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Pion Properties in Chiral Perturbation Theory at Next-to-Leading Order

Judith Plenter

Tutor : Antonio Pich Zardoya

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Abstract

Chiral Perturbation Theory is a low-energy effective field theory for the strong interaction. It is based on the spontaneous breaking of the approximate chiral symmetry in Quantum Chromodynamics and allows to study the dynamics of strong interactions at long distances in terms of the pseudoscalar mesons. Within this formalism, the mass, the decay constant, the electromagnetic form factor and the charge radius of pions are calculated at next-to-leading order and the low-energy constants L_5^r and L_9^r are determined.

Resumen

La Teoría Quiral de Perturbaciones es una teoría efectiva de las interacciones fuertes a baja energía basada en la ruptura espontánea de la simetría quiral, que es una simetría aproximada del lagrangiano de la Cromodinámica Cuántica. Esta teoría efectiva permite describir las interacciones fuertes a grandes distancias a través de los mesones pseudoescalares. Mediante este formalismo se calcula la masa, la constante de desintegración, el factor de forma electromagnético y el radio de carga del pion a segundo orden y se determinan los acoplamientos L_5^r y L_9^r.

Contents

1. Chiral Perturbation Theory

Until the middle of the 20th century the most fundamental particles known were the constituents of the atoms, namely the electron, the proton and the neutron and additionally the positron as well as the muon. Beginning with the discovery of the pion in 1947 the amount of discovered particles multiplied rapidly due to the advancements in accelerator technology. The multitude of particles created the need for a systematic explanation.

Since the discovery of the neutron as part of the atomic nuclei, it had become apparent that the physical description in terms of only the electromagnetic interaction was insufficient. Many of the newly discovered particles also appeared to be sensitive to this strong interaction that was seemingly causing the binding of the nucleus. It was then possible to group these so-called hadrons into multiplets with the same spin and same transformation properties under parity. Thus those multiplets could be identified with the irreducible representations of the symmetry group SU(3). In this system, proposed by Gell-Mann and Ne'eman as the Eightfold Way, hadrons are built from the more fundamental quarks. Originally the quarks were introduced as fictitious constituents in order to make the group-theoretical classification of the hadron spectrum possible [GM64]. Only after the substructure of the hadrons was experimentally probed were their components identified with the quarks (and the gluons mediating the interaction between them) [Sch03].

The implementation of the strong interaction between the quarks as a gauge theory is called Quantum Chromodynamics (QCD). Together with the unified theory of the electroweak interaction it makes up the Standard Model of particle physics. The elementary particles of the Standard Model besides the quarks are the leptons (electron, muon, tau and the corresponding neutrinos) and the gauge bosons mediating their interactions (photon, gluon, Z^0- and $W^\pm$- boson) as well as the Higgs boson responsible for the masses of fermions and gauge bosons.

1.1. Strong Interaction in the Standard Model

The quarks in the Standard Model are fermions that appear in six different flavors (up, down, charm, strange, top, bottom). The detected mesons are identified with the bound state of a quark and an antiquark while the (anti-)baryons correspond to a state of three (anti-)quarks. Applying this concept to all of the hadronic spectrum one needs to introduce a new quantum number, the color charge, in order to not violate Fermi-Dirac statistics in the baryonic sector. Color itself cannot be observed since all asymptotic states are color singlets leading to the first experimental requirement for QCD: Due to their color charge, quarks can only appear in color-neutral bound states. This concept is called *Confinement*. From measuring the ratio between the hadronic and leptonic τ decay widths the number of colors can be determined to be three. Further it has been experimentally established through the measurement of the proton form factors in deep inelastic scattering experiments that quarks behave as nearly free particles at very short distances, leading to another experimental requirement which is called *Asymptotic Freedom* [Pic12] [Pic99].

Since all asymptotic states are color singlets the most relevant symmetry of QCD is the invariance under rotations in the three-dimensional color space. Particularly this $SU(3)_c$ symmetry, acting on the quark vector in color space, should also hold when being promoted to a local one

$$q \longrightarrow \exp\left\{-ig_s\ \theta^a(x)\frac{\lambda_c^a}{2}\right\}\ q \tag{1.1}$$

and can thus be used to construct the gauge theory of Quantum Chromodynamics. Here $\theta^a(x)$ is an arbitrary function depending on space-time, the $T_c^a = \lambda_c^a/2$ are the generators of $SU(3)_c$ with the λ_c^a being the 8 Gell-Mann matrices and g_s is the strong coupling constant. Starting with the Dirac Lagrangian and requiring the invariance under the gauge symmetry, the QCD Lagrangian becomes

$$\mathcal{L}_{\text{QCD}} = \sum_{\text{flavor}} \bar{q}_f \left(i\not{D} - m\right) q_f - \frac{1}{4} G_a^{\mu\nu} G_{\mu\nu}^a, \tag{1.2}$$

where the covariant derivative is given by

$$D_\mu q = \left[\partial_\mu - ig_s\ \frac{\lambda_c^a}{2} G_a^\mu(x)\right] q, \tag{1.3}$$

with the gluon fields $G_a^\mu(x)$ and the gluon field-strength tensor $G_a^{\mu\nu}$ [Pic99].

Due to the fact that $SU(3)_c$ is a non-abelian Lie group, $\mathcal{L}_{\text{QCD}}$ generates gluon self-interaction diagrams leading to the strong scale dependence of the QCD running coupling which to leading order is given by

$$\alpha_s(\mu^2) = \frac{\alpha_s(\mu_0^2)}{1 - \frac{\beta_1}{2}\frac{\alpha_s(\mu_0^2)}{\pi}\ln\left(\frac{\mu^2}{\mu_0^2}\right) + \ldots}. \tag{1.4}$$

The constant β_1 is negative leading to the observed asymptotic behaviour of the QCD coupling: Increasing the energy the strength of the strong interaction decreases, resulting in the phenomenon of asymptotic freedom and making the application of perturbation theory possible. In this perturbative region the predictions of QCD are consistent with all experimental data. In the low-energy regime the coupling increases thus confining the quarks and gluons inside color-neutral hadrons. Since in this regime QCD is non-perturbative the direct connection between QCD and the relevant hadronic degrees of freedom (mesons and baryons) cannot be analytically calculated [Pic99] [Sch03] [Kub07].

A description of strong interaction in the low-energy regime can be achieved either by lattice computations or through the construction of an effective field theory depending on the particles appearing at this energy scale. This can generally be done by finding the most general effective Lagrangian that respects the same symmetries as the fundamental $\mathcal{L}_{\text{QCD}}$. With the quantum number color not showing up explicitly in any hadron, a theory based on them will automatically be invariant under rotations in color space. However, besides the local gauge symmetry QCD has more global symmetries that need to be respected. On the one hand QCD is invariant under separate $U(1)$ phase transformations for each flavor leading to flavor (and baryon) number conservation. On the other hand QCD has an approximate symmetry that will be the basis for the construction of Chiral Perturbation Theory: Ignoring the quark mass term $\mathcal{L}_{\text{QCD}}$ decouples into separate parts for the left-handed and right-handed quarks $q_{L/R} = \frac{1}{2}(1 \mp \gamma_5)q$ as

$$\mathcal{L}_{\text{QCD}} = \bar{\mathbf{q}}_L i\not{D}\mathbf{q}_L + \bar{\mathbf{q}}_R i\not{D}\mathbf{q}_R - \frac{1}{4} G_a^{\mu\nu} G_{\mu\nu}^a + \mathcal{O}(m) = \mathcal{L}_{\text{QCD}}^0 + \mathcal{O}(m). \tag{1.5}$$

