ; \rho) \rightarrow \delta\left(\eta^{2}-q^{2}-m_{\pi}^{2}\right)$ one finds the alternative notation

$$
\begin{align*}
\operatorname{Im} G_{\pi N}\left(P^{0}, \mathbf{P}, \rho\right) & =-\frac{1}{8 \pi} \int_{0}^{\infty} d q \frac{q M_{N}}{P \omega} \theta\left(1-x_{\text {pole }}^{2}\right) \\
& \times \theta\left(\left|\left(P^{0}-\omega\right)^{2}-M_{N}^{2}\right|-k_{F}\right) \theta\left(q_{\mathrm{cm}}^{\max }-\left|\mathbf{q}_{c m}\right|\right) \tag{B.5}
\end{align*}
$$

with $\omega^{2}=q^{2}+m_{\pi}^{2}$. It is instructive to consider several limiting cases for Eq. (B.1) such as $\left(P^{0}=m_{\pi}, P \rightarrow 0\right)$ and/or $k_{F} \rightarrow 0$ and/or $S(\omega, q ; \rho) \rightarrow$ $\delta\left(\omega^{2}-q^{2}-m_{\pi}^{2}\right)$ which results in well known expressions such as Eq. (3.5).

## Appendix C

## $1 / 2^{-}$meson $1 / 2^{+}$baryon $3 / 2^{+}$ baryon ( $M B B^{*}$ ) interaction

In the chapters 4 to 7 , the $M B B^{*}$ couplings are needed (see, e.g., Fig. 6.3). Some more details of the formalism are given here. For completeness, Feynman rules for the $M B B^{*}$ interaction for all channels allowed by the Lagrangian Eq. (4.56) are given, although only some are used in Sec. 6.3. The $\operatorname{SU}(4)$ breaking in the decuplet leads to different coupling strengths for different isospin-multiplets. Some results from the literature are summarized and values for the couplings $\mathcal{C}$ are provided from a fit to partial decay widths.

Before turning to the flavor structure of the interaction given by the roman symbols in Eq. (4.56) let us deduce the Feynman rules for the Lorentz structure given by the index $\mu$. The Lorentz structure of the RaritaSchwinger fields of the Lagrangian in Eq. (4.56) is given by [44]

$$
\begin{align*}
u_{\Delta, \mu}\left(p, s_{\Delta}\right) & =\sum_{\lambda, s} C\left(1 \frac{1}{2} \frac{3}{2} ; \lambda s s_{\Delta}\right) e_{\mu}(p, \lambda) u(p, s), \\
\bar{u}_{\Delta, \mu}\left(p, s_{\Delta}\right) & =\sum_{\lambda, s} C\left(1 \frac{1}{2} \frac{3}{2} ; \lambda s s_{\Delta}\right) e_{\mu}^{\star}(p, \lambda) \bar{u}(p, s) \tag{C.1}
\end{align*}
$$

where $e_{\mu}=(0, \hat{\mathbf{e}})$ in the $B^{*}$ rest frame,

$$
\hat{\mathbf{e}}_{+}=-\frac{1}{\sqrt{2}}\left(\begin{array}{c}
1  \tag{C.2}\\
i \\
0
\end{array}\right), \quad \hat{\mathbf{e}}_{-}=\frac{1}{\sqrt{2}}\left(\begin{array}{c}
1 \\
-i \\
0
\end{array}\right), \quad \hat{\mathbf{e}}_{0}=\left(\begin{array}{l}
0 \\
0 \\
1
\end{array}\right)
$$

and $\lambda=(+1,-1,0)$ is the spherical base.
In the next step we show the relation

$$
\begin{equation*}
\left\langle\frac{3}{2} s_{\Delta}\right| \mathbf{S}^{\dagger}\left|\frac{1}{2} s\right\rangle=\sum_{\lambda} C\left(1 \frac{1}{2} \frac{3}{2} ; \lambda s s_{\Delta}\right) \hat{\mathbf{e}}_{\lambda}^{\star} \tag{C.3}
\end{equation*}
$$

Writing this component-wise and using the definition

$$
\begin{equation*}
\left\langle\frac{3}{2} s_{\Delta}\right| \mathbf{S}_{\lambda}^{\dagger}\left|\frac{1}{2} s\right\rangle=C\left(1 \frac{1}{2} \frac{3}{2} ; \lambda s s_{\Delta}\right)\left\langle\frac{3}{2}\left\|\mathbf{S}^{\dagger}\right\| \frac{1}{2}\right\rangle \tag{C.4}
\end{equation*}
$$

for $\mathbf{S}^{\dagger}$ in the spherical base, we obtain:

$$
\begin{equation*}
S_{x}^{\dagger}=\frac{1}{\sqrt{2}}\left(S_{-}^{\dagger}-S_{+}^{\dagger}\right), \quad S_{y}^{\dagger}=\frac{i}{\sqrt{2}}\left(S_{-}^{\dagger}+S_{+}^{\dagger}\right), \quad S_{z}^{\dagger}=S_{0}^{\dagger} \tag{C.5}
\end{equation*}
$$

using the definition of the polarization vectors from Eq. (C.2). This shows indeed Eq. (C.3) which we can now plug in Eq. (C.1), writing additionally the other field from the ordinary baryon of the vertex, $\bar{u}\left(p, s_{B}^{\prime}\right)$, and the meson momentum from the derivative, $k_{M}^{\mu}$ :

$$
\begin{align*}
& \bar{u}\left(p, s_{B}^{\prime}\right) k_{M}^{\mu} u_{\Delta, \mu}\left(p, s_{\Delta}\right) \rightarrow-\bar{u}\left(p, s_{B}^{\prime}\right) \vec{k}_{M} \cdot \vec{u}_{\Delta}\left(p, s_{\Delta}\right) \\
& \rightarrow-\bar{u}\left(p, s_{B}^{\prime}\right) \sum_{s} \mathbf{S} \cdot \vec{k}_{M} u(p, s) \rightarrow-\mathbf{S} \cdot \vec{k}_{M} \tag{C.6}
\end{align*}
$$

which is indeed the structure we expect (note that there is one additional "-") and which results in the Feynman rule Eq. (6.8). For the other term in Eq. (C.1), the one with $\bar{u}_{\Delta, \mu}\left(p, s_{\Delta}\right)$, we get the corresponding structure $\mathbf{S}^{\dagger} \cdot \vec{k}_{M}$.

The Feynman rule Eq. (6.8) has been formulated for the process $B \rightarrow$ $M(\mathbf{q}) B^{*}$ with the momentum $q$ of the pion outgoing. If one does not use the identification from Eq. (6.7) which only holds when the $\Delta$ is involved, we obtain

$$
\begin{equation*}
(-i t)_{B \rightarrow M(\mathbf{q}) B^{*}}=\mathcal{C} \frac{d}{\sqrt{2} f_{\pi}} \mathbf{S}^{\dagger} \cdot \mathbf{q} \tag{C.7}
\end{equation*}
$$

For the process $B^{*} \rightarrow M(\mathbf{q}) B$ with $q$ again defined as outgoing we obtain

$$
\begin{equation*}
(-i t)_{B^{*} \rightarrow M(\mathbf{q}) B}=\mathcal{C} \frac{d^{\prime}}{\sqrt{2} f_{\pi}} \mathbf{S} \cdot \mathbf{q} \tag{C.8}
\end{equation*}
$$

Table C.1: Coefficients $d=d^{\prime}$ for the Feynman rules Eq. (C.7) and Eq. (C.8) for all channels provided by the Lagrangian.

| $\Delta^{-}$ | $\rightarrow \pi^{-} n$ | -1 | $\Sigma^{*-}$ | $\rightarrow K^{-} n$ | $-\sqrt{1 / 3}$ | $\Sigma^{*+}$ | $\rightarrow \eta \Sigma^{+}$ | $+\sqrt{1 / 2}$ |
| :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- |
| $\Delta^{-}$ | $\rightarrow K^{0} \Sigma^{-}$ | +1 | $\Sigma^{*-}$ | $\rightarrow \eta \Sigma^{-}$ | $-\sqrt{1 / 2}$ | $\Xi^{*-}$ | $\rightarrow \pi^{-} \Xi^{0}$ | $+\sqrt{1 / 3}$ |
| $\Delta^{0}$ | $\rightarrow \pi^{0} n$ | $-\sqrt{2 / 3}$ | $\Sigma^{*-}$ | $\rightarrow K^{0} \Xi^{-}$ | $+\sqrt{1 / 3}$ | $\Xi^{*-}$ | $\rightarrow \pi^{0} \Xi^{-}$ | $+\sqrt{1 / 6}$ |
| $\Delta^{0}$ | $\rightarrow \pi^{-} p$ | $-\sqrt{1 / 3}$ | $\Sigma^{* 0}$ | $\rightarrow \pi^{0} \Lambda$ | $+\sqrt{1 / 2}$ | $\Xi^{*-}$ | $\rightarrow K^{-} \Lambda$ | $+\sqrt{1 / 2}$ |
| $\Delta^{0}$ | $\rightarrow K^{0} \Sigma^{0}$ | $+\sqrt{2 / 3}$ | $\Sigma^{* 0}$ | $\rightarrow \pi^{-} \Sigma^{+}$ | $+\sqrt{1 / 6}$ | $\Xi^{*-}$ | $\rightarrow K^{-} \Sigma^{0}$ | $-\sqrt{1 / 6}$ |
| $\Delta^{0}$ | $\rightarrow K^{+} \Sigma^{-}$ | $+\sqrt{1 / 3}$ | $\Sigma^{* 0}$ | $\rightarrow \pi^{+} \Sigma^{-}$ | $-\sqrt{1 / 6}$ | $\Xi^{*-}$ | $\rightarrow \bar{K}^{0} \Sigma^{-}$ | $-\sqrt{1 / 3}$ |
| $\Delta^{+}$ | $\rightarrow \pi^{+} n$ | $+\sqrt{1 / 3}$ | $\Sigma^{* 0}$ | $\rightarrow K^{-} p$ | $-\sqrt{1 / 6}$ | $\Xi^{*-}$ | $\rightarrow \eta \Xi^{-}$ | $-\sqrt{1 / 2}$ |
| $\Delta^{+}$ | $\rightarrow \pi^{0} p$ | $-\sqrt{2 / 3}$ | $\Sigma^{* 0}$ | $\rightarrow K^{0} n$ | $+\sqrt{1 / 6}$ | $\Xi^{* 0}$ | $\rightarrow \pi^{+} \Xi^{-}$ | $-\sqrt{1 / 3}$ |
| $\Delta^{+}$ | $\rightarrow K^{0} \Sigma^{+}$ | $-\sqrt{1 / 3}$ | $\Sigma^{* 0}$ | $\rightarrow \eta \Sigma^{0}$ | $-\sqrt{1 / 2}$ | $\Xi^{* 0}$ | $\rightarrow \pi^{0} \Xi^{0}$ | $+\sqrt{1 / 6}$ |
| $\Delta^{+}$ | $\rightarrow K^{+} \Sigma^{0}$ | $+\sqrt{2 / 3}$ | $\Sigma^{* 0}$ | $\rightarrow K^{+} \Xi^{-}$ | $+\sqrt{1 / 6}$ | $\Xi^{* 0}$ | $\rightarrow \bar{K}^{0} \Lambda$ | $-\sqrt{1 / 2}$ |
| $\Delta^{++}$ | $\rightarrow \pi^{+} p$ | +1 | $\Sigma^{* 0}$ | $\rightarrow K^{0} \Xi^{0}$ | $-\sqrt{1 / 6}$ | $\Xi^{* 0}$ | $\rightarrow \bar{K}^{0} \Sigma^{0}$ | $-\sqrt{1 / 6}$ |
| $\Delta^{++}$ | $\rightarrow K^{+} \Sigma^{+}$ | -1 | $\Sigma^{*+}$ | $\rightarrow \pi^{+} \Lambda$ | $-\sqrt{1 / 2}$ | $\Xi^{* 0}$ | $\rightarrow K^{-} \Sigma^{+}$ | $+\sqrt{1 / 3}$ |
| $\Sigma^{*-}$ | $\rightarrow \pi^{-} \Lambda$ | $+\sqrt{1 / 2}$ | $\Sigma^{*+}$ | $\rightarrow \pi^{0} \Sigma^{+}$ | $+\sqrt{1 / 6}$ | $\Xi^{* 0}$ | $\rightarrow \eta \Xi^{0}$ | $+\sqrt{1 / 2}$ |
| $\Sigma^{*-}$ | $\rightarrow \pi^{-} \Sigma^{0}$ | $-\sqrt{1 / 6}$ | $\Sigma^{*+}$ | $\rightarrow \pi^{+} \Sigma^{0}$ | $-\sqrt{1 / 6}$ | $\Omega^{-}$ | $\rightarrow K^{-} \Xi^{0}$ | +1 |
| $\Sigma^{*-}$ | $\rightarrow \pi^{0} \Sigma^{-}$ | $+\sqrt{1 / 6}$ | $\Sigma^{*+}$ | $\rightarrow \bar{K}^{0} p$ | $+\sqrt{1 / 3}$ | $\Omega^{-}$ | $\rightarrow \bar{K}^{0} \Xi^{-}$ | -1 |
|  |  |  | $\Sigma^{*+}$ | $\rightarrow K^{+} \Xi^{0}$ | $-\sqrt{1 / 3}$ |  |  |  |

Although the rules Eq. (C.7) and Eq. (C.8) come from different parts of the Lagrangian Eq. (4.56) one obtains for all processes

$$
\begin{equation*}
d=d^{\prime} . \tag{C.9}
\end{equation*}
$$

In Tab. C. 1 all combinations of $M B B^{*}$ interactions are listed that are allowed by the Lagrangian from Eq. (4.56). For clarity, in the table the processes have been labeled for the $B^{*} \rightarrow M(\mathbf{q}) B$ process from Eq. (C.8). In order to find $d$ if the $B \rightarrow M(\mathbf{q}) B^{*}$ process from Eq. (C.7) is considered one takes the value for the reaction that has the same $1 / 2^{+}$and $3 / 2^{+}$baryons in Tab. C.1. Example: The $p \rightarrow \pi^{+} \Delta^{0}$ reaction corresponds to the fourth entry in Tab. C. 1 and the vertex is given by Eq. (C.7) with $d=-\sqrt{1 / 3}$.

For processes where the meson is incoming the Feynman rules are straightforward: One simply chooses the process from Tab. C. 1 which has the same $B$ and $B^{*}$ and changes sign. Example: The $p \pi^{-} \rightarrow \Delta^{0}$ reaction corresponds to the fourth entry in Tab. C. 1 where $d=-\sqrt{1 / 3}$. The rule is then:
$(-i t)=-(-1 / \sqrt{3})\left(\frac{c}{\sqrt{2} f_{\pi}}\right) \mathbf{S}^{\dagger} \cdot \mathbf{k}$ with $k$ the incoming $\pi^{-}$momentum. Exam$p l e:$ The $\Delta^{0} \pi^{+} \rightarrow p$ reaction also corresponds to the fourth entry in Tab. C.1. The rule is then: $(-i t)=-(-1 / \sqrt{3})\left(\frac{C}{\sqrt{2} f_{\pi}}\right) \mathbf{S} \cdot \mathbf{k}$.