This means that for vanishing quark masses (the "chiral limit") the QCD Lagrangian is invariant under separate $SU(3)$ transformations in flavor space. Since the masses of the up and down quark, and also to a lesser degree the mass of the strange quark, are very small compared to the typical hadronic scale of about 1 GeV it is reasonable to treat the masses of these light quarks as a perturbation. This implies that (1.5) may be used for the light quark sector, with exemplary the left-handed quark vector in flavor space transforming as a $(3,1)$ multiplet under the chiral symmetry group $G = SU(3)_L \times SU(3)_R$ [Sch03] [Pic98]:

$$\mathbf{q}_L = \begin{pmatrix} u \\ d \\ s \end{pmatrix}_L \xrightarrow{G} g_L\ \mathbf{q}_L, \qquad g_L = \exp\left\{-i\theta^a \frac{\lambda_L^a}{2}\right\} \in SU(3)_L. \tag{1.6}$$

Due to the unitarity of the elements of $SU(3)$, $\mathcal{L}_{\text{QCD}}$ is invariant under chiral transformations when the quark masses can be neglected.

1.2. Chiral Symmetry Breaking

Chiral symmetry is certainly explicitly broken through the non-vanishing quark masses but this is expected to be a small effect and can thus be treated as a perturbation. Conceptually more relevant is on the other hand that the doubled $SU(3)$ symmetry is not represented in the hadronic spectrum. The linear combination $T_V^a = T_R^a + T_L^a$ generates the invariant subgroup $H \in G$ which is realized in the Wigner-Weyl mode meaning that the irreducible representations of H can be related to the physical states: The mesonic sector of the hadron spectrum contains an octet of pseudoscalar mesons with the pions, kaons and η in a mass range of (135–548) MeV and quantum numbers $J^P = 0^-$. The remaining generators of G, which are given by $T_A^a = T_R^a - T_L^a$, should create degenerate multiplets (same mass, same spin) of opposite parity if one assumes the axial generators T_A^a to be unbroken. Those multiplets have not been observed, leading to the assumption that the ground state is indeed not invariant under the full symmetry group G but instead only under the subgroup H. The full symmetry group is thus spontaneously broken as

$$G = SU(3)_L \times SU(3)_R \xrightarrow{SSB} SU(3)_V. \tag{1.7}$$

According to the Goldstone theorem this spontaneous symmetry breaking produces one massless Goldstone boson for each broken generator. The bosons' transformation properties under parity must be the same as those of the first component of the current associated with the generator, meaning that due to the T_A^a producing an axial vector current the Goldstone bosons must be pseudoscalars. Although there exists no octet of *massless* pseudoscalars the Goldstone bosons may be associated with the octet formed by the pions, kaons and the η since their masses are very small compared to the typical hadronic scale. Especially the pions, for whom the effect of the explicit symmetry breaking due to the quark masses is the smallest, are with less than 140 MeV by far the lightest particles in the hadronic spectrum.

1.3. Transformation Properties of Goldstone Bosons

The effective theory in terms of the Goldstone bosons is built based on the spontaneous breaking of the chiral symmetry. Therefore it is imperative to determine the transformation properties of the meson fields under the elements of the chiral group, which are defined only for the quark fields, to construct the effective Lagrangian invariant under it.

Defining a vector of the Goldstone fields as $\vec{\phi} = (\phi^1, ..., \phi^8)$ where every element is a real scalar field $\phi^a : M^4 \to \mathbb{R}$ set in Minkowski spacetime M^4, the action of the symmetry group on $\vec{\phi}$ is generally given by a mapping

$$\vec{\phi} \xrightarrow{G} \vec{\phi}' = \vec{f}(g, \vec{\phi}), \tag{1.8}$$

depending both on the field vector and the group element $g \in G$. Although this mapping is generally not a representation of the group G the action of the identity e should be given by

$$\vec{f}(e, \vec{\phi}) = \vec{\phi} \tag{1.9}$$

and the composition law (group-homomorphism property)

$$\vec{f}(g_1, \vec{f}(g_2, \vec{\phi})) = \vec{f}(g_1 g_2, \vec{\phi}) \tag{1.10}$$

can be required.

The origin of the field vector $\vec{\phi} = \vec{0}$ may be identified with the ground state of the system, which has to be invariant under the unbroken subgroup H. Therefore the image of the origin with respect to elements of h is given by $\vec{f}(h, \vec{0}) = \vec{0}$ leading to the relation for the mapping induced by a general group element:

$$\vec{f}(gh, \vec{0}) = \vec{f}(g, \vec{0}) \quad \forall g \in G \quad \forall h \in H. \tag{1.11}$$

This relation means in particular that the function $\vec{f}$ maps the origin onto the same vector in the Goldstone space for a specific group element of G combined with any element of H, which is just the definiton of the (left-) coset of g: $gH = \{gh \mid g \in G\}$. Cosets either completely overlap or are completely disjoint such that the set of all distinct cosets is denoted by G/H and called the quotient group. Due to H being an invariant subgroup, $G/H = \{gH \mid g \in G\}$ can be seen as a group itself and is called coset space.

Two arbitrary elements of the coset space $g_1 H$ and $g_2 H$ are either equivalent or are mapped onto different vectors in the Goldstone space, which can be seen by assuming the opposite and mapping both $\vec{f}(g_i, \vec{0})$ and $\vec{f}(g_j, \vec{0})$ with respect to g_i^{-1} leading to $\vec{f}(e, \vec{0}) = \vec{0} = \vec{f}(g_i^{-1} g_j, \vec{0})$. Since only the elements of H leave the ground state invariant it would follow that $g_i^{-1} g_j \in H$ and therefore $g_j \in g_i H$ which is contradictory to the assumption of two disjoint cosets. Therefore the mapping is isomorphic and invertible leading to the interpretation that the elements of the quotient space (the disjoint sets of cosets) produce a mapping of the meson ground state onto different vectors in the Goldstone space leading to the possibility of identifying the Goldstone fields with the elements of G/H.

Specifically for each coset $\tilde{g}H$ and therefore for each Goldstone field an arbitrary group element $r = \tilde{g} h_r \in \tilde{g}H$ is chosen to be the coset representative. Through the action in the mapping $\vec{f}$ the effect of the group elements on the Goldstone bosons can be seen:

$$\vec{\phi} = \vec{f}(r, \vec{0}) \xrightarrow{G} \vec{f}(g, \vec{\phi}) = \vec{f}(g\tilde{g}h_r, \vec{0}) = \vec{f}(\tilde{g}' h_r, \vec{0}) \tag{1.12}$$

So to obtain the transformed $\vec{\phi}'$ from a given $\vec{\phi}$ one needs to multiply the coset $\tilde{g}H$ representing the original field by the group element g to obtain the coset representing the transformed field. Although the transformed field is sufficiently identified by any member of the new coset, the found one is not necessarily from the set of chosen coset representatives, making a correcting transformation necessary such that the coset representative identifying the transformed meson field is given by $r' = g\tilde{g}h_r h^{-1} = grh^{-1}$.

In the chiral symmetry the transformation is produced by the separate left- and righthanded group elements $g = (g_L, g_R)$. The convention for the choice of the coset representative is to rewrite $g = (\mathbb{E}, g_R g_L^{-1})(g_L, g_L)$ characterising each Goldstone boson by the unitary matrix defined through the representatives of the left- and righthanded coset

$$U(\phi) = r_R r_L^{-1} = \exp\left(i\frac{\phi^a \lambda^a}{f}\right) \tag{1.13}$$

transforming linearly under the chiral group as [Kub07] [Sch03] [Jon90]

$$U(\phi) \to (g_R r_R h^{-1})(g_L r_L h^{-1})^{-1} = g_R U(\Phi) g_L^{-1}. \tag{1.14}$$

1.4. Low-energy Effective Field Theory for the Strong Interaction

In the formalism of the effective field theory called Chiral Perturbation Theory (ChPT) the Goldstone fields are parametrized in flavor space as

$$U(\Phi) = e^{i\frac{\sqrt{2}}{f}\Phi} \qquad \Phi(x) = \frac{1}{\sqrt{2}}\phi^a\lambda^a = \begin{pmatrix} \frac{1}{\sqrt{2}}\pi^0 + \frac{1}{\sqrt{6}}\eta_8 & \pi^+ & K^+ \\ \pi^- & -\frac{1}{\sqrt{2}}\pi^0 + \frac{1}{\sqrt{6}}\eta_8 & K^0 \\ K^- & \bar{K}^0 & -\frac{2}{\sqrt{6}}\eta_8 \end{pmatrix}, \tag{1.15}$$

where f is a dimensionful constant, that can be related to the pion decay constant, and the fields of the pseudoscalar mesons are linear combinations of the real fields ϕ^a.

From this matrix the most general Lagrangian invariant under the chiral symmetry, parity and continuous Lorentz transformations has to be constructed. Being an effective field theory the Lagrangian can be written as an expansion in terms of powers of the momenta, which appear in the Lagrangian through derivatives. Since the derivative of a field $\partial_\mu \phi$ is odd under parity only terms with an even number of derivatives may appear. A term without any derivatives can only be constant since $UU^\dagger = 1$ and is therefore of no interest, leading to the general expansion

$$\mathcal{L}_{\text{ChPT}}(U) = \sum_n \mathcal{L}_{2n} = \mathcal{L}_2 + \mathcal{L}_4 + \mathcal{O}(p^6). \tag{1.16}$$

An isolated theory of pseudoscalar mesons is only useful for calculating pure meson processes. Much more interesting is the study of the hadronic part of the Standard Model which includes also couplings to the electroweak part. At the same time it is not reasonable to completely neglect the explicit breaking of the chiral symmetry through the non-vanishing quark masses considering that especially the kaons and the η masses cannot be considered a very small perturbation. Both of these concepts can be incorporated into the theory by introducing couplings to external classical fields, both in $\mathcal{L}_{\text{QCD}}$ and in $\mathcal{L}_{\text{ChPT}}$, in the most general (and therefore equivalent) way. Organizing potential external fields into groups based on their properties under Lorentz transformations one can introduce the generalized external fields $v_\mu = \frac{1}{2}(r_\mu + l_\mu), a_\mu = \frac{1}{2}(r_\mu - l_\mu), s$ and p which are Hermitian matrices coupling to the quark fields as

$$\mathcal{L}_{\text{QCD}} = \mathcal{L}^0_{\text{QCD}} + \bar{q}\gamma^\mu (v_\mu + \gamma_5 a_\mu) q - \bar{q}(s - i\gamma_5 p) q. \tag{1.17}$$

Here the electromagnetic field enters for example as $v_\mu \supset e\mathcal{Q}A_\mu$ where the quark-charge matrix $\mathcal{Q} = \frac{1}{3}\text{diag}(2,-1,-1)$ introduces an explicit breaking of chiral symmetry. The quark-mass matrix appears in $s \supset \mathcal{M} = \text{diag}(m_u, m_d, m_s)$.

As expected the explicit symmetry breaking has the effect that after having introduced the external fields $\mathcal{L}_{\text{QCD}}$ turns out to not be invariant any more under the chiral symmetry group. This can be fixed if one defines the external fields to also transform under a local $SU(3)_L \times SU(3)_R$ transformation as

$$\begin{aligned} s + ip &\longrightarrow g_R(s+ip)g_L^\dagger, \\ l_\mu &\longrightarrow g_L l_\mu g_L^\dagger + i g_L \partial_\mu g_L^\dagger, \\ \text{and} \quad r_\mu &\longrightarrow g_R r_\mu g_R^\dagger + i g_R \partial_\mu g_R^\dagger. \end{aligned} \tag{1.18}$$

The left- and righthanded external currents can be introduced into the effective Lagrangian by defining the covariant derivative

$$D_\mu U = \partial_\mu U - i r_\mu U + i U l_\mu \tag{1.19}$$

and the field-strength tensors $F^{\mu\nu}_{L/R}$ as

$$F^{\mu\nu}_L = \partial^\mu l^\nu - \partial^\nu l^\mu - i[l^\mu, l^\nu]. \tag{1.20}$$

In leading order the fields a_μ and v_μ contribute to the effective Lagrangian in $\mathcal{L}_2$ only through the covariant derivatives since the field-strength tensors contracted with derivatives of U would automatically be of $\mathcal{O}(p^4)$. The most general Lagrangian invariant under the assumed symmetries is then given by

$$\mathcal{L}_2 = \frac{f^2}{4}\langle D_\mu U^\dagger D^\mu U + U^\dagger \chi + \chi^\dagger U\rangle \quad \text{where} \quad \chi = 2B_0(s+ip). \tag{1.21}$$

The trace $\langle ... \rangle$ is necessary to obtain a scalar Lagrangian from the field and mass matrices and the two appearing constants f and B_0 can be related to the pion decay constant and the quark condensate [Pic98].

For any given order in p^2 the matrix U needs to be expanded in order to find the Lagrangian in terms of the meson fields. In $\mathcal{O}(p^2)$ an expansion up to order ϕ^4 in the fields is sufficient for finding kinetic, mass and interaction terms. To find these terms, while considering the explicit symmetry breaking through the quark masses but only meson-meson interactions, one sets $s = \mathcal{M}$ and $r_\mu = l_\mu = p = 0$. Inserting the expansion of U (see appendix A.1) one finds that

$$\begin{aligned} \mathcal{L}_2^0 &= \frac{f^2}{4}\langle \partial_\mu U^\dagger \partial^\mu U\rangle + \frac{f^2}{2} B_0 \langle \mathcal{M}(U^\dagger + U)\rangle \\ &= f^2 B_0(m_u + m_d + m_s) + \frac{1}{2}\langle \partial_\mu \Phi \partial^\mu \Phi\rangle - B_0 \left\langle \mathcal{M}\Phi^2 \right\rangle \\ &\quad + \frac{1}{12 f^2}\left\langle \left(\Phi \overset{\leftrightarrow}{\partial_\mu} \Phi\right)\left(\Phi \overset{\leftrightarrow}{\partial^\mu} \Phi\right)\right\rangle + B_0 \frac{1}{6f^2}\left\langle \mathcal{M}\Phi^4\right\rangle + \mathcal{O}\left(\frac{\Phi^6}{f^4}\right). \end{aligned} \tag{1.22}$$

While the first term is an irrelevant constant, the second one delivers the kinetic terms as expected for the mesons as real or complex Klein-Gordon fields

$$\frac{1}{2}\langle \partial_\mu \Phi \partial^\mu \Phi\rangle = \frac{1}{2}\partial_\mu \pi^0 \partial^\mu \pi^0 + \partial_\mu \pi^- \partial^\mu \pi^+ + \partial_\mu K^0 \partial^\mu \bar{K}^0 + \partial_\mu K^- \partial^\mu K^+ + \frac{1}{2}\partial_\mu \eta_8 \partial^\mu \eta_8 \tag{1.23}$$

and the second one determines the meson masses as functions of the quark masses :

$$-B_0\langle\mathcal{M}\Phi^2\rangle = -\underbrace{2B_0 m}_{M^2_{\pi^+}}\ \pi^-\pi^+ - \underbrace{B_0(m_d+m_s)}_{M^2_{K^0}}\ K^0\bar{K}^0 - \underbrace{B_0(m_u+m_s)}_{M^2_{K^+}}\ K^-K^+$$
$$- B_0 m\ (\pi^0)^2 - B_0\frac{1}{3}(m+2m_s)\eta_8^2 - B_0\frac{m_u-m_d}{\sqrt{3}}\pi^0\eta_8, \qquad (1.24)$$

where $m \equiv (m_u+m_d)/2$ has been defined. The interaction terms can be found in the appendix A.2.