## The value of $\mathcal{C}$

Due to $S U(4)$ breaking the coupling of the members of the decuplet to octet mesons and baryons is not universal. For a given process, we can determine $\mathcal{C}$ if we compare, e.g., the process $p \rightarrow \Delta^{++} \pi^{-}$, described by the present Lagrangian, with a calculation in the usual way, which leads to the relation

$$
\begin{equation*}
\frac{f_{\Delta \pi N}^{\star}}{m_{\pi}} \equiv \frac{\mathcal{C}_{\Delta \pi N}}{\sqrt{2} f_{\pi}} \tag{C.10}
\end{equation*}
$$

that fixes $\mathcal{C}=2.03$. For other decays of the baryon decuplet into meson and baryon octet it is worth summarizing some results from the literature:

- $\Sigma^{*} \rightarrow \pi \Sigma$ : In Ref. [192] the value

$$
\begin{equation*}
\frac{g_{\Sigma^{*}}}{2 M}=\frac{2 \sqrt{6}}{5} \frac{D+F}{2 f_{\pi}} \hat{=} \frac{\mathcal{C}_{\Sigma^{*} \pi \Sigma}}{\sqrt{6} f_{\pi}} \tag{C.11}
\end{equation*}
$$

for the $\Sigma^{*} \rightarrow \pi \Sigma$ decay is derived. Note that the expression with " $D+F$ " is some $9 \%$ too small which leads to a partial decay width which is $20 \%$ too small.

- $\underline{\Sigma}^{*} \rightarrow \pi \Lambda$ : For the $\Sigma^{*} \rightarrow \pi \Lambda$ decay, we use from Ref. [61]

$$
\begin{equation*}
\frac{f_{\Sigma^{*} \pi \Lambda}}{m_{\pi}}=\frac{1.3}{m_{\pi}} \simeq 1.15 \frac{6}{5} \frac{D+F}{2 f_{\pi}} \hat{=} \frac{\mathcal{C}_{\Sigma^{*} \pi \Lambda}}{2 f_{\pi}} \tag{C.12}
\end{equation*}
$$

where the first value is the phenomenological one, and the second uses $\mathrm{SU}(3)$ arguments and a phenomenological correction factor of 1.15. The value in Tab. C. 2 is obtained from the latter expression.

- $\underline{\Sigma}^{*} \rightarrow \bar{K} N$ : For this (closed) decay channel Ref. [192] gives:

$$
\begin{equation*}
\frac{g_{\Sigma^{*}}}{2 M}=\frac{2 \sqrt{6}}{5} \frac{D+F}{2 f_{\pi}} \hat{=} \frac{\mathcal{C}_{\Sigma^{*} \bar{K} N}}{\sqrt{6} f_{\pi}} . \tag{C.13}
\end{equation*}
$$

Table C.2: Partial decay widths $[\mathrm{MeV}]$ from experiment, from $S U(6)$ arguments, $\mathcal{C}$ from Ref. [190] fitted to data at one loop level, and $\mathcal{C}$ fitted to data at tree level.

|  | $\Gamma_{i}\left(M_{B^{*}}\right)$, exp. | $\Gamma_{i}\left(M_{B^{*}}\right), S U(6)$ | $\mathcal{C}$, Ref. [190] | $\mathcal{C}$, from exp. |
| :--- | :--- | :--- | :--- | :--- |
| $\Delta \rightarrow \pi N$ | $115-125$ | 113.1 | 1.5 | $2.05-2.14$ |
| $\Sigma^{*} \rightarrow \Sigma \pi$ | 4.45 | 3.76 | 1.4 | 1.64 |
| $\Sigma^{*} \rightarrow \Lambda \pi$ | 32.6 | 33.5 | 1.4 | 1.71 |
| $\Xi^{*} \rightarrow \Xi \pi$ | 9.5 | - | 1.3 | 1.41 |
| $\Omega^{-} \rightarrow B M$ | 0 | - | - | 1.07 (extrapol.) |

One can determine $\mathcal{C}$ for the different decuplet decays by comparing the width from

$$
\begin{equation*}
\Gamma_{i}(\sqrt{s})=\frac{1}{6 \pi} g_{i}^{2} \frac{M}{\sqrt{s}} Q^{3} \Theta(\sqrt{s}-m-M) \tag{C.14}
\end{equation*}
$$

with experiment. In Eq. (C.14) $m(M)$ signifies meson (baryon) masses of the decay products and $Q \equiv Q_{C M}$ their CM momentum. The Feynman rules derived from Eq. (4.56) lead to the following correspondences between the values of the $\mathcal{C}$ 's and the couplings $g_{i}$ of Eq. (C.14):

$$
\begin{align*}
& g_{\Delta \pi N}^{2}=\frac{1}{2} \mathcal{C}_{\Delta \pi N}^{2}, \quad g_{\Sigma^{*} \pi \Sigma}^{2}=\frac{1}{6} \mathcal{C}_{\Sigma^{*} \pi \Sigma}^{2}, \\
& g_{\Sigma^{*} \pi \Lambda}^{2}=\frac{1}{4} \mathcal{C}_{\Sigma^{*} \pi \Lambda}^{2}, \quad g_{\Xi^{*} \pi \Xi}^{2}=\frac{1}{4} \mathcal{C}_{\Xi^{*} \pi \Xi}^{2} . \tag{C.15}
\end{align*}
$$

The fractions $1 / 2,1 / 3$ come from the sum over the squares of the isospin coefficients of a decay mode, e.g., $\Xi^{*-} \rightarrow \Xi^{-} \pi^{0} \rightarrow 1 / \sqrt{6}$ and $\Xi^{*-} \rightarrow \Xi^{0} \pi^{-} \rightarrow$ $1 / \sqrt{3}$ and $1 / 6+1 / 3=1 / 2$.

The results are summarized in Tab. C.2. For the values in Tab. C. 2 we have taken $f_{\pi}=93 \mathrm{MeV}, D+F=1.257$, and averaged particle masses from the latest edition of the PDG Ref. [57]. The first column shows the experimental partial decay width. The second column shows the resulting partial widths using Eqs. (C.10) to (C.13). The third and fourth columns show values of $\mathcal{C}$ from Ref. [190] and from Eq. (C.14), Eq. (C.15), respectively, meaning the values for $\mathcal{C}$ extracted from the experimental partial decay width.

As Tab. C. 2 shows, there are some differences for $\mathcal{C}$ between the last column and Ref. [190]. However, the values for $\mathcal{C}$ have been fitted in Ref. [190] to the decuplet partial decay widths at the one-loop level, together with the $B^{*} B^{*} M$ term with strength $\mathcal{H}$ and others. Different values for $\mathcal{C}$ are, thus, no surprise.

The $\Omega^{-}$cannot decay into $M B$ because the channels $K^{-} \Xi^{0}$ and $\bar{K}^{0} \Xi^{-}$, which are provided by the Lagrangian, are physically closed. Nevertheless, we can determine the coupling by linear extrapolation. For the other decay channels, the function $\mathcal{C}\left(M_{B^{*}}\right)$, with $\mathcal{C}$ from the last column in Tab. C.2, is linear as a function of the mass in good approximation, and we fit it with $\mathcal{C}\left(M_{B^{*}}\right)=4.80075-0.00223166 M_{B^{*}}, M_{B^{*}}$ in $[\mathrm{MeV}]$, which results in $\mathcal{C}\left(M_{\Omega^{-}}\right)=1.07$.

## Kroll-Ruderman terms

The Feynman rules for the Kroll-Ruderman terms can be easily constructed by considering the tree level process $B \rightarrow B^{*} M$ and coupling the $\gamma$ to the meson $M$. By requiring gauge invariance of the process a contact counterterm of the $\gamma B B^{*} M$ form is necessary and can be easily derived.

For this, we start with the tree level diagram, where the baryon emits a meson at momentum $q-k$ and transform into a $B^{*}$. The meson at $q-k$ absorbs a photon that enters with momentum $k$, and the outgoing meson momentum is $q$. The amplitude for this process is then given by

$$
\begin{align*}
(-i \vec{T} \cdot \vec{\epsilon}) & =\left(-i t_{B \rightarrow M B^{*}}\right) \frac{i}{(q-k)^{2}-m^{2}+i \epsilon}\left(-i t_{\gamma M \rightarrow M}\right) \\
& =-\frac{e c d \mathcal{C}}{\sqrt{2} f_{\pi}} \frac{1}{-2 p q} \mathbf{S}^{\dagger} \cdot(\mathbf{q}-\mathbf{k})(2 \mathbf{q}-\mathbf{k}) \cdot \vec{\epsilon} \\
& \xrightarrow{\epsilon_{\mu} \rightarrow k_{\mu}} \\
& -\frac{e c d \mathcal{C}}{\sqrt{2} f_{\pi}} \mathbf{S}^{\dagger} \cdot(\mathbf{q}-\mathbf{k})  \tag{C.16}\\
& \frac{e c d \mathcal{C}}{\sqrt{2} f_{\pi}} \mathbf{S}^{\dagger} \cdot \mathbf{k} .
\end{align*}
$$

Here, we have put the photon and external meson on-shell, $k^{2}=0, q^{2}=m^{2}$, and taken the limit $\mathbf{q} \rightarrow 0$. In that limit, diagrams where the photon couples directly to the $\left(1 / 2^{+}\right)$baryon component ( $\gamma B B$ vertex) vanish. From Eq.

Table C.3: Coefficients $c$ for the Kroll-Ruderman vertex (C.17).

|  |
| :--- |$\pi^{+}$| $\pi^{-}$ | $K^{-}$ |  |
| :---: | :---: | :---: |
| $c, \gamma(k) M(q-k) \rightarrow M(q)$ | 1 | -1 |

(C.16), we can directly read off that the vertex rule we are looking for is

$$
\begin{equation*}
(-i \vec{T} \cdot \vec{\epsilon})=-\frac{e c d \mathcal{C}}{\sqrt{2} f_{\pi}} \mathbf{S}^{\dagger} \cdot \vec{\epsilon} \tag{C.17}
\end{equation*}
$$

with $c$ from Tab. C. 3 and $d$ from Tab. C.1. . If one wishes to eliminate $\mathcal{C}$, one can use the relations (C.10)-(C.12). If one keeps using $\mathcal{C}$, one should use the values in the last column of Tab. C. 2 as we have seen in the last section.

## Appendix D

## Charge fluctuations

## D. 1 From charge fluctuations to photon selfenergy in sQED

In this section, an outline for the proof of Eq. (9.3) for scalar QED is given. The argument follows Ref. [265] where a similar connection is made for QED. If $\pi \pi$ contact interactions are included according to Eq. (8.14), the steps outlined below are similar, but lengthier, and Ward identities for four-point functions have to be determined.

CF are defined as $\left\langle\delta Q^{2}\right\rangle=\left\langle Q^{2}\right\rangle-\langle Q\rangle^{2}$, and the expectation values are calculated via the statistical operator of the grand canonical ensemble with the charge chemical potential $\mu \equiv \mu_{Q}$. One obtains immediately:

$$
\begin{equation*}
\left\langle\delta Q^{2}\right\rangle=e^{2} T V \frac{\partial}{\partial \mu}\left\langle\hat{\mathrm{~J}}_{0}\right\rangle \tag{D.1}
\end{equation*}
$$

with $\hat{\mathrm{J}}_{0}$ the zero-component of the conserved current, $\hat{Q}=\int \hat{\mathrm{J}}_{0}=V \hat{\mathrm{~J}}_{0}$. The expectation value of $\hat{\mathrm{J}}_{0}=i\left(\phi^{\star}\left(\partial^{0}+i e A^{0}\right) \phi-\phi\left(\partial^{0}-i e A^{0}\right) \phi^{\star}\right)$ can be expressed in terms of the propagator

$$
\begin{equation*}
\left(\frac{\partial\left\langle\hat{\mathrm{J}}_{0}\right\rangle}{\partial \mu}\right)_{T}=-\frac{\partial}{\partial \mu} T \sum_{\omega_{n}=-\infty}^{\infty} \int \frac{d^{3} \mathbf{p}}{(2 \pi)^{3}} 2\left(p^{0}-\mu\right) \mathcal{G}\left(p^{0}, \mathbf{p}\right) \tag{D.2}
\end{equation*}
$$

where we have used $\mu=e A^{0}$ and the definition of the imaginary time propagator

$$
\begin{equation*}
\mathcal{G}_{\alpha \beta}\left(\mathbf{x} \tau ; \mathbf{x}^{\prime} \tau^{\prime}\right)=-\operatorname{Tr}\left[\hat{\rho}_{G} \mathrm{~T}_{\tau}\left[\phi_{K \alpha}(\mathbf{x} \tau) \phi_{K \beta}^{\dagger}\left(\mathbf{x}^{\prime} \tau^{\prime}\right)\right]\right] \tag{D.3}
\end{equation*}
$$

where $T_{\tau}$ is the $\tau$-ordered product in the modified Heisenberg picture, see, e.g., Ref. [289], and the Fourier transform is at equal times $\tau, \tau^{+}$and position $\mathbf{x}=\mathbf{x}^{\prime}$. The $\mu$-dependence of the propagator is given by $p^{0}=i \omega_{n}-\mu$ where $\omega_{n}=2 \pi i n T$. With this, the derivative can be rewritten as

$$
\begin{equation*}
\left(\frac{\partial\left\langle\left\langle\hat{\mathrm{j}}_{0}\right\rangle\right.}{\partial \mu}\right)_{T}=-\sum_{\omega_{n}=-\infty}^{\infty} T \int \frac{d^{3} \mathbf{p}}{(2 \pi)^{3}}\left(-2 \mathcal{G}\left(p^{0}, \mathbf{p}\right)-2 p^{0} \frac{\partial}{\partial p_{0}} \mathcal{G}\left(p^{0}, \mathbf{p}\right)\right) \tag{D.4}
\end{equation*}
$$

at zero chemical potential $\mu=0$. Using $\partial / \partial p^{0} \mathcal{G}=-\mathcal{G}\left(\partial / \partial p^{0} \mathcal{G}^{-1}\right) \mathcal{G}$, the Ward identity in the differential form for scalar QED can be applied. The Ward identity connects the inverse propagator with the fully dressed vertex $\Gamma^{\mu}$ according to

$$
\begin{align*}
& e^{2} T V\left(\frac{\partial\left\langle\hat{\mathbf{j}}_{0}\right\rangle}{\partial \mu}\right)_{T} \\
= & T^{2} V \sum_{\omega_{n}=-\infty}^{\infty} \int \frac{d^{3} \mathbf{p}}{(2 \pi)^{3}}\left[2 e^{2} \mathcal{G}\left(p^{0}, \mathbf{p}\right)-e\left(2 p^{0}\right) \mathcal{G}\left(p^{0}, \mathbf{p}\right) \Gamma^{0}\left(\mathbf{p}, \omega_{n}\right) \mathcal{G}\left(p^{0}, \mathbf{p}\right)\right] \\
= & T V(\underbrace{\Pi_{D \text { mat }}^{00}\left(k_{0}=0, \mathbf{k} \rightarrow \mathbf{0}\right)}+\underbrace{\Pi_{C \text { mat }}^{00}\left(k_{0}=0, \mathbf{k} \rightarrow \mathbf{0}\right)}_{\text {(D.5 }}) . \tag{D.5}
\end{align*}
$$

Factors of $e$ and $p^{0}$ have been identified here with the bare $\gamma \gamma \pi \pi$ and $\gamma \pi \pi$ vertices. In the step from Eq. (D.4) to Eq. (D.5) we have generated three propagators from one, and it should be noted that this takes place inside the momentum integral and summation. Therefore, the limit in Eq. (9.3) has to be taken before summation and integration.

## D. 2 Pion-pion interaction

## D.2.1 Chiral $\pi \pi$ interaction and vector exchange

In this section the effective pion-pion contact interaction from Sec. 8.3 and its connection to the chiral Lagrangian is discussed in more detail. For four pion fields the kinetic term of $\mathcal{L}_{\pi \pi}^{(2)}$ in Eq. (8.6) and the effective interaction in Eq. (8.12) have identical isospin and momentum structure. Comparing
the overall coefficients leads to the result in Eq. (8.13) which differs from the KSFR relation by a factor of $3 / 2$. Studying the low energy behavior of both theories helps solve this puzzle of the obvious violation of the phenomenologically well-fulfilled KSFR relation. The $\pi \pi$ amplitude at threshold from the LO chiral Lagrangian Eq. (8.6) and the effective interaction Eq. (8.12) is given by

$$
\begin{equation*}
T_{\pi \pi}^{(2)}=-\frac{2 m_{\pi}^{2}}{f_{\pi}^{2}}, \quad T_{\mathrm{eff}}=-\frac{4 g^{2} m_{\pi}^{2}}{m_{\rho}^{2}} \tag{D.6}
\end{equation*}
$$

respectively which leads to the correct KSFR relation

$$
\begin{equation*}
2 f_{\pi}^{2} g^{2}=m_{\rho}^{2} \tag{D.7}
\end{equation*}
$$

This is due to the mass correction term proportional to $\mathcal{M}$ in Eq. (8.6). This term, however, does not have any momentum structure and immediately becomes small at finite pion momenta compared to the kinetic term. It has no influence in the results of this study.

For finite pion momenta, higher order partial waves have to be included. We concentrate on the quantum numbers of the $\rho$-meson and obtain for $\pi \pi$ scattering via the LO chiral interaction in isospin $I=1$ :

$$
\begin{equation*}
T_{\pi \pi}^{1}=\frac{-1}{f_{\pi}^{2}}(t-u) \tag{D.8}
\end{equation*}
$$

which should be compared to the result from $\rho$-exchange from Eq. (8.8):

$$
\begin{align*}
T^{1}(\text { dyn. } \rho) & =g^{2}\left(\frac{s-u}{t-m_{\rho}^{2}}+2 \frac{t-u}{s-m_{\rho}^{2}+i m_{\rho} \Gamma(s)}+\frac{t-s}{u-m_{\rho}^{2}}\right) \\
T^{2}(\text { dyn. } \rho) & =g^{2}\left(\frac{u-s}{t-m_{\rho}^{2}}+\frac{t-s}{u-m_{\rho}^{2}}\right) \tag{D.9}
\end{align*}
$$

where we have also given the result for $T^{2}$ for completeness, and $T^{0}$ is immediately obtained by crossing symmetry, $T^{0}=-2 T^{2}$. Projecting out the $p$-wave in both results (D.8) and (D.9) by using

$$
\begin{equation*}
T_{\ell}^{I}(s)=\frac{1}{64 \pi} \int_{-1}^{1} d(\cos \theta) P_{\ell}(\cos \theta) T^{I}(s, t, u) \tag{D.10}
\end{equation*}
$$

for $(I, \ell)=(1,1)$, making an expansion in $\mathbf{p}_{\mathrm{cm}}^{2}$, and comparing the coefficients, leads to the relation $m_{\rho}^{2}-4 m_{\pi}^{2}=3 f_{\pi}^{2} g^{2}$ which shows again the


Figure D.1: $p$-wave isovector $\pi \pi$ interaction. Dots: Partial wave analysis from Ref. [290]. Dashed line: $\mathcal{L}_{\pi \pi}^{(2)}$ calculation. Dashed-dotted line: $\mathcal{L}_{\pi \pi}^{(4)}$ calculation from Ref. [291]. Solid line: Effective interaction from Eq. (8.12). Thin solid lines: Explicit $\rho$ exchange from Eq. (8.8) with and without (momentum dependent) width for the $\rho$.
deviation of $3 / 2$ from the KSFR relation up to a correction from the pion mass. However, taking only the $s$-channel vector exchange, which is given by the second term of $T^{1}$ Eq. (D.9), we obtain after projection to the $p$-wave:

$$
\begin{equation*}
m_{\rho}^{2}-4 m_{\pi}^{2}=2 f_{\pi}^{2} g^{2} \tag{D.11}
\end{equation*}
$$

This is indeed the KSFR relation in Eq. (D.7) with some small correction which vanishes when $s$ is neglected against $m_{\rho}^{2}$ in the denominator of Eq. (D.9). Concluding, the restriction to $s$-channel vector exchange in $\pi \pi$ scattering restores the KSFR relation in the $p$-wave expansion of the scattering amplitude. However, $t$ - and $u$-channel vector exchange is also present, and this leads to the effective interaction in Eq. (8.12) which is $3 / 2$ times stronger than the interaction from the LO chiral Lagrangian.

Fig. D. 1 illustrates the behavior of the different theories together with data from Ref. [290]: The LO chiral Lagrangian underpredicts the strength of the experimental $T_{1}^{1}$ amplitude. In contrast, the interaction up to $\mathcal{L}_{\pi \pi}^{(4)}$ and the effective interaction from Eq. (8.12) describe better the data at low energies. The explicit $\rho$ exchange with width (thin line) delivers a good data description even beyond the $\rho$-mass.

One more remark is appropriate in the framework of this section: In the treatment of the $\rho$-meson as a heavy gauge field, the covariant derivative introduces the $\pi \rho$ interaction as we have seen in Sec. 8.3. Additionally, the original $\pi \pi$ interaction from Eq. (8.6) remains in this derivative. In the present model, we have omitted this term, as has been also done, e.g., in Ref. [268]. This leads to better agreement with the data in the $T_{1}^{1}$-channel and ensures the KSFR relation. It is possible to keep the original chiral interaction, but then additional refinements have to be added added as, e.g., in Ref. [157].

## D.2.2 Unitarization of the $\pi \pi$-amplitude with the $K$ matrix

The $K$-matrix is defined via the $S$-matrix as

$$
\begin{equation*}
S_{K}(E)=\frac{1+i Q K}{1-i Q K}, \quad K=-\frac{T_{\text {tree }}}{16 \pi E} \tag{D.12}
\end{equation*}
$$

with the tree level amplitude $T_{\text {tree }}$ from Eq. (D.8) and the c.m. momentum $Q$ from Eq. (8.32). The unitarized amplitude $T_{(u), 1}^{1}$ which is given by

$$
\begin{equation*}
T_{(u), 1}^{1}=\frac{T_{1}^{1}}{1+2 i Q T_{1}^{1} / E}, \quad T_{1}^{1}=-\frac{E^{2}-4 m_{\pi}^{2}}{96 \pi f_{\pi}^{2}} \tag{D.13}
\end{equation*}
$$

can be parametrized via phase shift as

$$
\begin{equation*}
\delta_{1}^{1}=\frac{1}{2} \arctan \frac{-\operatorname{Re} T_{(u), 1}^{1}}{\frac{E}{4 Q}+\operatorname{Im} T_{(u), 1}^{1}} \tag{D.14}
\end{equation*}
$$

## D. 3 The $\rho$-meson in the heatbath

## D.3.1 Analytic results

The analytical expressions and numerical contributions from the set of gauge invariant diagrams in Fig. 8.3 are given which are obtained from the interactions from Sec. 8.3. With
$L_{ \pm}(a, b):=\log \left|\frac{\left[m_{\rho}^{2}+(p \pm q)^{2}-\left(b \omega-a \omega^{\prime}\right)^{2}\right]\left[m_{\rho}^{2}+(p-q)^{2}-\left(b \omega+a \omega^{\prime}\right)^{2}\right]}{\left[m_{\rho}^{2}+(p \mp q)^{2}-\left(b \omega-a \omega^{\prime}\right)^{2}\right]\left[m_{\rho}^{2}+(p+q)^{2}-\left(b \omega+a \omega^{\prime}\right)^{2}\right]}\right|$,
where $\omega^{2}=q^{2}+m_{\pi}^{2}$ and $\omega^{\prime 2}=p^{2}+m_{\pi}^{2}$, we obtain for the real parts of the diagrams in Fig. 8.3, left column:

$$
\begin{aligned}
& \Pi_{(1 a 1)}^{00}\left(k_{0}=0, \mathbf{k} \rightarrow \mathbf{0}\right) \\
& =\left.\frac{-e^{2} g^{2}}{(2 \pi)^{4} m_{\rho}^{2}} \frac{\partial}{\partial \alpha} \frac{\partial}{\partial \beta}\right|_{\alpha=\beta=1} \int_{0}^{\infty} d p \int_{0}^{\infty} d q \frac{p q}{\alpha \beta \omega \omega^{\prime 3}} n[\omega] n\left[\alpha / \beta \omega^{\prime}\right] \\
& \left.\times\left[-8 m_{\rho}^{2} p q+\left[\left(2 m_{\pi} m_{\rho}\right)^{2}-\left(m_{\rho}^{2}-\left((\alpha / \beta)^{2}-1\right) \omega^{\prime 2}\right)^{2}\right] L_{-}(\alpha / \beta, 1)\right)\right] \text {, } \\
& \Pi_{(1 \mathrm{a} 2)}^{00}\left(k_{0}=0, \mathbf{k} \rightarrow \mathbf{0}\right) \\
& =\left.\frac{-e^{2} g^{2}}{2(2 \pi)^{4}} \frac{\partial}{\partial \alpha} \frac{\partial}{\partial \beta}\right|_{\alpha=\beta=1} \int_{0}^{\infty} d p \int_{0}^{\infty} d q \frac{p q}{\left(\omega \omega^{\prime}\right)^{2}} n[\beta \omega] n\left[\alpha \omega^{\prime}\right] \\
& \times\left[4 m_{\pi}^{2}-m_{\rho}^{2}+2\left(\left(\beta^{2}-1\right) \omega^{2}+\left(\alpha^{2}-1\right) \omega^{\prime 2}\right)\right. \\
& \left.-\frac{1}{m_{\rho}^{2}}\left(\left(\alpha^{2}-1\right) \omega^{\prime 2}-\left(\beta^{2}-1\right) \omega^{2}\right)^{2}\right] L_{+}(\alpha, \beta), \\
& \Pi_{(4 \mathrm{a})}^{00}\left(k_{0}=0, \mathbf{k} \rightarrow \mathbf{0}\right) \\
& =\left.\frac{-e^{2} g^{2}}{(2 \pi)^{4} m_{\rho}^{2}} \frac{\partial}{\partial \alpha}\right|_{\alpha=1} \int_{0}^{\infty} d p \int_{0}^{\infty} d q \frac{p q}{\alpha \omega \omega^{\prime 3}} n[\omega] n\left[\alpha \omega^{\prime}\right] \\
& \times\left[-8 m_{\rho}^{2} p q+\left[\left(2 m_{\pi} m_{\rho}\right)^{2}-\left(m_{\rho}^{2}-\left(\alpha^{2}-1\right) \omega^{\prime 2}\right)^{2}\right] L_{-}(\alpha, 1)\right] \text {, } \\
& \Pi_{(5 \mathrm{a})}^{00}\left(k_{0}=0, \mathbf{k} \rightarrow \mathbf{0}\right) \\
& =\left.\frac{6 e^{2} g^{2}}{(2 \pi)^{4} m_{\rho}^{2}} \frac{\partial}{\partial \alpha}\right|_{\alpha=1} \int_{0}^{\infty} d p \int_{0}^{\infty} d q \frac{p q}{\omega \omega^{\prime 2}} n[\omega] n\left[\alpha \omega^{\prime}\right] \\
& \times\left[-\alpha \omega^{\prime}\left(m_{\rho}^{2}+\left(1-\alpha^{2}\right) \omega^{\prime 2}\right) L_{-}(\alpha, 1)+\omega\left(m_{\rho}^{2}+\left(\alpha^{2}-1\right) \omega^{\prime 2}\right) L_{+}(\alpha, 1)\right] \text {, } \\
& \Pi_{(6 \mathrm{a})}^{00}\left(k_{0}=0, \mathbf{k} \rightarrow \mathbf{0}\right) \\
& =\frac{-3 e^{2} g^{2}}{(2 \pi)^{4} m_{\rho}^{2}} \int_{0}^{\infty} d p \int_{0}^{\infty} d q \frac{p q}{\omega \omega^{\prime}} n[\omega] n\left[\omega^{\prime}\right] \\
& \times\left[-\left(m_{\rho}^{2}-\omega^{2}-\omega^{\prime 2}\right) L_{-}(1,1)+2 \omega \omega^{\prime} L_{+}(1,1)\right],
\end{aligned}
$$

$$
\begin{align*}
& \log Z_{(\mathrm{en} 1)} \\
= & \left.\frac{3 g^{2} V}{4 T(2 \pi)^{4}} \int_{0}^{\infty} d p \int_{0}^{\infty} d q \frac{p q}{\omega \omega^{\prime}} n[\omega] n\left[\omega^{\prime}\right]\left[-8 p q+\left(4 m_{\pi}^{2}-m_{\rho}^{2}\right) L_{-}(1,1)\right)\right] . \tag{D.16}
\end{align*}
$$

In these expressions the poles from the $\rho$-propagator have been omitted as discussed in Appendix D.3.2. The use of derivatives in Eq. (D.16) cures infrared divergences which occur (see Appendix D.3.2). The logarithmic pole in the numerically relevant integration regions for $p$ and $q$ is in all cases is given by

$$
\begin{equation*}
m_{\rho}^{2}+(p-q)^{2}-\left(b \omega+a \omega^{\prime}\right)^{2}=0 \tag{D.17}
\end{equation*}
$$

where $a, b$ take values according to the arguments of $L_{ \pm}(a, b)$ of Eqs. (D.15,D.16). The singularity leads to an imaginary part which we neglect. The issue of imaginary parts is discussed in Sec. 8.6.1. The diagrams from the second column of Fig. 8.3 are calculated straightforward with the results

$$
\begin{equation*}
\Pi_{(1 c 1)}^{00}=-\frac{e^{2} g^{2}}{m_{\rho}^{2}} C^{2}, \quad \Pi_{(5 c 1)}^{00}=-\frac{2 e^{2} g^{2}}{m_{\rho}^{2}} C D, \quad \Pi_{(6 c)}^{00}=-\frac{e^{2} g^{2}}{m_{\rho}^{2}} D^{2} \tag{D.18}
\end{equation*}
$$

in the static limit $\left(k_{0}=0, \mathbf{k} \rightarrow \mathbf{0}\right)$.
Fig. D. 2 shows the numerical results. For every diagram, the contribution of the dynamical $\rho$-meson at its physical mass of $m_{\rho}=770 \mathrm{MeV}$ (indicated with " 770 ") is displayed. Additionally, the amplitudes for $\rho$-masses of $m_{\rho}^{i}=1070,1770,2770,10000 \mathrm{MeV}$ are evaluated, multiplying the result with $\left(m_{\rho}^{i} / m_{\rho}\right)^{2}$ (gray lines). This would correspond to a $\rho$-meson with mass $m_{\rho}^{i}$ whose strong coupling $g$ is increased by $\left(m_{\rho}^{i} / m_{\rho}\right)$. This is indeed equivalent to the heavy $\rho$ limit from Sec. 8.3 and convergence of the results from Eq. (D.16) towards the heavy $\rho$ limit of Sec. 8.4.2 (dashed lines) is observed. This convergence is, on the other hand, a useful tool to check the results from Eq. (D.16).