Since the π^0 and the η_8 have the same quantum numbers a mixing between the interaction eigenstates takes place so that their masses cannot be immediately read off the Lagrangian. Since finally the mixing is small this effect will be neglected in the following. Further, since the difference between the up and down quark mass is small compared to the difference to the mass of the strange quark, most calculations will be done in the isospin limit $m_u = m_d = m$. In this sense quark masses appearing in $\mathcal{O}(p^4)$ results will be substituted by the following expressions for the meson masses:

$$2B_0 m = M_\pi^2 \qquad B_0(m+m_s) = M_K^2 \qquad (1.25)$$

The particle η_8 appearing in ChPT is not a physical particle since it mixes with the η_1 stemming from the pseudoscalar singlet[1] forming the physical particles η and η'. Since this process is not negligible with a considerably big mixing angle of at least $\theta = -11.5°$ [B+12, p. 200] it is not justified to use the mass of the η meson. Instead the parameter M_{η_8} may be written in terms of the pion and kaon masses as

$$M^2_{\eta_8} \approx \frac{2}{3}B_0\,(m+2m_s) = \frac{4M_K^2 - M_\pi^2}{3}, \qquad (1.26)$$

which is just the Gell-Mann-Okubo mass relation.

For calculations at $\mathcal{O}(p^4)$ the next-order chiral Lagrangian is necessary, which is given in terms of ten measurable low-energy constants as [Pic98]

$$\begin{aligned}\mathcal{L}_4 = {} & L_1\langle D_\mu U^\dagger D^\mu U\rangle^2 + L_2\langle D_\mu U^\dagger D_\nu U\rangle\langle D^\mu U^\dagger D^\nu U\rangle \\ & + L_3\langle D_\mu U^\dagger D^\mu U D_\nu U^\dagger D^\nu U\rangle + L_4\langle D_\mu U^\dagger D^\mu U\rangle\langle U^\dagger\chi + \chi^\dagger U\rangle \\ & + L_5\langle D_\mu U^\dagger D^\mu U(U^\dagger\chi + \chi^\dagger U)\rangle + L_6\langle U^\dagger\chi + \chi^\dagger U\rangle^2 \\ & + L_7\langle U^\dagger\chi - \chi^\dagger U\rangle^2 + L_8\langle\chi^\dagger U\chi^\dagger U + U^\dagger\chi U^\dagger\chi\rangle \\ & - iL_9\langle F_R^{\mu\nu} D_\mu U D_\nu U^\dagger + F_L^{\mu\nu} D_\mu U^\dagger D_\nu U\rangle + L_{10}\langle U^\dagger F_R^{\mu\nu} U F_{L\mu\nu}\rangle \\ & + H_1\langle F_{R\mu\nu}F_R^{\mu\nu} + F_{L\mu\nu}F_L^{\mu\nu}\rangle + H_2\langle\chi^\dagger\chi\rangle.\end{aligned} \qquad (1.27)$$

Since the last two terms in this Lagrangian do not contain pseudoscalar fields they are not directly measurable.

Of these constants the renormalized couplings $L_5^r(\mu)$ and $L_9^r(\mu)$ will be determined in the following through the calculation of the pion decay constant and the pion vector form factor, respectively.

A general amplitude at $\mathcal{O}(p^4)$ will include tree-level contributions from both $\mathcal{L}_2$ and $\mathcal{L}_4$ as well as loop-diagrams from $\mathcal{L}_2$. The divergences contained in these expressions have to be absorbed by redefinitions of the low-energy couplings appearing in $\mathcal{L}_4$, which is the typical procedure for an effective field theory: Every order in the momentum expansion is renormalized entirely within itself.

[1] An effective field theory containing the η_1 is given by a version of ChPT which includes an expansion in the inverse of the number of colors $1/N_c$ to extend $SU(3)_R \times SU(3)_L \times U(1)_V$ to $U(3)_L \times U(3)_R$.

2. Derivation of Currents

Chiral Perturbation Theory is not an isolated theory of pseudoscalar mesons instead it is a low-energy effective field theory of the strongly interacting part of the Standard Model. Therefore the ability to calculate processes involving only those mesons cannot be sufficient. The interactions of the non-hadronic part of the Standard Model with the quarks are clearly given by $\mathcal{L}_{\mathrm{SM}}$, although the part involving quarks and gluons cannot be calculated for low energies. The relevant Green's functions involving both non-hadronic fields as well as quark and gluon fields can be separated into the solvable electroweak part and the strong part. This strong part will generally be the vacuum expectation value of an operator written in terms of quark and gluon fields, which cannot be calculated perturbatively in QCD. In order to calculate this expression in ChPT the action of the operator on the ChPT-vacuum must be known and therefore the form of the operator in terms of the Goldstone fields needs to be found. This can be systematically done using the path integral formalism.

2.1. Derivation of Operators at $\mathcal{O}(p^4)$ in the Chiral Expansion

The general expression for the derivation of n-point correlation functions using the path integral formalism is [PS95]

$$\begin{aligned} \langle\Omega| T\left[\phi(x_1)...\phi(x_n)\right]|\Omega\rangle &= \frac{\int \mathcal{D}\phi\ \phi(x_1)...\phi(x_n)\ \exp\left\{i\int \mathrm{d}^4x\ \mathcal{L}\right\}}{\int \mathcal{D}\phi\ \exp\left\{i\int \mathrm{d}^4x\ \mathcal{L}\right\}} \\ &= \frac{\delta}{\delta ij(x_1)}...\frac{\delta}{\delta ij(x_n)} iZ[j]\Big|_{j=0}, \end{aligned} \tag{2.1}$$

where the generating functional is defined as

$$W[j] = e^{iZ[j]} = \int \mathcal{D}\phi\ \exp\left\{i\int \mathrm{d}^4x\ (\mathcal{L}+j\phi)\right\}. \tag{2.2}$$

In the case of strong interaction the generating functional can be written both in terms of the fundamental QCD as well as using the effective theory:

$$\begin{aligned} &W[v_\mu, a_\mu, s, p] = e^{iZ[v_\mu,a_\mu,s,p]} \\ &= \int \mathcal{D}q\ \mathcal{D}\bar{q}\ \mathcal{D}G_\mu\ \exp\left\{i\int \mathrm{d}^4x\ \left(\mathcal{L}^0_{\mathrm{QCD}} + \bar{q}\gamma^\mu(v_\mu+\gamma_5 a_\mu)q - \bar{q}(s-i\gamma_5 p)q\right)\right\} \\ &= \int \mathcal{D}U\ \exp\left\{i\int \mathrm{d}^4x\ (\mathcal{L}_2+\mathcal{L}_4+...)\right\}. \end{aligned} \tag{2.3}$$