The large difference of both models at $m_{\rho}=770 \mathrm{MeV}$ in case of the diagrams (4a) and (eff3) is due to terms that partially cancel: diagram (eff3) $\sim D(2 D-C)$. For the calculation of the entropy in Eq. (8.4), the correction to $\log Z$ is needed which is very different for diagram (eneff) and diagram


Figure D.2: Numerical results for the diagrams from Fig. 8.3, Eq. (D.16) as a function of $T[\mathrm{MeV}]$. Selfenergy $\Pi /\left(e^{2} T^{4}\right)$ in $\left[\mathrm{MeV}^{-2}\right]$ for all plots, except the correction to $Z: \log Z_{(\text {en1 })} /\left(V T^{4}\right)$ in $[\mathrm{MeV}]^{-1}$. Results for different $m_{\rho}$ with solid lines. Dashed lines: Corresponding diagrams from the heavy $\rho$ limit, see Tab. 8.1 and Eq. (8.25).
(en1) as Fig. D. 2 shows. The discrepancy can be traced back to the different high energy behavior of the amplitudes. In any case, the total size of the entropy correction, compared to the result of the free pion gas, Eq. (8.20), is small and of no relevance for the final results.

## D.3.2 Calculation of diagram (1a2)

The calculation of one of the diagrams from Fig. 8.3 is outlined in more detail. The evaluation of the other diagrams is carried out in an analog way, with the results given in Eqs. (D.16,D.18). For diagram (1a2) it is most convenient to treat the vertex correction first, that is given by the left side of the diagram. The external photon momentum has to be set to zero from the beginning of the calculation as has been shown in Appendix D.1; the matter part of the vertex correction reads for an external $\pi^{+}$:

$$
\begin{align*}
& \Gamma^{0}\left[k^{0}=0, \mathbf{k} \rightarrow \mathbf{0}\right] \\
= & \frac{2 e g^{2} p^{0}}{\pi^{2}} \int_{0}^{\infty} d q q^{2} \int_{-1}^{1} d x \frac{1}{2 \pi i} \int_{-i \infty+\epsilon}^{i \infty+\epsilon} d q^{0} n\left[q^{0}\right] \frac{\left(q^{0}\right)^{2}}{\left(\left(q^{0}\right)^{2}-\omega^{2}\right)^{2}} \\
\times & \frac{4 m_{\pi}^{2}-m_{\rho}^{2}+2\left(\left(p^{0}\right)^{2}+\left(q^{0}\right)^{2}-\omega^{2}-\omega^{\prime 2}\right)-\left(1 / m_{\rho}^{2}\right)\left(\left(p^{0}\right)^{2}-\left(q^{0}\right)^{2}+\omega^{2}-\omega^{\prime 2}\right)^{2}}{\left(\left(p^{0}+q^{0}\right)^{2}-\eta^{2}\right)\left(\left(p^{0}-q^{0}\right)^{2}-\eta^{2}\right)} \tag{D.19}
\end{align*}
$$

where the contour integration method of Ref. [265] is used for the summation over Matsubara frequencies. In Eq. (D.19), $\omega^{2}=q^{2}+m_{\pi}^{2}$ and $\eta^{2}=p^{2}+q^{2}-$ $2 p q x+m_{\rho}^{2}$. A problem occurs when closing the integration contour in the right $q^{0}$ half plane: The residue at $\omega$ from the double pole of the two pion propagators at the same energy is given by

$$
\begin{equation*}
\text { Res }\left.f(z)\right|_{z=\omega}=\lim _{z \rightarrow \omega} \frac{1}{(m-1)!} \frac{d^{m-1}}{d z^{m-1}}\left[f(z)(z-\omega)^{m}\right] \tag{D.20}
\end{equation*}
$$

at $m=2$. The derivative also applies to the denominator in the second line of Eq. (D.19) from the $\rho$ propagator. The integrand exhibits then a divergence of the type

$$
\begin{equation*}
\Gamma\left[k^{0}=0, \mathbf{k} \rightarrow \mathbf{0}\right] \sim \int d q \frac{1}{a-q^{2}}, \quad \text { at } \quad p^{0}=0 \tag{D.21}
\end{equation*}
$$

The divergence affects only the zero-mode $p^{0}=0$, but when the pion lines are closed later on, in order to obtain diagram (1a2), the integrals in Eq. (D.19)
are not defined any more, and one finds poles of the type $1 /\left(a-\mathbf{q}^{2}\right)$ in the three-momentum integration. This infrared divergence, for the external photon at $k=0$, occurs in diagrams that contain, besides two or more propagators at the same momentum, an additional propagator as in this case the one of the $\rho$-meson.

The complication can be most easily overcome with the introduction of additional parameters according to

$$
\begin{equation*}
\left.\frac{1}{2 \omega^{2}} \frac{\partial}{\partial \beta}\right|_{\beta=1} \frac{1}{\left(q^{0}\right)^{2}-(\beta \omega)^{2}}=\frac{1}{\left(\left(q^{0}\right)^{2}-\omega^{2}\right)^{2}}, \tag{D.22}
\end{equation*}
$$

and performing the derivative numerically after the three-momentum integration. Still, singularities of the $1 / q$ type remain, but they are well-defined by the $\epsilon$-prescription in the $q^{0}$-integral of Eq. (D.19). We can in this case, as well as in all other diagrams from Fig. 8.3, integrate the angle $x=\cos (\mathbf{p}, \mathbf{q})$ analytically, thus being left with logarithmic singularities, that are easily treated numerically with the help of Eq. (D.17).

It has been checked for all diagrams in Fig. 8.3 that the poles of the $\rho$ meson can be omitted: In Eq. (D.19) the denominator of the second line from the $\rho$-propagator produces two single poles in the right $q^{0}$ half plane. Taking these residues into account in the contour integration leads to deviations of less than $1 \%$ of the result for the vertex correction, for all values of $\left(p^{0}, \mathbf{p}\right)$ and up to temperatures $T \sim 200 \mathrm{MeV}$. Intuitively, this is clear since these poles produce a strong Bose-Einstein suppression $\sim n\left[m_{\rho}\right]$ and extra powers of $m_{\rho}$ in the denominator compared to the pion pole. This approximation is made for all results of Eq. (D.16). See also Sec. 8.6.1 where the approximation is again tested.

The rest of the evaluation of diagram (1a2) is straightforward up to the introduction of an additional derivative parameter in the same manner as above. As one can see in Fig. 8.3, a topologically different structure, diagram (1c1), is possible for the combination of two $\gamma \pi \pi$ and two $\rho \pi \pi$-vertices. This diagram is easily evaluated and has to be added.

## D.3.3 The $\gamma \pi \rho$ system at finite $\mu$.

Explicit results for $\log Z$ from the diagrams (b), (c), and (d) from Fig. 8.5 are given from which the electric mass can be directly calculated using Eq. (8.18). As argued in the main text, the diagrams (b,c,d) from Fig. 8.5 lead to the same CF as all diagrams with dynamical $\rho$ from Figs. 8.3 and 8.4. For diagram (b), the result is

$$
\begin{align*}
& \log Z_{(\mathrm{b})}^{\pi \pi}(\mu) \\
= & \frac{-g^{2} \beta V}{32}\left(U_{+}+U_{-}\right)\left(U_{+}+U_{-}+4 D\right)+\frac{g^{2} \beta V}{128 \pi^{4}} \int_{0}^{\infty} d p \int_{0}^{\infty} d q \frac{p q\left(4 m_{\pi}^{2}-m_{\rho}^{2}\right)}{\omega \omega^{\prime}} \\
\times & {\left[\left(n_{+} n\left[\omega^{\prime}-\mu\right]+n_{-} n\left[\omega^{\prime}+\mu\right]\right) \log _{1}+\left(n_{+} n\left[\omega^{\prime}+\mu\right]+n_{-} n\left[\omega^{\prime}-\mu\right]\right) \log _{2}\right] } \tag{D.23}
\end{align*}
$$

with $U_{ \pm}$and $V_{ \pm}$from Eq. (8.27), $n_{ \pm}=n[\omega \pm \mu]+2 n[\omega]$, and

$$
\begin{align*}
& \log _{1}=\log \left[\frac{m_{\rho}^{2}+(p-q)^{2}-\left(\omega+\omega^{\prime}\right)^{2}}{m_{\rho}^{2}+(p+q)^{2}-\left(\omega+\omega^{\prime}\right)^{2}}\right] \\
& \log _{2}=\log \left[\frac{m_{\rho}^{2}+(p-q)^{2}-\left(\omega-\omega^{\prime}\right)^{2}}{m_{\rho}^{2}+(p+q)^{2}-\left(\omega-\omega^{\prime}\right)^{2}}\right] \tag{D.24}
\end{align*}
$$

with $\omega^{2}=q^{2}+m_{\pi}^{2}, \omega^{\prime 2}=p^{2}+m_{\pi}^{2}$, and $n$ the Bose-Einstein distribution. We have checked that $\log Z_{(\mathrm{b})}(\mu=0)=\log Z_{(\text {en1 })}$ from Eq. (D.16). The diagram (c) in Fig. 8.5 with

$$
\begin{equation*}
\log Z_{(c)}^{\pi \pi}(\mu)=-\frac{g^{2} \beta V}{8 m_{\rho}^{2}}\left(V_{+}-V_{-}\right)^{2} \tag{D.25}
\end{equation*}
$$

is zero for $\mu=0$ and therefore $\log Z_{\text {(c) }}$ does not contribute to the entropy but only to the CF.

The diagram (d) in Fig. 8.5 contains a $\rho \rho \pi \pi$ vertex that comes from Eq. (8.9). This interaction is required by the gauge invariance of the $\rho$ with the contribution to $m_{\text {el }}$ given by

$$
\begin{equation*}
\log Z_{(\mathrm{d})}^{\pi \pi}(\mu)=-\frac{3 \beta V g^{2}}{16}\left[\left(U_{+}^{\rho}+U_{-}^{\rho}+2 D^{\rho}\right)\left(U_{+}^{\pi}+U_{-}^{\pi}+2 D^{\pi}\right)-4 D^{\rho} D^{\pi}\right] . \tag{D.26}
\end{equation*}
$$

The upper index for $U, D$ indicates which mass has to be used in the meson energy $\omega$ of Eqs. (8.17) and (8.27).


Figure D.3: Elementary diagrams to be saturated with an additional photon in order to construct the selfenergy from Fig. 8.3.

## D.3.4 Charge conservation

In a calculation of CF the conservation of charge is essential and therefore gauge invariance of the diagrams must be ensured. The set of diagrams in Fig. 8.2 has been constructed using the Ward identity following the procedure outlined in Appendix D.1. They ought to be charge conserving by construction. Nevertheless, it is desirable to have an explicit proof. The diagrams from Fig. 8.2 represent the heavy $\rho$ limit of the ones with dynamical $\rho$-mesons in Fig. 8.3 as shown in Appendix D.3.1. Therefore, it is enough to show charge conservation for the latter.

From Ref. [292] we utilize the part of the proof that concerns closed loops. The main statement extracted from Ref. [292] is, adapted to the current situation: Define a diagram with one external photon at momentum $k$, not necessarily on-shell. By inserting another photon in all possible ways in the diagram, a set of new diagrams of photon selfenergy type emerges: For example, the four diagrams in Fig. D. 3 lead to the photon self energies in the two left columns of Fig. 8.3 plus the (vanishing) diagrams (1c2), (4c), and (5c2) from Fig. 8.4, once saturated with an additional photon (we do not allow direct $\gamma \rho \rho$ and $\gamma \gamma \rho \rho$ vertices). The selfenergy diagrams $\Pi_{i}^{\mu \nu}$ constructed in this way are charge conserving, and $k_{\mu} \sum_{i} \Pi_{i}^{\mu \nu}=0$ for the sum over all diagrams.

For this statement, it has to be shown first that indeed the diagrams from Fig. 8.3, including all symmetry and isospin factors, turn out from the ones of Fig. D.3. This short exercise reveals that there are two classes of selfenergy diagrams: one comes from inserting photons in diagrams (1) and (2) of Fig. D. 3 and the other one from inserting photons in (3) and (4). Thus, there are two separate gauge-invariant classes. In a second step, one has to show the statement from Ref. [292] for the current theory which is different from QED and richer in vertices of different type:
(I) The $\gamma \pi \pi$ couplings in Fig. D. 3 can be transformed into $\gamma \gamma \pi \pi$ couplings by inserting an additional photon. The $\rho \pi \pi$ vertex can be transformed into a $\gamma \rho \pi \pi$ vertex. These transformations which are a consequence of the momentum dependence of the vertices are essential for the proof.
(II) For this proof we do not allow direct $\gamma \rho \rho$ and $\gamma \gamma \rho \rho$ couplings. However, diagrams which include these couplings as in Fig. 8.4 form a disjoint gauge class anyway.
(III) The gauge invariance of the diagrams with dynamical $\rho$ in Fig. 8.3 survives in the heavy $\rho$ limit: According to Appendix D.3.1, the amplitudes at a $\rho$-mass of $m_{\rho}^{i}$ are multiplied by $\left(m_{\rho}^{i} / m_{\rho}\right)^{2}$, with $m_{\rho}$ the physical mass. Then, the limit $m_{\rho}^{i} \rightarrow \infty$ is taken and the effective diagrams of Fig. 8.2 turn out. The gauge invariance of these diagrams follows.

This simple graphical proof demonstrates the charge conservation, and, turning the argument around, provides a useful tool to ensure that the amplitudes, including symmetry and isospin factors, have been correctly determined in Eqs. (D.16,D.18) and Tab. 8.1.

## D. 4 Solutions for the resummations

An additional technical complication appears in the evaluation of Eq. (8.38) for the summation (n) when the structure of the vertices between $\pi^{ \pm} \pi^{ \pm}$-loops or $\pi^{0} \pi^{ \pm}$-loops is inspected: The interaction of Eq. (8.12) leads to a Feynman rule of the form $\left(p^{2}+q^{2}+6 p q\right)$ for the vertex between two charged pion loops of momenta $p$ and $q$, and of the form $\left(p^{2}+q^{2}\right)$ between a charged and a $\pi^{0}$-loop, always implying the corresponding shift $p^{0} \rightarrow p^{0} \pm \mu\left(q^{0} \rightarrow q^{0} \pm \mu\right)$
for the inclusion of finite chemical potential. Therefore, the loops can not be factorized easily in the way Eq. (8.38) suggests. In order to cast the resummations in a manageable from, we introduce for every term of the sum $\left(p^{2}+q^{2}+6 p q\right)$ an entry in an additional index that runs from 1 to 3 . The Eq. (8.38) is then to be read as a matrix equation in its variables. With the definitions

$$
\begin{equation*}
W_{ \pm}=\frac{1}{\pi^{2}} \int_{0}^{\infty} d q \omega n[\omega \pm \mu], \quad X_{ \pm}=\frac{1}{\pi^{2}} \int_{0}^{\infty} d q \omega^{2} n[\omega \pm \mu], \tag{D.27}
\end{equation*}
$$

additional to the ones of Eq. (8.17) and (8.27), the entries of the Faddeev-like equations (8.38) can be cast in the form

$$
\begin{align*}
& a_{0}=\left(D, m_{\pi}^{2} D, 0\right), \quad a_{ \pm}=\frac{1}{4}\left(U_{+}+U_{-}, m_{\pi}^{2}\left(U_{+}+U_{-}\right), \sqrt{6}\left(V_{-}-V_{+}\right)\right) \\
& c_{0}=\left(\begin{array}{l}
m_{\pi}^{2} D \\
D \\
0
\end{array}\right), \quad c_{ \pm}=\frac{1}{4}\left(\begin{array}{l}
m_{\pi}^{2}\left(U_{+}+U_{-}\right) \\
U_{+}+U_{-} \\
\sqrt{6}\left(V_{-}-V_{+}\right)
\end{array}\right), \\
& l_{0}=\left(\begin{array}{lll}
C-3 D & m_{\pi}^{2}(C-5 D) & 0 \\
\frac{1}{m_{\pi}^{2}}(C-D) & C-3 D & 0 \\
0 & 0 & 0
\end{array}\right), \\
& l_{ \pm}=\frac{1}{8}\left(\begin{array}{ll}
W_{+}-3 U_{+}+W_{-}-3 U_{-} & m_{\pi}^{2}\left(W_{+}-5 U_{+}+W_{-}-5 U_{-}\right) \\
\frac{1}{m_{2}^{2}}\left(W_{+}-U_{+}+W_{-}-U_{-}\right) & W_{+}-3 U_{+}+W_{-}-3 U_{-} \\
\frac{\sqrt{6}}{m_{\pi}^{2}}\left(X_{-}-V_{-}-X_{+}+V_{+}\right) & \sqrt{6}\left(3 V_{+}-X_{+}-3 V_{-}+X_{-}\right)
\end{array}\right. \\
& \left.\begin{array}{l}
\quad \sqrt{6}\left(3 V_{+}-X_{+}-3 V_{-}+X_{-}\right) \\
\times \\
\frac{\sqrt{6}}{m_{\pi}^{2}}\left(X_{-}-V_{-}-X_{+}+V_{+}\right) \\
6\left(W_{+}+W_{-}\right)
\end{array}\right) . \tag{D.28}
\end{align*}
$$

With this extension Eq. (8.38) is easily solved. In order to check for bulk errors, one can expand the result in the coupling constant, and at order $g^{2} / m_{\rho}^{2}$ Eq. (8.26) indeed turns out. At order $g^{4} / m_{\rho}^{4}$ the expansion gives the


Figure D.4: Resummation (f) from the expansion of the LO chiral Lagrangian to all orders.
linear chain of three loops which also emerges from the diagram (r) at that order, and the results are identical.

The ring diagram (r) from Fig. 8.11 with $N$ "small" loops is given by

$$
\begin{align*}
\log Z_{(\mathrm{r}), N}(\mu)= & \frac{-(-1)^{N} \beta V}{2 N \pi^{2}} \int_{0}^{\infty} d p p^{2} \operatorname{Res}\left[\left(\frac{\Pi_{ \pm}\left(p^{0}\right)}{\left(p^{0}+\mu\right)^{2}-\omega^{2}}\right)^{N} n\left[p^{0}\right]\right. \\
& \left.+\left(\frac{\Pi_{ \pm}\left(-p^{0}\right)}{\left(p^{0}-\mu\right)^{2}-\omega^{2}}\right)^{N} n\left[p^{0}\right]+\left(\frac{\Pi_{0}\left(p^{0}\right)}{\left(p^{0}\right)^{2}-\omega^{2}}\right)^{N} n\left[p^{0}\right]\right] \tag{D.29}
\end{align*}
$$

for $N \geq 3$. The residue is taken for the variable $p^{0}$ at the poles of order $N$ in the right $p^{0}$ half-plane. The tadpole selfenergies $\Pi_{ \pm}$and $\Pi_{0}$ for the charged and neutral pion propagator in Eq. (D.29) are given by

$$
\begin{align*}
\Pi_{ \pm}\left(p^{0}\right) & =-\frac{g^{2}}{4 m_{\rho}^{2}}\left(\left[\left(p^{0}+\mu\right)^{2}-\omega^{2}+2 m_{\pi}^{2}\right]\left[U_{+}+U_{-}+2 D\right]\right. \\
& \left.+6\left(p^{0}+\mu\right)\left(V_{-}-V_{+}\right)\right) \\
\Pi_{0}\left(p^{0}\right) & =-\frac{g^{2}}{2 m_{\rho}^{2}}\left(\left(p^{0}\right)^{2}-\omega^{2}+2 m_{\pi}^{2}\right)\left(U_{+}+U_{-}\right) \tag{D.30}
\end{align*}
$$

which is immediately obtained from $a_{ \pm}$and $a_{0}$ in Eq. (D.28).
There is an additional possible resummation scheme displayed as (f) in Fig. D.4. The interaction is obtained from the kinetic term of the LO chiral Lagrangian Eq. (8.6) by expanding it to all orders in the pion fields which
means an exact calculation of the exponentials $U=\exp \left(i \Phi / f_{\pi}^{2}\right)$ in Eq. (8.6). The mass correction with $\mathcal{M}$ from Eq. (8.6) is tiny (see Sec. 8.3) and can be safely neglected. The Lagrangian for $2 n$ fields is then given by $(n \geq 2)$ :

$$
\begin{align*}
\mathcal{L}_{2 n \pi}^{(2)} & =\frac{(-1)^{n} 4^{n-1} f_{\pi}^{2(1-n)}}{(2 n)!}\left(\left(\pi^{0}\right)^{2}+2 \pi^{+} \pi^{-}\right)^{n-2} \\
& \times\left(\left(\pi^{+} \stackrel{\leftrightarrow}{\partial_{\mu}} \pi^{-}\right)^{2}-2\left(\pi^{-} \overleftrightarrow{\partial_{\mu}} \pi^{0}\right)\left(\pi^{+} \overleftrightarrow{\partial^{\mu}} \pi^{0}\right)\right) \tag{D.31}
\end{align*}
$$

At $\mu=0$, the grand canonical partition function from this interaction, summed over all $n$, results in a surprisingly simple expression,

$$
\begin{equation*}
\log Z_{(\mathrm{f})}(\mu=0)=\frac{\beta V m_{\pi}^{2}}{2}\left(f_{\pi}^{2}\left(1-e^{-\frac{D}{f_{\pi}^{2}}}\right)-D\right) \tag{D.32}
\end{equation*}
$$

with $D$ from Eq. (8.17). This is the special case of the result for finite $\mu$ ( $n \geq 2$ ),

$$
\left.\begin{array}{rl} 
& \log Z_{(\mathrm{f})}(\mu) \\
= & \beta V \sum_{n=2}^{\infty} \frac{(-1)^{n} 4^{n-1} f_{\pi}^{2(1-n)}}{(2 n)!} \sum_{k=0}^{n-2} 2^{k-n-1}(-D)^{k} \sqrt{\pi}\left(U_{+}+U_{-}\right)^{n-k-2} \\
\times & \binom{n-2}{k}\left(\frac{4 D(n-k-2)!\left(m_{\pi}^{2}\left(U_{+}+U_{-}\right)^{2}-(2+k-n)\left(V_{+}-V_{-}\right)^{2}\right)}{\Gamma(-1 / 2-k)\left(U_{-}+U_{+}\right)}\right. \\
+ & \frac{\Gamma(n-k)\left(-m_{\pi}^{2}\left(U_{+}+U_{-}\right)\left(2 D+U_{+}+U_{-}\right)+(1+k-n)\left(V_{+}-V_{-}\right)^{2}\right)}{-\Gamma(1-k+n)\left(V_{+}-V_{-}\right)^{2}} \\
\Gamma(1 / 2-k) \tag{D.33}
\end{array}\right) .
$$

The sum over $k$ comes from the expansion of the polynomial of order $n-2$ in Eq. (D.31). The possibilities of contracting $2 k$ neutral pion fields have been rewritten, $\prod_{i=0, k-1}(2 k-1-2 i)=(-2)^{k} \Gamma(1 / 2) / \Gamma(1 / 2-k)$.

The structure of the Lagrangian in Eq. (D.31) resembles the vertex structure of resummation ( t ) from Sec. 8.7 with the $\rho \rho \pi \pi$ interaction: At any order $n \geq 2$ in the interaction, there are only two derivative couplings. Also, the diagrammatic representation of resummation ( t ) has the same topology as diagram (f) once the heavy $\rho$ limit is taken. Indeed, we observe a close numerical correspondence between the resummations ( t ) and (f). Thus, it is interesting to note that the $\rho$ tadpole resummation is well described by an
expansion of $\mathcal{L}_{\pi \pi}^{(2)}$ to all orders. The resummation (f) is not included in the final numerical results due to these potential double counting problems with ( t ).

## D. 5 Extension to $S U(3)$

It is straightforward to extend the study of CF and other thermodynamical observables to $S U(3)$. Compared to the pion, the other members of the meson octet have higher masses which simplifies the selection of relevant processes in a thermal heat bath: we regard diagrams which do not contain any pion as kinematically suppressed. The contribution to $\log Z$ at $g^{2}$ then consists of diagram (b) in Fig. 8.5 with one pion line replaced by a kaon and the $\rho$ replaced by the $K^{*}(892)$. The $\pi K K^{*}$ interaction follows from Eq. (8.8) in the $S U(3)$ version by extending the representation in Eq. (8.7) to the full meson and vector meson octet in the standard way [269, 270]. The result reads

$$
\begin{align*}
& \log Z_{(b)}^{\pi K}(\mu) \\
= & -\frac{g^{2} \beta V}{32}\left[\left(U_{+}^{\pi}+U_{-}^{\pi}\right)\left(U_{+}^{K}+U_{-}^{K}+2 D^{K}\right)+D^{\pi}\left(U_{+}^{K}+U_{-}^{K}\right)\right] \\
+ & \frac{g^{2} \beta V}{128 \pi^{4}}\left(2 m_{\pi}^{2}+2 m_{K}^{2}-m_{K^{*}}^{2}\right) \int_{0}^{\infty} d p \int_{0}^{\infty} d q \frac{p q}{\omega \omega^{\prime}} \\
{[ } & \left(n_{+} n\left[\omega^{\prime}-\mu\right]+n_{-} n\left[\omega^{\prime}+\mu\right]+\frac{1}{2} n\left[\omega^{\prime}\right](n[\omega+\mu]+n[\omega-\mu])\right) \log _{1} \\
& \left.+\left(n_{+} n\left[\omega^{\prime}+\mu\right]+n_{-} n\left[\omega^{\prime}-\mu\right]+\frac{1}{2} n\left[\omega^{\prime}\right](n[\omega+\mu]+n[\omega-\mu])\right) \log _{2}\right] \tag{D.34}
\end{align*}
$$

where $\omega^{2}=q^{2}+m_{K}^{2}, \omega^{\prime 2}=p^{2}+m_{\pi}^{2}, n_{ \pm}=n[\omega \pm \mu]+n[\omega]$, and the upper index specifies the mass that has to be used in the definitions of $D$ and $U$ from Eqs. (8.17) and (8.27). The expressions $\log _{1}$ and $\log _{2}$ are given by Eq. (D.24) with the replacement $m_{\rho} \rightarrow m_{K^{*}}$ and $\omega, \omega^{\prime}$ defined as in Eq. (D.34).

Diagram (c) from Fig. 8.5 with $\pi, \rho$, and $K$ is possible. Also, the $K^{*} K^{*} \pi \pi$ term from Eq. (8.9) is present, shown in Fig. 8.5 (d) with the $\rho$ replaced by
a $K^{*}(892)$. The corresponding contributions are

$$
\begin{align*}
\log Z_{(c)}^{\pi K}(\mu) & =-\frac{g^{2} \beta V}{8 m_{\rho}^{2}}\left(V_{+}^{\pi}-V_{-}^{\pi}\right)\left(V_{+}^{K}-V_{-}^{K}\right) \\
\log Z_{(d)}^{\pi K}(\mu) & =-\frac{3 g^{2} \beta V}{32}\left[\left(U_{+}^{K^{*}}+U_{-}^{K^{*}}+2 D^{K^{*}}\right)\left(U_{+}^{\pi}+U_{-}^{\pi}+D^{\pi}\right)-2 D^{K^{*}} D^{\pi}\right] . \tag{D.35}
\end{align*}
$$

The electric mass from Eqs. (D.34) and (D.35) is plotted as " $\pi K$ dynamical" in Fig. 8.13.

The $\pi K$ interaction can be alternatively described by the LO chiral Lagrangian from Eq. (8.6) in the $S U(3)$ version (we do not try to construct an effective, point-like, $\pi K$ interaction from $K^{*}$ (892) exchange as it has been done for $\pi \pi$ via $\rho$ exchange). Using similar arguments as above, the calculation is reduced to diagram (a) in Fig. 8.5, with one pion replaced by a kaon. Taking only the kinetic part of Eq. (8.6) - contributions from the mass term are tiny - one obtains

$$
\begin{align*}
\log Z_{(a)}^{\pi K}(\mu) & =\frac{-\beta V}{96 f_{\pi} f_{K}}\left[6\left(V_{+}^{\pi}-V_{-}^{\pi}\right)\left(V_{+}^{K}-V_{-}^{K}\right)\right. \\
& \left.+\left(m_{\pi}^{2}+m_{K}^{2}\right)\left(U_{+}^{K}+U_{-}^{K}+2 D^{K}\right)\left(U_{+}^{\pi}+U_{-}^{\pi}+D^{\pi}\right)\right] \tag{D.36}
\end{align*}
$$

with $f_{K}=1.22 f_{\pi}$ taken from chiral perturbation theory [10]. The contribution from Eq. (D.36) is plotted as " $\pi K$ contact" in Fig. 8.13 with the dotted line.