Thus operators and amplitudes from the the two formalisms can be related, with J

being the appropriate combination of functional derivatives:[1]

$$\begin{aligned}\langle\Omega|\,\mathcal{O}(q,\bar{q},G_\mu)\,|\Omega\rangle &= \frac{\int \mathcal{D}q\;\mathcal{D}\bar{q}\;\mathcal{D}G_\mu\;\mathcal{O}(q,\bar{q},G_\mu)\;\exp\left\{i\int \mathrm{d}^4x\;\mathcal{L}_{\mathrm{QCD}}\right\}}{\int \mathcal{D}q\;\mathcal{D}\bar{q}\;\mathcal{D}G_\mu\;\exp\left\{i\int \mathrm{d}^4x\;\mathcal{L}_{\mathrm{QCD}}\right\}} \\ &= \frac{\delta}{\delta iJ}\;Z[v_\mu,a_\mu,s,p]\Big|_{j=j_0} \\ &= \frac{\int \mathcal{D}U\;\frac{\delta}{\delta iJ}\;\exp\left\{i\int \mathrm{d}^4x\;\left(\mathcal{L}_2+\mathcal{L}_4+...\right)\right\}}{\int \mathcal{D}U\;\exp\left\{i\int \mathrm{d}^4x\;\left(\mathcal{L}_2+\mathcal{L}_4+...\right)\right\}}\Bigg|_{j=j_0} \\ &= \frac{\int \mathcal{D}U\;\mathcal{O}(\Phi)\;\exp\left\{i\int \mathrm{d}^4x\;\left(\mathcal{L}_2+\mathcal{L}_4+...\right)\right\}}{\int \mathcal{D}U\;\exp\left\{i\int \mathrm{d}^4x\;\left(\mathcal{L}_2+\mathcal{L}_4+...\right)\right\}} \\ &= \langle 0|\,\mathcal{O}(\Phi)\,|0\rangle\,. \end{aligned} \tag{2.4}$$

Here and in the following $j = j_0$ is used as an abbreviation for $v_\mu = a_\mu = p = 0$ and $s = \mathcal{M}$.

Due to the nature of the ChPT Lagrangian as an expansion in the momentum the operator $\mathcal{O}(\Phi)$ will generally be an infinite sum of products of Goldstone fields and their derivatives. Since the ChPT vacuum is perturbative the vacuum expectation value of the operator in terms of the Goldstone fields can be determined by expanding around the classical solution, which involves solving the path integrals as of $\mathcal{O}(p^4)$.
As another possibility the operator can be inserted into transition amplitudes directly and evaluated including loop contributions depending on the desired degree of accuracy. This second method will be pursued in the following.
Up to $\mathcal{O}(p^4)$ the necessary terms of the operator are

$$\mathcal{O}(\Phi) \approx \int \mathrm{d}^4x\;\left[\frac{\delta\mathcal{L}_2}{\delta J}+\frac{\delta\mathcal{L}_4}{\delta J}\right] \equiv \mathcal{O}_2(\Phi)+\mathcal{O}_4(\Phi). \tag{2.5}$$

Of these, $\mathcal{O}_2(\Phi)$ needs to be derived up to an order in the fields that allows loop calculations while for $\mathcal{O}_4(\Phi)$ the leading order contribution is sufficient.

2.2. The Axialvector Current

The axial vector current is the operator derived from the generating functional by taking the functional derivative with respect to the external field a_μ. Using the QCD Lagrangian (1.17) it takes the simple form $\bar{q}\gamma^\mu\gamma_5 q$. Using the formalism described above its shape in terms of the Goldstone fields up to $\mathcal{O}(p^4)$ is derived through derivatives of the first two terms of the ChPT Lagrangian:

[1] The interacting vacuum of ChPT will be denoted as $|0\rangle$, while the non-perturbativity of the QCD vacuum will be emphasized by the use of $|\Omega\rangle$.

$$
\begin{aligned}
\langle\Omega|\,\bar{q}\gamma^\mu\gamma_5 q\,|\Omega\rangle &= \frac{\delta}{\delta i a_\mu}\; Z[v_\mu, a_\mu, s, p]\Big|_{j=j_0} \\
&= \frac{\int \mathcal{D}U\ \frac{\delta}{\delta i a_\mu}\ \exp\left\{i\int \mathrm{d}^4x\ \left(\mathcal{L}_2+\mathcal{L}_4+...\right)\right\}}{\int \mathcal{D}U\ \exp\left\{i\int \mathrm{d}^4x\ \left(\mathcal{L}_2+\mathcal{L}_4+...\right)\right\}}\Bigg|_{j=j_0} \\
&= \frac{\int \mathcal{D}U\ A^\mu(\Phi)\ \exp\left\{i\int \mathrm{d}^4x\ \left(\mathcal{L}_2+\mathcal{L}_4+...\right)\right\}}{\int \mathcal{D}U\ \exp\left\{i\int \mathrm{d}^4x\ \left(\mathcal{L}_2+\mathcal{L}_4+...\right)\right\}} \\
&= \langle 0|\, A^\mu(\Phi)\,|0\rangle\,.
\end{aligned}
\tag{2.6}
$$

This results in the expression

$$
A^\mu(\Phi) = A_2^\mu(\Phi) + A_4^\mu(\Phi) + ... = \int \mathrm{d}^4x\ \left[\frac{\delta\mathcal{L}_2}{\delta a_\mu} + \frac{\delta\mathcal{L}_4}{\delta a_\mu} + ...\right]_{j=j_0}. \tag{2.7}
$$

Specifically only the component $[A^\mu(\Phi)]^{12}$ will be given here in terms of the Goldstone fields since it will be used in the calculation of the pion decay constant later on.

2.2.1. $\mathcal{L}_2$ Contribution

The external field a_μ enters the chiral Lagrangian through the covariant derivative:

$$
D_\mu U\big|_{v_\mu=0} = \partial_\mu U - i a_\mu U - i U a_\mu \qquad D_\mu U^\dagger\Big|_{v_\mu=0} = \partial_\mu U^\dagger + i a_\mu U^\dagger + i U^\dagger a_\mu \tag{2.8}
$$

Taking into account that a_μ is a matrix, the functional derivative of these expressions reads

$$
\begin{aligned}
\frac{\delta\left(D_\nu U(x)\right)_{\alpha\beta}(x)}{\delta a_\mu(y)} = \frac{\delta\left(D_\nu U\right)_{\alpha\beta}}{\delta(a_\mu)_{\rho\sigma}} &= -i\delta_{\nu\mu}\left(\delta_{\alpha\rho}U_{\sigma\beta} + \delta_{\beta\sigma}U_{\alpha\rho}\right)\delta(x-y) \\
\text{and} \qquad \frac{\delta\left(D_\nu U^\dagger(x)\right)_{\alpha\beta}(x)}{\delta a_\mu(y)} &= i\delta_{\nu\mu}\left(\delta_{\alpha\rho}U^\dagger_{\sigma\beta} + \delta_{\beta\sigma}U^\dagger_{\alpha\rho}\right)\delta(x-y).
\end{aligned}
\tag{2.9}
$$

Thus the $\mathcal{L}_2$ contribution of the axial vector current becomes

$$
\begin{aligned}
A_2^\mu(\Phi) &= \frac{f^2}{4}\int \mathrm{d}^4x\ \frac{\delta}{\delta a_\mu}\left\langle D_\nu U^\dagger D^\nu U\right\rangle\Big|_{j=j_0} \\
&= \frac{f^2}{4}\int \mathrm{d}^4x\ \left[\frac{\delta\left(D_\nu U^\dagger\right)_{\alpha\beta}}{\delta a_\mu}\ \left(D^\nu U\right)_{\beta\alpha} + \left(D^\nu U^\dagger\right)_{\beta\alpha}\ \frac{\delta\left(D_\nu U\right)_{\alpha\beta}}{\delta a_\mu}\right]_{j=j_0} \\
&= i\frac{f^2}{4}\left[\left(\delta_{\alpha\rho}U^\dagger_{\sigma\beta} + \delta_{\beta\sigma}U^\dagger_{\alpha\rho}\right)\ \left(\partial^\mu U\right)_{\beta\alpha} - \left(\partial^\mu U^\dagger\right)_{\beta\alpha}\left(\delta_{\alpha\rho}U_{\sigma\beta} + \delta_{\beta\sigma}U_{\alpha\rho}\right)\right] \\
&= i\frac{f^2}{4}\left[U^\dagger\cdot\partial^\mu U + \partial^\mu U\cdot U^\dagger - U\cdot\partial^\mu U^\dagger - \partial^\mu U^\dagger\cdot U\right]_{\sigma\rho} \\
&= i\frac{f^2}{2}\left[\partial^\mu U\cdot U^\dagger - \partial^\mu U^\dagger\cdot U\right]
\end{aligned}
\tag{2.10}
$$

This expressions can be written in terms of the Goldstone fields using the expansion of the matrix U as given in the equations (A.1):

$$
A_2^\mu(\Phi) = -\sqrt{2}f\left[\partial^\mu\Phi - \frac{1}{3f^2}\left(\partial^\mu\Phi\ \Phi^2 - 2\Phi\ \partial^\mu\Phi\ \Phi + \Phi^2\ \partial^\mu\Phi\right) + \mathcal{O}\left(\frac{\Phi}{f}\right)^5\right] \tag{2.11}
$$

Inserting the expression for Φ (1.15) the relevant component of the axial vector current reads

$$
\begin{aligned}
[A_2^\mu(\Phi)]^{12} &= -\sqrt{2} f \left[1 - \frac{1}{3f^2} \left(2\pi^0\pi^0 + 2\pi^+\pi^- + K^+K^- + K^0\bar{K}^0 \right) \right] \partial^\mu \pi^+ \\
&+ \frac{\sqrt{2}}{3f} \pi^+ \left(K^0 \partial^\mu \bar{K}^0 - 2\bar{K}^0 \partial^\mu K^0 + K^- \partial^\mu K^+ - 2K^+ \partial^\mu K^- - 2\pi^0 \partial^\mu \pi^0 - 2\pi^+ \partial^\mu \pi^- \right) \\
&- \frac{1}{\sqrt{3} f} (K^+ \eta \partial^\mu \bar{K}^0 + \bar{K}^0 \eta \partial^\mu K^+ - 2K^+ \bar{K}^0 \partial^\mu \eta) + \frac{1}{f} \pi^0 \left(K^+ \partial^\mu \bar{K}^0 - \bar{K}^0 \partial^\mu K^+ \right).
\end{aligned}
\tag{2.12}
$$

2.2.2. $\mathcal{L}_4$ Contribution

Of all the terms in the $\mathcal{L}_4$ Lagrangian only a few can contribute. To find out which ones it is necessary to consider that with respect to the expansion of the effective Lagrangian in momenta the leading order contribution of $\mathcal{L}_4$ is of the same order as the next-to-leading order contribution of $\mathcal{L}_2$. Since A_2^μ was derived until cubic order in the fields, allowing for one-loop diagrams when contracted with a single external field, the comparable A_4^μ expression need only be of first order in the fields to deliver corresponding tree-level amplitudes.

Another general observation is that only Lagrangian terms of first order in the external field a_μ can contribute to the axial vector current since after the functional derivative all external fields except for the mass matrix are set to zero. This directly means that the term proportional to L_{10} starting at order a^2 does not contribute.

The first three terms of $\mathcal{L}_4$ all contain the covariant derivative $D_\mu U^{(\dagger)}$ four times. The only terms that are of first order in a_μ contain three derivatives of the meson matrix $\partial_\mu U^{(\dagger)}$ and are hence already of order ϕ^3, meaning that they will not contribute to the leading-order $\mathcal{L}_4$ contribution to the axial vector current. On the other hand the terms proportional to the couplings L_6, L_7 and L_8 do not have any covariant derivatives, are thus independent of a_μ and do not contribute at any order.

The term L_9 contains the a_μ already in the field strength tensor so in any potential contribution the covariant derivatives reduces to the actual derivatives $\partial_\mu U^{(\dagger)}$. Since there are two of them any contribution from L_9 would at least be of quadratic order in the fields (or considering parity in case of the axial vector current of cubic order) and can therefore also be neglected in a tree-level calculation.

The terms that remain are the ones proportional to the low-energy constants L_4 and L_5, whose contribution to the axial vector current is calculated very similarly to the one from $\mathcal{L}_2$ seen before. For the L_4 part the functional derivative gives

$$
\begin{aligned}
A^\mu_{4,L_4}(\Phi) &= \frac{\delta}{\delta i a_\mu} \int \mathrm{d}^4x \; i L_4 \left\langle D_\nu U^\dagger D^\nu U \right\rangle \left\langle U^\dagger \chi + \chi^\dagger U \right\rangle \Big|_{j=j_0} \\
&= 2B_0 L_4 \underbrace{\int \mathrm{d}^4x \; \frac{\delta}{\delta a_\mu} \left\langle D_\nu U^\dagger D^\nu U \right\rangle}_{4/f^2 \;\; A_0^\mu} \left\langle \mathcal{M}(U^\dagger + U) \right\rangle \Big|_{v_\mu = a_\mu = 0} \\
&= \frac{8 B_0 L_4}{f^2} \left\langle \mathcal{M}(U^\dagger + U) \right\rangle \left[A_0^\mu + \mathcal{O}\left(\frac{\Phi^2}{f^2} \right) \right],
\end{aligned}
\tag{2.13}
$$

where $A_0^\mu = -\sqrt{2} f \partial^\mu \Phi$ is introduced as a shorthand for the leading-order contribution to $A_2^\mu(\Phi)$. The remaining trace is, apart from the prefactors, the mass-dependent term of the chiral Lagrangian at $\mathcal{O}(p^2)$:

$$\left\langle \mathcal{M}(U^\dagger + U)\right\rangle = 2(m_u + m_d + m_s) - \frac{2}{f^2}\left\langle \mathcal{M}\Phi^2\right\rangle + \frac{1}{3f^4}\left\langle \mathcal{M}\Phi^4\right\rangle + \mathcal{O}\left(\frac{\Phi^6}{f^6}\right). \tag{2.14}$$

Since only the leading-order expression is needed nothing but the constant mass term is relevant here. Inserting the expressions found for the meson masses in the isospin limit (1.25) the expression can be simplified to give

$$A^\mu_{4,L_4}(\Phi) = 2\,\frac{8M_K^2 + 4M_\pi^2}{f^2}\,L_4\,\left[A_0^\mu + \mathcal{O}\left(\frac{\Phi^3}{f^3}\right)\right]. \tag{2.15}$$