In a similar way as in Sec. 8.6, it is possible to establish a density expansion for the $\pi K$ interaction that respects the Bose-Einstein statistics of the asymptotic states in $\pi K$ scattering. Following the same steps as in Sec. 8.6, we obtain, again assuming elastic unitarity (the $50 \%$ inelasticity in the $\delta_{2}^{1 / 2}$
partial wave changes the result only slightly),

$$
\begin{align*}
& B_{2}^{(\pi K), \text { Bose }}(\mu) \\
= & \frac{\beta}{4 \pi^{3}} \int_{m_{\pi}+m_{K}}^{\infty} d E \int_{-1}^{1} d x \int_{0}^{\infty} d k \frac{E k^{2}}{\sqrt{E^{2}+k^{2}}} \sum_{\ell=0,1,2, \cdots}(2 \ell+1) \\
\times & 2\left[\delta_{\ell}^{3 / 2}\left(n\left[\omega_{\pi}+\mu\right] n\left[\omega_{K}+\mu\right]+n\left[\omega_{\pi}-\mu\right] n\left[\omega_{K}-\mu\right]\right)\right. \\
& +\frac{1}{3}\left(\delta_{\ell}^{1 / 2}+2 \delta_{\ell}^{3 / 2}\right)\left(n\left[\omega_{\pi}\right] n\left[\omega_{K}+\mu\right]+n\left[\omega_{\pi}\right] n\left[\omega_{K}-\mu\right]\right) \\
& +\frac{2}{3}\left(\delta_{\ell}^{1 / 2}+2 \delta_{\ell}^{3 / 2}\right)\left(n\left[\omega_{\pi}+\mu\right] n\left[\omega_{K}\right]+n\left[\omega_{\pi}-\mu\right] n\left[\omega_{K}\right]+n\left[\omega_{\pi}\right] n\left[\omega_{K}\right]\right) \\
& \left.+\frac{1}{3}\left(2 \delta_{\ell}^{1 / 2}+\delta_{\ell}^{3 / 2}\right)\left(n\left[\omega_{\pi}-\mu\right] n\left[\omega_{K}+\mu\right]+n\left[\omega_{\pi}+\mu\right] n\left[\omega_{K}-\mu\right]\right)\right] . \tag{D.37}
\end{align*}
$$

The boosted Bose-Einstein factors are

$$
\begin{align*}
& n\left[\omega_{\pi, K} \pm \mu\right]=\frac{1}{e^{\beta\left(\omega_{\pi, K} \pm \mu\right)}-1} \\
& \omega_{\pi}=\gamma_{f}\left(E_{\pi}+\frac{k Q x}{\sqrt{E^{2}+k^{2}}}\right), \quad \omega_{K}=\gamma_{f}\left(E_{K}-\frac{k Q x}{\sqrt{E^{2}+k^{2}}}\right) \\
& \gamma_{f}=\left(1-\frac{k^{2}}{E^{2}+k^{2}}\right)^{-\frac{1}{2}}, \quad E_{\pi, K}=\sqrt{Q^{2}+m_{\pi, K}^{2}}=\frac{E^{2}+m_{\pi, K}^{2}-m_{K, \pi}^{2}}{2 E} \tag{D.38}
\end{align*}
$$

with the center-of-mass c.m. momentum of the particles,
$Q=1 /(2 E) \sqrt{\left(E^{2}-\left(m_{\pi}+m_{K}\right)^{2}\right)\left(E^{2}-\left(m_{\pi}-m_{K}\right)^{2}\right)}$. For $\mu=0$ and in the Boltzmann limit Eq. (D.37) reduces to the virial coefficient

$$
\begin{align*}
B_{2}^{(\pi K), \text { Boltz }}(\mu=0) & =\frac{1}{2 \pi^{3}} \int_{m_{\pi}+m_{K}}^{\infty} d E E^{2} K_{1}(\beta E) \\
& \times 4 \sum_{\ell=0,1,2, \cdots}(2 \ell+1)\left(4 \delta_{\ell}^{3 / 2}+2 \delta_{\ell}^{1 / 2}\right) \tag{D.39}
\end{align*}
$$

which shows the correct ratio of degeneracy between $\delta_{\ell}^{3 / 2}$ and $\delta_{\ell}^{1 / 2}$, but is an overall factor of 4 larger than one would expect - compare, e.g., to Eq. (8.30): Instead of a sum over isospin of the form $\sum_{I, \ell}(2 I+1)(2 \ell+1) \delta_{\ell}^{I}$,
the projection of charge channels of pions and kaons to the isospin channels leads to $4 \sum_{I, \ell}(2 I+1)(2 \ell+1) \delta_{\ell}^{I}$. In any case, the result Eq. (D.37) for $m_{\mathrm{el}}$, using the chiral $\pi K$ interaction at $1 /\left(f_{\pi} f_{K}\right)$, matches exactly the thermal loops in Eq. (D.36). This we have shown in the same way as in Sec. 8.6.1 by using Eq. (8.36) and the partial waves $T^{1 / 2}=(7 u-5 s-2 t) /\left(12 f_{\pi} f_{K}\right)$ and $T^{3 / 2}=(2 s-t-u) /\left(6 f_{\pi} f_{K}\right)$ (as in Eq. (D.36), we consider only the kinetic term of $\left.\mathcal{L}_{\pi K}^{(2)}\right)$. A similar test has been performed by starting from Eq. (D.39) and using the partial waves from above. From this the pressure has been calculated and results are identical to the pressure obtained from Eq. (D.36) by taking the Boltzmann limit of the statistical factors $n$ in the definition of $D, U$, and $V$. Additionally, an independent check for the Lorentz structure of Eq. (D.37) has been performed in the same way as in Sec. 8.6.1, this time for a $\phi_{1}^{2} \phi_{2}^{2}$ interaction of uncharged bosons with different masses $m_{\phi_{1}}$ and $m_{\phi_{2}}$.

## Bibliography

[1] M. Döring, E. Oset and M. J. Vicente Vacas, Phys. Rev. C 70, 045203 (2004)
[2] M. Döring, E. Oset and M. J. Vicente-Vacas, Nucl. Phys. A 755, 673 (2005).
[3] M. Döring, E. Oset and D. Strottman, Phys. Rev. C 73, 045209 (2006).
[4] M. Döring, E. Oset and D. Strottman, Acta Phys. Slov. 56, 221 (2005)
[5] M. Döring, E. Oset, and D. Strottman, AIP Conference Proceedings 842 (PANIC05 conference, Santa Fe), ISBN 0-7354-0338-4
[6] M. Döring, E. Oset and D. Strottman, Phys. Lett. B 639, 59 (2006)
[7] M. Döring, E. Oset and S. Sarkar, Phys. Rev. C 74, 065204 (2006)
[8] M. Döring and V. Koch, arXiv:nucl-th/0609073.
[9] M. Döring and V. Koch, Acta Phys. Polon. B 33, 1495 (2002)
[10] J. Gasser and H. Leutwyler, Nucl. Phys. B 250, 465 (1985).
[11] U.-G. Meißner, Rept. Prog. Phys. 56, 903 (1993).
[12] V. Bernard, N. Kaiser and U.-G. Meißner, Int. J. Mod. Phys. E 4, 193 (1995).
[13] G. Ecker, Prog. Part. Nucl. Phys. 35, 1 (1995).
[14] V. Koch, Int. J. Mod. Phys. E 6, 203 (1997)
[15] A. Pich, Rept. Prog. Phys. 58, 563 (1995)
[16] J. A. Oller, E. Oset and A. Ramos, Prog. Part. Nucl. Phys. 45, 157 (2000)
[17] J. A. Oller and E. Oset, Phys. Rev. D 60, 074023 (1999).
[18] G. F. Chew and S. Mandelstam, Phys. Rev. 119, 467 (1960).
[19] U.-G. Meißner and J. A. Oller, Nucl. Phys. A 673, 311 (2000).
[20] A. D. Lahiff and I. R. Afnan, Phys. Rev. C 66, 044001 (2002)
[21] S. Kondratyuk, A. D. Lahiff and H. W. Fearing, Phys. Lett. B 521, 204 (2001)
[22] E. Oset and A. Ramos, Nucl. Phys. A 635, 99 (1998).
[23] D. Jido, E. Oset and A. Ramos, Phys. Rev. C 66, 055203 (2002)
[24] J. A. Oller and U.-G. Meißner, Phys. Lett. B 500, 263 (2001).
[25] F. Gross and Y. Surya, Phys. Rev. C 47, 703 (1993).
[26] Y. Surya and F. Gross, Phys. Rev. C 53, 2422 (1996).
[27] A. V. Sarantsev, V. A. Nikonov, A. V. Anisovich, E. Klempt and U. Thoma, Eur. Phys. J. A 25 (2005) 441
[28] A. V. Anisovich, A. Sarantsev, O. Bartholomy, E. Klempt, V. A. Nikonov and U. Thoma, Eur. Phys. J. A 25 (2005) 427
[29] D. M. Manley and E. M. Saleski, Phys. Rev. D 45, 4002 (1992).
[30] T. Feuster and U. Mosel, Phys. Rev. C 58 (1998) 457
[31] T. P. Vrana, S. A. Dytman and T. S. H. Lee, Phys. Rept. 328 (2000) 181
[32] T. Sato and T. S. H. Lee, Phys. Rev. C 54 (1996) 2660
[33] C. Schütz, J. Haidenbauer, J. Speth and J. W. Durso, Phys. Rev. C 57 (1998) 1464.
[34] O. Krehl, C. Hanhart, S. Krewald and J. Speth, Phys. Rev. C 62 (2000) 025207
[35] A. M. Gasparyan, J. Haidenbauer, C. Hanhart and J. Speth, Phys. Rev. C 68, 045207 (2003)
[36] G. Ecker, J. Gasser, H. Leutwyler, A. Pich and E. de Rafael, Phys. Lett. B 223 (1989) 425.
[37] V. Bernard, N. Kaiser and U.-G. Meißner, Nucl. Phys. A 615, 483 (1997).
[38] A. Dobado and J. R. Pelaez, Phys. Rev. D 47 (1993) 4883
[39] J. A. Oller and E. Oset, Nucl. Phys. A 620 (1997) 438 [Erratum-ibid. A 652 (1999) 407]
[40] N. Kaiser, T. Waas and W. Weise, Nucl. Phys. A 612, 297 (1997).
[41] N. Kaiser, P. B. Siegel and W. Weise, Phys. Lett. B 362 (1995) 23.
[42] J. A. Oller, E. Oset and J. R. Pelaez, Phys. Rev. D 59 (1999) 074001 [Erratum-ibid. D 60 (1999) 099906]
[43] E. E. Kolomeitsev and M. F. M. Lutz, Phys. Lett. B 585, 243 (2004)
[44] S. Sarkar, E. Oset and M. J. Vicente Vacas, Nucl. Phys. A 750, 294 (2005)
[45] T. Inoue, E. Oset and M. J. Vicente Vacas, Phys. Rev. C 65, 035204 (2002).
[46] R.H. Dalitz and S.F. Tuan, Ann. Phys. (N.Y.) 10, 307 (1960)
[47] B. K. Jennings, Phys. Lett. B 176, 229 (1986).
[48] J. C. Nacher, A. Parreno, E. Oset, A. Ramos, A. Hosaka and M. Oka, Nucl. Phys. A 678 (2000) 187
[49] E. Oset, A. Ramos and C. Bennhold, Phys. Lett. B 527, 99 (2002) [Erratumibid. B 530, 260 (2002)]
[50] C. Garcia-Recio, J. Nieves, E. Ruiz Arriola and M. J. Vicente Vacas, Phys. Rev. D 67 (2003) 076009
[51] D. Jido, J. A. Oller, E. Oset, A. Ramos and U.-G. Meißner, Nucl. Phys. A 725 (2003) 181
[52] C. Garcia-Recio, M. F. M. Lutz and J. Nieves, Phys. Lett. B 582 (2004) 49
[53] B. Borasoy, R. Nissler and W. Weise, Eur. Phys. J. A 25, 79 (2005)
[54] J. A. Oller, J. Prades and M. Verbeni, Phys. Rev. Lett. 95, 172502 (2005)
[55] J. A. Oller, arXiv:hep-ph/0603134.
[56] S. Prakhov et al. [Crystall Ball Collaboration], Phys. Rev. C 70, 034605 (2004).
[57] S. Eidelman et al. [Particle Data Group], Phys. Lett. B 592 (2004) 1.
[58] D. W. Thomas, A. Engler, H. E. Fisk and R. W. Kraemer, Nucl. Phys. B 56, 15 (1973).
[59] V. K. Magas, E. Oset and A. Ramos, Phys. Rev. Lett. 95, 052301 (2005)
[60] L. Roca, S. Sarkar, V. K. Magas and E. Oset, Phys. Rev. C 73, 045208 (2006)
[61] S. Sarkar, E. Oset and M. J. Vicente Vacas, Phys. Rev. C 72, 015206 (2005)
[62] N. Isgur and G. Karl, Phys. Rev. D 18 (1978) 4187.
[63] S. Capstick and W. Roberts, Phys. Rev. D 49 (1994) 4570
[64] T. W. Chiu and T. H. Hsieh, Nucl. Phys. A 755, 471 (2005)
[65] N. Nakajima, H. Matsufuru, Y. Nemoto and H. Suganuma, arXiv:heplat/0204014.
[66] E. Jenkins and A. V. Manohar, Phys. Lett. B 259, 353 (1991).
[67] G. F. Chew and F. E. Low, Phys. Rev. 101, 1570 (1956).
[68] L. Castillejo, R. H. Dalitz and F. J. Dyson, Phys. Rev. 101, 453 (1956).
[69] J. R. Pelaez, Phys. Rev. Lett. 92, 102001 (2004)
[70] H. C. Schroder et al., Phys. Lett. B 469 (1999) 25.
[71] D. Sigg et al., Nucl. Phys. A 609, 269 (1996).
[72] D. Sigg et al., Phys. Rev. Lett. 75, 3245 (1995).
[73] H. C. Schroder et al., Eur. Phys. J. C 21 (2001) 473.
[74] P. Hauser et al., Phys. Rev. C 58 (1998) 1869.
[75] T. E. O. Ericson, B. Loiseau and A. W. Thomas, Phys. Rev. C 66, 014005 (2002).
[76] S. Weinberg, Phys. Rev. Lett. 17 (1966) 616.
[77] I. R. Afnan and A. W. Thomas, Phys. Rev. C 10 (1974) 109.
[78] T. Mizutani and D. S. Koltun, Annals Phys. 109 (1977) 1.
[79] C. Fayard, G. H. Lamot and T. Mizutani, Phys. Rev. Lett. 45, 524 (1980).
[80] A. W. Thomas and R. H. Landau, Phys. Rept. 58, 121 (1980).
[81] B. Borasoy and H. W. Griesshammer, Int. J. Mod. Phys. E 12, 65 (2003).
[82] V. V. Baru, A. E. Kudryavtsev and V. E. Tarasov, Phys. Atom. Nucl. 67, 743 (2004) [Yad. Fiz. 67, 764 (2004)]
[83] A. Bahaoui, C. Fayard, T. Mizutani and B. Saghai, Phys. Rev. C 68, 064001 (2003).
[84] V. V. Baru and A. E. Kudryavtsev, Phys. Atom. Nucl. 60 (1997) 1475.
[85] V. E. Tarasov, V. V. Baru and A. E. Kudryavtsev, Phys. Atom. Nucl. 63 (2000) 801.
[86] S. R. Beane, V. Bernard, E. Epelbaum, U.-G. Meißner and D. R. Phillips, Nucl. Phys. A 720, 399 (2003).
[87] N. Fettes, U.-G. Meißner and S. Steininger, Nucl. Phys. A 640, 199 (1998).
[88] N. Fettes, U.-G. Meißner and S. Steininger, Phys. Lett. B 451, 233 (1999).
[89] W. R. Gibbs, L. Ai and W. B. Kaufmann, Phys. Rev. Lett. 74, 3740 (1995).
[90] W. R. Gibbs, L. Ai and W. B. Kaufmann, Phys. Rev. C 57, 784 (1998).
[91] E. Matsinos, Phys. Rev. C 56, 3014 (1997).
[92] J. Piekarewicz, Phys. Lett. B 358, 27 (1995).
[93] N. Fettes and U.-G. Meißner, Nucl. Phys. A 679, 629 (2001).
[94] S. S. Kamalov, E. Oset and A. Ramos, Nucl. Phys. A 690, 494 (2001).
[95] A. Deloff, Phys. Rev. C 61, 024004 (2000).
[96] A. Deloff, Phys. Rev. C 64, 065205 (2001)
[97] V. Baru, C. Hanhart, A. E. Kudryavtsev and U.-G. Meißner, Phys. Lett. B 589, 118 (2004)
[98] O. V. Maxwell, W. Weise and M. Brack, Nucl. Phys. A 348, 388 (1980).
[99] C. Garcia-Recio, E. Oset and L. L. Salcedo, Phys. Rev. C 37, 194 (1988).
[100] D.S. Koltun, A. Reitan, Phys. Rev. 141 (1966) 1413.
[101] R. Machleidt, Phys. Rev. C 63, 024001 (2001).
[102] M. Lacombe, B. Loiseau, R. Vinh Mau, J. Cote, P. Pires and R. de Tourreil, Phys. Lett. B 101, 139 (1981).
[103] O. Schori et al., Phys. Rev. C 35, 2252 (1987).
[104] D. Kharzeev, E. Levin and M. Nardi, Nucl. Phys. A 730, 448 (2004).
[105] G. Faldt, Phys. Scripta 16, 81 (1977).
[106] T. E. O. Ericson and W. Weise, Pions and Nuclei, Oxford, UK: Clarendon (1988) 479 p.
[107] U.-G. Meißner, E. Oset and A. Pich, Phys. Lett. B 353, 161 (1995).
[108] E. Oset, C. Garcia-Recio and J. Nieves, Nucl. Phys. A 584 (1995) 653.
[109] S. R. Beane, V. Bernard, T. S. H. Lee and U.-G. Meißner, Phys. Rev. C 57, 424 (1998).
[110] V. Lensky, V. Baru, J. Haidenbauer, C. Hanhart, A. E. Kudryavtsev and U.-G. Meißner, arXiv:nucl-th/0608042.
[111] M. Pavan, private communication, L. M. Simons, on the "HadAtom03" conference,
in: J. Gasser, A. Rusetsky and J. Schacher, arXiv:hep-ph/0401204.
[112] T. E. O. Ericson, B. Loiseau and S. Wycech, arXiv:hep-ph/0310134.
[113] R. A. Arndt, W. J. Briscoe, I. I. Strakovsky, R. L. Workman and M. M. Pavan, Phys. Rev. C 69, 035213 (2004)
CNS Data Analysis Center, web: http://gwdac.phys.gwu.edu/analysis/pin_analysis.html
[114] V. E. Lyubovitskij and A. Rusetsky, Phys. Lett. B 494, 9 (2000).
[115] J. Gasser, M. A. Ivanov, E. Lipartia, M. Mojzis and A. Rusetsky, Eur. Phys. J. C 26, 13 (2002).
[116] N. Fettes and U.-G. Meißner, Nucl. Phys. A 693, 693 (2001).
[117] R. Koch, Nucl. Phys. A 448, 707 (1986).
[118] R. A. Arndt, I. I. Strakovsky, R. L. Workman and M. M. Pavan, Phys. Rev. C 52, 2120 (1995).
[119] R. A. Arndt, R. L. Workman and M. M. Pavan, Phys. Rev. C 49, 2729 (1994).
[120] M. Ericson and T. E. O. Ericson, Annals Phys. 36, 323 (1966).
[121] A. B. Migdal, E. E. Saperstein, M. A. Troitsky and D. N. Voskresensky, Phys. Rept. 192, 179 (1990).
[122] L. L. Salcedo, K. Holinde, E. Oset and C. Schutz, Phys. Lett. B 353, 1 (1995)
[123] C. J. Batty, E. Friedman and A. Gal, Phys. Rept. 287, 385 (1997).
[124] N. Kaiser and W. Weise, Phys. Lett. B 512, 283 (2001)
[125] W. Weise, Nucl. Phys. A 690, 98c (2001).
[126] E. E. Kolomeitsev, N. Kaiser and W. Weise, Phys. Rev. Lett. 90, 092501 (2003)
[127] G. Chanfray, M. Ericson and M. Oertel, Phys. Lett. B 563, 61 (2003).
[128] L. Girlanda, A. Rusetsky and W. Weise, Annals Phys. 312, 92 (2004)
[129] E. Friedman et al., Phys. Rev. Lett. 93, 122302 (2004)
[130] T. Yamazaki et al., Z. Phys. A 355, 219 (1996).
[131] T. Yamazaki et al., Phys. Lett. B 418, 246 (1998).
[132] K. Itahashi et al., Phys. Rev. C 62, 025202 (2000).
[133] H. Gilg et al., Phys. Rev. C 62, 025201 (2000).
[134] J. Nieves, E. Oset and C. Garcia-Recio, Nucl. Phys. A 554, 509 (1993).
[135] J. A. Oller, Phys. Rev. C 65, 025204 (2002)
[136] U.-G. Meißner, J. A. Oller and A. Wirzba, Annals Phys. 297, 27 (2002)
[137] K. Suzuki et al., Phys. Rev. Lett. 92, 072302 (2004)
[138] L. Tauscher and W. Schneider, Z. Phys. 271, 409 (1974).
[139] K. Stricker, J. A. Carr and H. Mcmanus, Phys. Rev. C 22, 2043 (1980).
[140] R. Seki and K. Masutani, Phys. Rev. C 27, 2799 (1983).
[141] R. Seki, K. Masutani and K. Yazaki, Phys. Rev. C 27, 2817 (1983).
[142] M. J. Vicente, E. Oset, L. L. Salcedo and C. Garcia-Recio, Phys. Rev. C 39, 209 (1989).
[143] C. Garcia-Recio and E. Oset, Phys. Rev. C 40, 1308 (1989).
[144] J. Nieves, E. Oset and C. Garcia-Recio, Nucl. Phys. A 554, 554 (1993).
[145] E. Oset and W. Weise, Nucl. Phys. A 319, 477 (1979).
[146] U.-G. Meißner, U. Raha and A. Rusetsky, Eur. Phys. J. C 41, 213 (2005)
[147] U.-G. Meißner, U. Raha and A. Rusetsky, Phys. Lett. B 639, 478 (2006)
[148] T. E. O. Ericson, B. Loiseau and S. Wycech, Int. J. Mod. Phys. A 20, 1650 (2005)
[149] A. Ramos and E. Oset, Nucl. Phys. A 671, 481 (2000)
[150] T. Inoue and E. Oset, Nucl. Phys. A 710, 354 (2002)
[151] E. Oset, P. Fernandez de Cordoba, L. L. Salcedo and R. Brockmann, Phys. Rept. 188, 79 (1990).
[152] M. Urban and J. Wambach, Phys. Rev. C 65, 067302 (2002)
[153] F. Mandl, G. Shaw, Quantum Field Theory, Wiley 1984
[154] V. Lensky, V. Baru, J. Haidenbauer, C. Hanhart, A. E. Kudryavtsev and U.-G. Meißner, Eur. Phys. J. A 27, 37 (2006)
[155] M. M. Kaskulov, E. Oset and M. J. Vicente Vacas, Phys. Rev. C 73, 014004 (2006)
[156] E. Oset, H. Toki, M. Mizobe and T. T. Takahashi, Prog. Theor. Phys. 103, 351 (2000)
[157] D. Cabrera, E. Oset and M. J. Vicente Vacas, Nucl. Phys. A 705, 90 (2002)
[158] D. Cabrera, E. Oset and M. J. Vicente Vacas, Phys. Rev. C 72, 025207 (2005)
[159] V. Thorsson and A. Wirzba, Nucl. Phys. A 589, 633 (1995)
[160] D. Jido, T. Hatsuda and T. Kunihiro, Phys. Rev. D 63, 011901 (2001)
[161] L. Alvarez-Ruso, E. Oset and E. Hernandez, Nucl. Phys. A 633, 519 (1998)
[162] S. Hirenzaki, P. Fernandez de Cordoba and E. Oset, Phys. Rev. C 53, 277 (1996)
[163] V. Bernard, N. Kaiser and U.-G. Meißner, Nucl. Phys. B 457, 147 (1995)
[164] R. Machleidt, K. Holinde and C. Elster, Phys. Rept. 149, 1 (1987).
[165] J. K. Ahn [LEPS Collaboration], Nucl. Phys. A 721 (2003) 715.
[166] J. C. Nacher, E. Oset, H. Toki and A. Ramos, Phys. Lett. B 455 (1999) 55
[167] V. Metag and M. Nanova, private communication.
[168] J. C. Nacher, E. Oset, M. J. Vicente and L. Roca, Nucl. Phys. A 695, 295 (2001)
[169] J. A. Gomez Tejedor and E. Oset, Nucl. Phys. A 571, 667 (1994).
[170] J. A. Gomez Tejedor and E. Oset, Nucl. Phys. A 600, 413 (1996)
[171] J. Nieves and E. Ruiz Arriola, Phys. Rev. D 64, 116008 (2001)
[172] M. F. M. Lutz, G. Wolf and B. Friman, Nucl. Phys. A 706, 431 (2002) [Erratum-ibid. A 765, 431 (2006)]
[173] Current (06/2005) fit of the SAID solution of $\eta$ photoproduction, http://gwdac.phys.gwu.edu/
[174] B. Borasoy, E. Marco and S. Wetzel, Phys. Rev. C 66, 055208 (2002)
[175] J. Weiss et al., Eur. Phys. J. A 16, 275 (2003)
[176] B. Krusche and S. Schadmand, Prog. Part. Nucl. Phys. 51, 399 (2003)
[177] V. Crede et al. [CB-ELSA Collaboration], Phys. Rev. Lett. 94, 012004 (2005)
[178] T. S. H. Lee, J. A. Oller, E. Oset and A. Ramos, Nucl. Phys. A 643 (1998) 402
[179] F. Gross and D. O. Riska, Phys. Rev. C 36, 1928 (1987).
[180] A. N. Kvinikhidze and B. Blankleider, Phys. Rev. C 60, 044003 (1999)
[181] C. H. M. van Antwerpen and I. R. Afnan, Phys. Rev. C 52, 554 (1995)
[182] H. Haberzettl, Phys. Rev. C 56, 2041 (1997)
[183] B. Borasoy, P. C. Bruns, U.-G. Meißner and R. Nissler, Phys. Rev. C 72, 065201 (2005)
[184] U.-G. Meißner and S. Steininger, Nucl. Phys. B 499 (1997) 349
[185] D. Jido, A. Hosaka, J. C. Nacher, E. Oset and A. Ramos, Phys. Rev. C 66 (2002) 025203
[186] A. Fix and H. Arenhovel, Eur. Phys. J. A 25, 115 (2005)
[187] Pions and Nuclei, T. E. O. Ericson and W. Weise, Oxford Science Publications, 1988.
[188] T. R. Hemmert, B. R. Holstein and J. Kambor, J. Phys. G 24 (1998) 1831
[189] A. Ramos, E. Oset and C. Bennhold, Phys. Rev. Lett. 89 (2002) 252001
[190] M. N. Butler, M. J. Savage and R. P. Springer, Nucl. Phys. B 399, 69 (1993)
[191] M. F. M. Lutz and E. E. Kolomeitsev, Nucl. Phys. A 700, 193 (2002)
[192] E. Oset and A. Ramos, Nucl. Phys. A 679, 616 (2001)
[193] M. Nanova at the "International Workshop On The Physics Of Excited Baryons (NSTAR 05)", 10-15 Oct 2005, Tallahassee, Florida
[194] O. I. Dahl, L. M. Hardy, R. I. Hess et.al., Phys. Rev. 163, 1337 (1967).
[195] L. J. Curtis, C. T. Coffin, D. I. Meyer, and K. M. Terwilliger, Phys. Rev. 132, 1771 (1963).
[196] P. Hanson, G. E. Kalmus and J. Louie, Phys. Rev. D 4, 1296 (1971).
[197] D. Grether, G. Gidal and G. Borreani, Phys. Rev. D 7, 3200 (1973).
[198] T. Hyodo, A. Hosaka, E. Oset, A. Ramos and M. J. Vicente Vacas, Phys. Rev. C 68, 065203 (2003)
[199] A. W. Thomas, S. Theberge and G. A. Miller, Phys. Rev. D 24 (1981) 216.
[200] R. Erbe et al. [Aachen-Berlin-Bonn-Hamburg-Heidelberg-Muenchen Collaboration,], Phys. Rev. 188, 2060 (1969).
[201] R. Erbe et al. [Aachen-Berlin-Bonn-Hamburg-Heidelberg-Muenchen Collaboration,], Nuovo Cimento 49A, 504 (1967).
[202] Cambridge Bubble Chamber Group, Phys. Rev. 156, 1426 (1966)
[203] J.M. Blatt and V.F. Weisskopff, Theoretical Nuclear Physics (Wiley, New York, 1952)
[204] D. M. Manley, R. A. Arndt, Y. Goradia and V. L. Teplitz, Phys. Rev. D 30, 904 (1984).
[205] O. Goussu, M. Sené, B. Ghidini et.al., Nuovo Cimento 42A, 606 (1966)
[206] F. W. Wieland, Die Untersuchung der Reaktionen $\gamma p \rightarrow K^{0} \Sigma(1385)^{+}$und $\gamma p \rightarrow K(892)^{+} \Lambda$, BONN-IR-2005-10 (2005),
electronic version available under:
http://lisa12.physik.uni-bonn.de/saphir/thesis.html
[207] I. Schulday, Untersuchung der Reaktion $\gamma p \rightarrow K^{+} \Sigma^{-} \pi^{+}$für Photonenergien bis 2.6 GeV mit dem SAPHIR-Detektor an ELSA, BONN-IR-04-15 (2004), electronic version available under:
http://lisa12.physik.uni-bonn.de/saphir/thesis.html
[208] F. W. Wieland, private communication
[209] S. Sarkar, L. Roca, E. Oset, V. K. Magas and M. J. V. Vacas, arXiv:nuclth/0511062.
[210] S. Taylor et al. [CLAS Collaboration], Phys. Rev. C 71, 054609 (2005) [Erratum-ibid. C 72, 039902 (2005)]
[211] E. Kaxiras, E. J. Moniz and M. Soyeur, Phys. Rev. D 32, 695 (1985).
[212] J. W. Darewych, M. Horbatsch and R. Koniuk, Phys. Rev. D 28, 1125 (1983).
[213] M. Warns, W. Pfeil and H. Rollnik, Phys. Lett. B 258, 431 (1991).
[214] Y. Umino and F. Myhrer, Nucl. Phys. A 529, 713 (1991).
[215] Y. Umino and F. Myhrer, Nucl. Phys. A 554, 593 (1993)
[216] T. Van Cauteren, D. Merten, T. Corthals, S. Janssen, B. Metsch, H. R. Petry and J. Ryckebusch, Eur. Phys. J. A 20, 283 (2004)
[217] T. Van Cauteren, J. Ryckebusch, B. Metsch and H. R. Petry, Eur. Phys. J. A 26, 339 (2005)
[218] R. Bijker, F. Iachello and A. Leviatan, Annals Phys. 284, 89 (2000)
[219] G. P. Gopal, R. T. Ross, A. J. Van Horn, A. C. McPherson, E. F. Clayton, T. C. Bacon and I. Butterworth [Rutherford-London Collaboration], Nucl. Phys. B 119, 362 (1977).
[220] M. Alston-Garnjost, R. W. Kenney, D. L. Pollard, R. R. Ross, R. D. Tripp, H. Nicholson and M. Ferro-Luzzi, Phys. Rev. D 18, 182 (1978).
[221] R. Bertini, Nucl. Phys. B 279 (1987) 49, R. Bertini et. al., SACLAY-DPh-N-2372 (unpublished).
[222] T. S. Mast, M. Alston-Garnjost, R. O. Bangerter, A. Barbaro-Galtieri, L. K. Gershwin, F. T. Solmitz and R. D. Tripp, Phys. Rev. Lett. 21 (1968) 1715.
[223] Y. M. Antipov et al. [SPHINX Collaboration], Phys. Lett. B 604, 22 (2004)
[224] Y. Oh, K. Nakayama and T. S. Lee, arXiv:hep-ph/0511198.
[225] D. V. Vavilov et al. [SPHINX Collaboration], Phys. Atom. Nucl. 68, 378 (2005)
[226] T. Hyodo, S. Sarkar, A. Hosaka and E. Oset, Phys. Rev. C 73, 035209 (2006)
[227] J. C. Nacher and E. Oset, Nucl. Phys. A 674, 205 (2000)
[228] W. M. Yao et al. [Particle Data Group], J. Phys. G 33 (2006) 1.
[229] T. Nakabayashi et al., Phys. Rev. C 74, 035202 (2006).
[230] R. A. Arndt, W. J. Briscoe, I. I. Strakovsky and R. L. Workman, Phys. Rev. C 74, 045205 (2006)
[231] L. Roca, A. Hosaka and E. Oset, arXiv:hep-ph/0611075.
[232] R. Koniuk and N. Isgur, Phys. Rev. D 21, 1868 (1980) [Erratum-ibid. D 23, 818 (1981)].
[233] F. Myhrer and J. Wroldsen, Z. Phys. C 25, 281 (1984).
[234] S. Capstick and W. Roberts, Prog. Part. Nucl. Phys. 45, S241 (2000)
[235] D. Merten, U. Loring, K. Kretzschmar, B. Metsch and H. R. Petry, Eur. Phys. J. A 14, 477 (2002)
[236] S. V. Afanasev et al. [NA49 Collaboration], Phys. Rev. Lett. 86, 1965 (2001)
[237] T. Anticic et al. [NA49 Collaboration], Phys. Rev. C 70, 034902 (2004)
[238] D. Adamova et al. [CERES Collaboration], Nucl. Phys. A 727, 97 (2003)
[239] J. Adams et al. [STAR Collaboration], Phys. Rev. C 71, 064906 (2005)
[240] S. S. Adler et al. [PHENIX Collaboration], Phys. Rev. Lett. 93, 092301 (2004)
[241] J. Adams et al. [STAR Collaboration], Phys. Rev. C 68, 044905 (2003)
[242] C. Alt et al. [NA49 Collaboration], Phys. Rev. C 70, 064903 (2004)
[243] K. Adcox et al. [PHENIX Collaboration], Phys. Rev. Lett. 89, 082301 (2002)
[244] H. Sako and H. Appelshaeuser [CERES/NA45 Collaboration], J. Phys. G 30, S1371 (2004)
[245] J. Adams et al. [STAR Collaboration], J. Phys. G 32, L37 (2006)
[246] B. B. Back et al. [PHOBOS Collaboration], Phys. Rev. C 74, 011901 (2006)
[247] S. Y. Jeon and V. Koch, Quark-Gluon Plasma 3, eds. R.C. Hwa and X.N Wang,
World Scientific, 2004. arXiv:hep-ph/0304012.
[248] M. A. Stephanov, K. Rajagopal and E. V. Shuryak, Phys. Rev. D 60, 114028 (1999)
[249] S. Jeon and V. Koch, Phys. Rev. Lett. 85, 2076 (2000)
[250] M. Asakawa, U. W. Heinz and B. Muller, Phys. Rev. Lett. 85, 2072 (2000)
[251] A. Bialas and V. Koch, Phys. Lett. B 456, 1 (1999)
[252] C. Adler et al. [STAR Collaboration], Phys. Rev. Lett. 90, 082302 (2003)
[253] S. S. Adler et al. [PHENIX Collaboration], Phys. Rev. Lett. 97, 052301 (2006)
[254] M. Bleicher, S. Jeon and V. Koch, Phys. Rev. C 62, 061902 (2000)
[255] E. V. Shuryak and M. A. Stephanov, Phys. Rev. C 63, 064903 (2001)
[256] Q. H. Zhang, V. Topor Pop, S. Jeon and C. Gale, Phys. Rev. C 66, 014909 (2002)
[257] P. Braun-Munzinger, K. Redlich and J. Stachel,
Quark-Gluon Plasma 3, eds. R.C. Hwa and X.N Wang,World Scientific, 2004. arXiv:nucl-th/0304013.
[258] F. Becattini, M. Gazdzicki, A. Keranen, J. Manninen and R. Stock, Phys. Rev. C 69, 024905 (2004)
[259] R. V. Gavai and S. Gupta, Phys. Rev. D 64, 074506 (2001)
[260] J. I. Kapusta, Phys. Rev. D 46, 4749 (1992).
[261] C. R. Allton et al., Phys. Rev. D 71, 054508 (2005)
[262] F. Karsch, K. Redlich and A. Tawfik, Phys. Lett. B 571, 67 (2003)
[263] V. L. Eletsky, J. I. Kapusta and R. Venugopalan, Phys. Rev. D 48, 4398 (1993)
[264] M. Prakash, R. Rapp, J. Wambach and I. Zahed, Phys. Rev. C 65, 034906 (2002)
[265] J.I. Kapusta: Finite Temperature Field Theory, Cambridge 1989
[266] M. Bando, T. Kugo, S. Uehara, K. Yamawaki and T. Yanagida, Phys. Rev. Lett. 54, 1215 (1985).
[267] O. Kaymakcalan and J. Schechter, Phys. Rev. D 31, 1109 (1985).
[268] F. Klingl, N. Kaiser and W. Weise, Z. Phys. A 356, 193 (1996)
[269] L. Alvarez-Ruso and V. Koch, Phys. Rev. C 65, 054901 (2002)
[270] E. Marco, S. Hirenzaki, E. Oset and H. Toki, Phys. Lett. B 470, 20 (1999)
[271] M. C. Birse, Z. Phys. A 355, 231 (1996)
[272] K. Kawarabayashi and M. Suzuki, Phys. Rev. Lett. 16, 255 (1966), Riazuddin and Fayyazuddin, Phys. Rev. 147, 1071 (1966).
[273] T. D. Cohen, Phys. Lett. B 233, 467 (1989).
[274] J. Schechter, Phys. Rev. D 34, 868 (1986).
[275] Dashen, Ma, and Bernstein, PR 187 (1969), 345-370
[276] R. Venugopalan and M. Prakash, Nucl. Phys. A 546, 718 (1992).
[277] G. M. Welke, R. Venugopalan and M. Prakash, Phys. Lett. B 245, 137 (1990).
[278] A. I. Bugrii, L. L. Jenkovszky and V. N. Shadura, arXiv:hep-th/9507101.
[279] R. F. Dashen and R. Rajaraman, Phys. Rev. D 10, 694 (1974).
[280] M. Döring and V. Koch, in preparation
[281] J. f. Liao, X. l. Zhu and P. f. Zhuang, Phys. Rev. D 67, 105023 (2003)
[282] S. Sarkar, J. e. Alam, P. Roy, A. K. Dutt-Mazumder, B. Dutta-Roy and B. Sinha, Nucl. Phys. A 634, 206 (1998)
[283] C. Gale and J. I. Kapusta, Nucl. Phys. B 357, 65 (1991).
[284] D. Aston et al., Nucl. Phys. B 296, 493 (1988).
[285] P. Estabrooks, R. K. Carnegie, A. D. Martin, W. M. Dunwoodie, T. A. Lasinski and D. W. G. Leith, Nucl. Phys. B 133, 490 (1978).
[286] A. Dobado, A. Gomez Nicola, F. J. Llanes-Estrada and J. R. Pelaez, Phys. Rev. C 66, 055201 (2002)
[287] A. Gomez Nicola, J. R. Pelaez, A. Dobado and F. J. Llanes-Estrada, AIP Conf. Proc. 660, 156 (2003) [arXiv:hep-ph/0212121].
[288] J. R. Pelaez, Phys. Rev. D 66, 096007 (2002)
[289] A. L. Fetter and J. D. Walecka: Quantum theory of many particle systems, McGraw-Hill 1971
[290] C. D. Froggatt and J. L. Petersen, Nucl. Phys. B 129, 89 (1977).
[291] J. F. Donoghue, E. Golowich and B. R. Holstein, Dynamics of the standard model, Camb. Monogr. Part. Phys. Nucl. Phys. Cosmol. 2, 1 (1992).
[292] M.E. Peskin, D.V. Schroeder: An Introduction to Quantum Field Theory, Cambridge 1995
[293] A. Dobado and J. R. Pelaez, Phys. Rev. D 56, 3057 (1997)
[294] H. A. Weldon, Phys. Rev. D 28, 2007 (1983).
[295] M. J. Losty et al., Nucl. Phys. B 69, 185 (1974).
[296] W. Hoogland et al., Nucl. Phys. B 126, 109 (1977).
[297] A. Gomez Nicola, F. J. Llanes-Estrada and J. R. Pelaez, Phys. Lett. B 550, 55 (2002)
[298] A. Schenk, Nucl. Phys. B 363, 97 (1991).