The other term of $\mathcal{L}_4$ that enters into the axial vector current is the one proportional to L_5:

$$\begin{aligned} A^\mu_{4,L_5}(\Phi) &= \frac{\delta}{\delta i a_\mu}\int \mathrm{d}^4x\; iL_5\left\langle D_\nu U^\dagger D^\nu U\left(U^\dagger\chi + \chi^\dagger U\right)\right\rangle\Big|_{j=j_0} \\ &= 2B_0\;L_5\int \mathrm{d}^4x\;\frac{\delta}{\delta a_\mu}\left(D_\nu U^\dagger D^\nu U\right)_{\alpha\beta}\left(U^\dagger\mathcal{M} + \mathcal{M}U\right)_{\beta\alpha}. \end{aligned} \tag{2.16}$$

Evaluating the functional derivative gives

$$\begin{aligned} \frac{\delta}{\delta a_\mu}\left(D_\nu U^\dagger D^\nu U\right)_{\alpha\beta} &= \frac{\delta(D_\nu U^\dagger)_{\alpha\gamma}}{\delta a_\mu}(\partial^\nu U)_{\gamma\beta} + (\partial^\nu U^\dagger)_{\alpha\gamma}\frac{\delta(D_\nu U)_{\gamma\beta}}{\delta a_\mu} \\ &= i\delta(x-y)\left[\delta_{\alpha\rho}(U^\dagger\partial^\mu U)_{\sigma\beta} + U^\dagger_{\alpha\rho}(\partial^\mu U)_{\sigma\beta}\right. \\ &\qquad\left. -(\partial^\mu U^\dagger)_{\alpha\rho}U_{\sigma\beta} - \delta_{\beta\sigma}(\partial^\mu U^\dagger U)_{\alpha\rho}\right]. \end{aligned} \tag{2.17}$$

Combining the terms yields the following expression:

$$\begin{aligned} A^\mu_{4,L_5}(\Phi) &= 2iB_0\;L_5\Bigg[\underbrace{U^\dagger\,\partial^\mu U}_{-\partial^\mu U^\dagger\;U}\;U^\dagger\mathcal{M} + \underbrace{U^\dagger\,\partial^\mu U}_{-\partial^\mu U^\dagger\;U}\;\mathcal{M}U + \partial^\mu U\;U^\dagger\mathcal{M}\;U^\dagger + \partial^\mu U\;\mathcal{M} \\ &\qquad -\mathcal{M}\partial^\mu U^\dagger - U\;\mathcal{M}U\partial^\mu U^\dagger - U^\dagger\mathcal{M}\;\underbrace{\partial^\mu U^\dagger U}_{-U^\dagger\partial^\mu U} - \mathcal{M}U\;\underbrace{\partial^\mu U^\dagger U}_{-U^\dagger\partial^\mu U}\Bigg] \\ &= 2iB_0\;L_5\left[\left(\partial^\mu U\;-\partial^\mu U^\dagger\right)\mathcal{M} - \partial^\mu U^\dagger\;U\;\mathcal{M}U + \partial^\mu U\;U^\dagger\mathcal{M}\;U^\dagger\right. \\ &\qquad\left. +\mathcal{M}\left(\partial^\mu U - \partial^\mu U^\dagger\right) - U\;\mathcal{M}U\partial^\mu U^\dagger + U^\dagger\mathcal{M}\;U^\dagger\partial^\mu U\right]. \end{aligned} \tag{2.18}$$

Since an expansion of U that is linear in Φ is sufficient,

$$U^\dagger\mathcal{M}U^\dagger\;\partial^\mu U - U\mathcal{M}U\;\partial^\mu U^\dagger = \mathcal{M}\left(\partial^\mu U - \partial^\mu U^\dagger\right) + \mathcal{O}\left(\frac{\Phi^3}{f^3}\right) \tag{2.19}$$

and thus the final expression for the L_5 contribution to the axial vector current:

$$A^\mu_{4,L_5}(\Phi) = \frac{8B_0}{f^2}\;L_5\left[A_0^\mu\mathcal{M} + \mathcal{M}A_0^\mu + \mathcal{O}\left(\mathcal{M}\frac{\Phi^3}{f^3}\right)\right] \tag{2.20}$$

Taking only the component that will be used for the calculation of the pion decay and defining $A^\mu_\pi = -\sqrt{2} f \partial^\mu \pi^+$ one gets

$$\left[A^\mu_{4,L_5}(\Phi)\right]^{12} = 2\,\frac{4M_\pi^2}{f^2}\;L_5\;\left[A^\mu_\pi + \mathcal{O}\left(\frac{\Phi^3}{f^3}\right)\right]. \tag{2.21}$$

Combining those results the complete axial vector current at $\mathcal{O}(p^4)$ is given by

$$\begin{aligned}
[A^\mu(\Phi)]^{12} &= \left[A^\mu_2(\Phi) + A^\mu_{4,L_4}(\Phi) + A^\mu_{4,L_5}(\Phi)\right]^{12} \\
&= [A^\mu_2(\Phi)]^{12} + \left(2\,\frac{8M_K^2 + 4M_\pi^2}{f^2}\;L_4\; + 2\,\frac{4M_\pi^2}{f^2}\;L_5\right)\left[A^\mu_\pi + \mathcal{O}\left(\frac{\Phi^3}{f^3}\right)\right].
\end{aligned} \tag{2.22}$$

2.3. The Vector Current

The steps to derive the vector current in terms of the Goldstone fields are analogous to the derivation seen above. The general expression reads

$$\langle\Omega|\,\bar{q}\gamma^\mu q\,|\Omega\rangle = \langle 0|\,V^\mu(\Phi)\,|0\rangle = \frac{\delta}{\delta i v_\mu}\;Z[v_\mu, a_\mu, s, p]\Big|_{j=j_0}\,, \tag{2.23}$$

which leads to

$$V^\mu(\Phi) = V^\mu_2(\Phi) + V^\mu_4(\Phi) + ... = \int \mathrm{d}^4x\;\left[\frac{\delta\mathcal{L}_2}{\delta v_\mu} + \frac{\delta\mathcal{L}_4}{\delta v_\mu} + ...\right]_{j=j_0}. \tag{2.24}$$

2.3.1. $\mathcal{L}_2$ Contribution

The expressions for the covariant derivative in terms of v_μ instead of a_μ differ only by a minus sign:

$$D_\mu U^{(\dagger)}\Big|_{a_\mu=0} = \partial_\mu U^{(\dagger)} + iU^{(\dagger)}v_\mu - iv_\mu U^{(\dagger)} \tag{2.25}$$

Taking the functional derivative of these expressions one gets

$$\frac{\delta\left(D_\nu U^{(\dagger)}(x)\right)_{\alpha\beta}(x)}{\delta v_\mu(y)} = i\delta_{\nu\mu}\left(\delta_{\beta\sigma}U^{(\dagger)}_{\alpha\rho} - \delta_{\alpha\rho}U^{(\dagger)}_{\sigma\beta}\right)\delta(x-y). \tag{2.26}$$

Thus the $\mathcal{L}_2$ contribution of the axial vector current becomes

$$\begin{aligned}
V^\mu_2(\Phi) &= \frac{f^2}{4}\int \mathrm{d}^4x\;\frac{\delta}{\delta v_\mu}\left\langle D_\nu U^\dagger D^\nu U\right\rangle\Big|_{j=j_0} \\
&= i\frac{f^2}{4}\left[\left(\delta_{\beta\sigma}U^\dagger_{\alpha\rho} - \delta_{\alpha\rho}U^\dagger_{\sigma\beta}\right)\left(\partial^\mu U\right)_{\beta\alpha} + \left(\partial^\mu U^\dagger\right)_{\beta\alpha}\left(-\delta_{\alpha\rho}U_{\sigma\beta} + \delta_{\beta\sigma}U_{\alpha\rho}\right)\right] \\
&= i\frac{f^2}{2}\left[\partial^\mu U\cdot U^\dagger + \partial_\mu U^\dagger\cdot U\right]
\end{aligned} \tag{2.27}$$