## Danksagung, Agradecimientos

An erster Stelle möchte ich den beiden Betreuern meiner Doktorarbeit danken, Prof. Oset und Prof. Vicente Vacas, die trotz ihrer vielen Verpflichtungen immer Zeit gefunden haben, Fragen zu beantworten und Zweifel zu klären. Für all die Fachkenntnis, die sie mir in dieser Zeit vermittelt haben und für die herzliche Aufnahme in unserer Gruppe. Bereichernd war auch der Austausch mit meinen Kollegen, die Diskussionen und Anregungen.

Außerdem gilt mein großer Dank Maria José, die mir immer zur Seite gestanden hat, auch in den komplizierten Momenten, die eine Promotion mit sich bringt. Für ihre Unterstützung, besonders auch für die Übersetzung der Einleitung.

Schließlich möchte ich meinen Eltern danken, ohne deren Hilfe ich niemals bis zu diesem Punkt gelangt wäre, und die mein Interesse an der Wissenschaft geweckt haben.

En primer lugar, deseo expresar mi más profundo agradecimiento a los directores de mi tesis, Prof. Oset y Prof. Vicente Vacas, quienes a pesar de sus ocupaciones, siempre encontraron tiempo para guiarme y ayudarme a disipar cualquier duda. Mi agradecimiento por los conocimientos que tan sabiamente me han transmitido y por acogerme en este equipo como a uno más. Enriquecedor ha resultado también, el intercambio de ideas con mis compañeros de grupo -los que están y los que se fueron- y el haber compartido con ellos todo este tiempo mis experiencias.

Quiero agradecer a María José su constante apoyo, sobre todo en los momentos complicados. Por estar ahí, por sus ánimos y, en especial, por su inestimable colaboración en la traducción al español de la introducción de esta tesis.

Por último, no puedo dejar de mencionar a mis padres, sin cuya ayuda, nunca hubiera llegado hasta aquí. Por despertar en mi el interés por el conocimiento.


[^0]:    ${ }^{1}$ Although Fig. 2.5 suggests that the present model agrees well with experiment, this is not the case, because around one third of the experimental value of $\operatorname{Im} a_{\pi^{-} d}$ comes from the radiative capture of the pion according to $\pi^{-} d \rightarrow \gamma n n$ [103]. Thus, the theoretical value is around one third too high. One of the reasons is that the cut-off of $\Lambda=1.72$ is certainly high, and as Fig. 2.5 shows, for $\Lambda$ around 1 GeV , the theoretical value is lower. Another reason is that the isoscalar and isovector coupling strengths $\lambda_{1}$ and $\lambda_{2}$ from [99] (see Eq. (2.21)) are large. In particular, the value of $\lambda_{2}$ corresponds to an isovector larger than the one of the Weinberg Tomozawa term. Anticipating the final results from Eq. (2.45) and using them for the $\pi N$ interaction, the value of $\operatorname{Im} a_{\pi^{-} d}$ is multiplied with a factor 0.76 . With this factor and taking a natural value for $\Lambda=1 \mathrm{GeV}$, the theoretical value is slighlty below the experimental one of $(2 / 3)(0.0063 \pm 0.0007) m_{\pi^{-}}^{-1}$. However, in a recent study [154] it has been shown that the influence of higher order effects can be effectively included by replacing the half-off-shell $\pi N$ vertex with the on-shell one. This would give another factor of $(4 / 3)^{2}$.

    Note that the use of large values for $\lambda_{1}$ and $\lambda_{2}$ has hardly consequences for the numerical results: one of the main conclusions, the smallness of the dispersive part, is still valid. Only

[^1]:    ${ }^{2}$ T.E.O. Ericson, private communication.

[^2]:    ${ }^{3} \mathrm{G}$. Höhler, private communication.

[^3]:    ${ }^{1} \mathrm{G}$. Höhler, private communication.

[^4]:    ${ }^{1}$ This is why cut-off scheme and dimensional regularization correspond so closely to each other over a wide range of energy. In the latter regularization scheme the subtraction constant plays the role of the additive constant.

[^5]:    ${ }^{2} \mathrm{M}$. Nanova, private communication

[^6]:    ${ }^{1}$ Of course in the $\pi \pi$-case resonances above the $\rho$-meson also contribute. While we have ignored these in the previous discussion, they will be included in the final analysis given in the following section 8.8.