Using the expansion of U in the fields (A.1) this leads to the expression

$$V^\mu_2(\Phi) = -i\left[\Phi\overleftrightarrow{\partial^\mu}\Phi + \frac{1}{6f^2}\left[\partial^\mu\Phi\;\Phi^3 - \Phi^3\;\partial^\mu\Phi\right] + \frac{1}{2f^2}\Phi\;(\Phi\overleftrightarrow{\partial^\mu}\Phi)\;\Phi\right] + \mathcal{O}(\Phi^6). \tag{2.28}$$

2.3.2. $\mathcal{L}_4$ Contribution

The vector current stemming from $\mathcal{L}_4$ has to be found up to quadratic order in the meson fields. With the same argument as in the case of the axial vector current most terms of $\mathcal{L}_4$ need not be considered. The only addition comes from the Lagrangian term containing L_9 which leads to a contribution at order Φ^2 as will be shown in the following.

The contributions of the terms of $\mathcal{L}_4$ proportional to the constants L_4 and L_5 are calculated just as in the axial vector case. The L_4 one is particularly simple and directly gives

$$\begin{aligned} V^\mu_{4,L_4}(\Phi) &= L_4 \underbrace{\int \mathrm{d}^4x \, \frac{\delta}{\delta v_\mu} \left\langle D_\nu U^\dagger D^\nu U \right\rangle}_{4/f^2 \; V_2^\mu} \quad \underbrace{\left\langle U^\dagger \chi + \chi^\dagger U \right\rangle\Big|_{j=j_0}}_{2B_0\langle \mathcal{M}(U^\dagger + U)\rangle = 4M_K^2 + 2M_\pi^2 + \mathcal{O}(\Phi^2)} \\ &= -i\frac{8L_4}{f^2} \, \left(2M_K^2 + M_\pi^2\right) \, \left[\Phi \overleftrightarrow{\partial^\mu} \Phi + \mathcal{O}\left(\frac{\Phi^4}{f^4}\right)\right] \end{aligned} \tag{2.29}$$

From the L_5-term one gets

$$\begin{aligned} V^\mu_{4,L_5}(\Phi) &= L_5 \int \mathrm{d}^4x \, \frac{\delta}{\delta v_\mu} \, \left\langle D_\nu U^\dagger D^\nu U \left[U^\dagger \chi + \chi^\dagger U\right] \right\rangle \Bigg|_{j=j_0} \\ &= 2B_0 L_5 \int \mathrm{d}^4x \, \left(U^\dagger \mathcal{M} + \mathcal{M} U\right)_{\beta\alpha} \frac{\delta}{\delta v_\mu} \left(D_\nu U^\dagger D^\nu U\right)_{\alpha\beta}\Big|_{j=j_0} \\ &= 2iB_0 L_5 \Big[\partial^\mu U \; U^\dagger \mathcal{M} U^\dagger + \partial^\mu U^\dagger \; U \mathcal{M} U - U^\dagger \mathcal{M} U^\dagger \; \partial^\mu U - U \mathcal{M} U \; \partial^\mu U^\dagger \\ &\qquad + \left(\partial^\mu U + \partial^\mu U^\dagger\right) \mathcal{M} - \mathcal{M} \left(\partial^\mu U + \partial^\mu U^\dagger\right)\Big] , \end{aligned} \tag{2.30}$$

which can be expanded to quadratic order in the meson fields to give

$$V^\mu_{4,L_5}(\Phi) = \frac{8iB_0 L_5}{f^2} \, \left([\partial^\mu \Phi \mathcal{M}, \Phi] + [\mathcal{M} \partial^\mu \Phi, \Phi]\right) . \tag{2.31}$$

The additional L_9 contribution is given by

$$V^\mu_{4,L_9}(\Phi) = -iL_9 \int \mathrm{d}^4x \, \frac{\delta}{\delta v_\mu} \left\langle F_R^{\rho\sigma} \; D_\rho U D_\sigma U^\dagger + F_L^{\rho\sigma} \; D_\rho U^\dagger D_\sigma U \right\rangle\Big|_{j=j_0} ,$$

where the derivative acting on the fields will not contribute due to the field strength tensors vanishing when setting the external fields to zero:

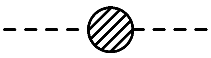
$$\begin{aligned} &= -iL_9 \int \mathrm{d}^4x \, \frac{\delta}{\delta v_\mu} \left(\partial_\rho v_\sigma - \partial_\sigma v_\rho\right)_{\alpha\beta} \, \left(\partial^\rho U \partial^\sigma U^\dagger + \; \partial^\rho U^\dagger \partial^\sigma U\right)_{\beta\alpha} \\ &= -iL_9 \; \partial_\nu \left(\partial^\mu U \; \partial^\nu U^\dagger - \partial^\nu U \; \partial^\mu U^\dagger + \partial^\mu U^\dagger \; \partial^\nu U - \partial^\nu U^\dagger \; \partial^\mu U\right) \end{aligned} \tag{2.32}$$

Expanding this to quadratic order in Φ leads the L_9 contribution to the vector current to be

$$V^\mu_{4,L_9}(\Phi) = -i\frac{4L_9}{f^2} \left([\partial^\nu \partial^\mu \Phi, \partial_\nu \Phi] + \left[\partial^\mu \Phi, \partial^2 \Phi\right] + \mathcal{O}\left(\frac{\Phi^4}{f^4}\right)\right) . \tag{2.33}$$

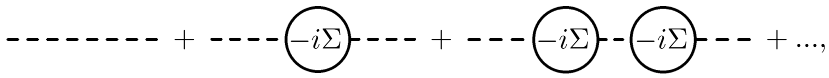
3. Self-energy: Mass and Wave-function Renormalization

The propagator of an interacting theory will always receive loop corrections and can therefore only to leading order be approximated with the standard Feynman propagator. In the case of an effective theory like ChPT in addition to the loops, higher orders in the momentum expansion will also create additional tree-level contributions that have to be included. The general propagator for a scalar field ϕ, with the vacuum of the interacting theory denoted by $|\Omega\rangle$, can be written as an expansion in the self-energy



$$= \int \mathrm{d}^4x \; e^{ipx} \; \langle\Omega|T\phi(x)\phi(0)|\Omega\rangle$$

$$=$$



where the self-energy $-\Sigma(p^2)$ includes all irreducible diagrams contributing to the propagator. In a quantum field theory that can be treated perturbatively the corrections through the self-energy should be small and so the expansion can be treated as a geometric series and rewritten as

$$\begin{aligned} &= \frac{i}{p^2 - m_0^2}\left[1 + \frac{\Sigma(p^2)}{p^2 - m_0^2} + \left(\frac{\Sigma(p^2)}{p^2 - m_0^2}\right)^2 + ...\right] \\ &= \frac{i}{p^2 - m_0^2 - \Sigma(p^2)}. \end{aligned} \tag{3.1}$$

The self-energy contains loop diagrams and will always be divergent. In order to calculate finite amplitudes those divergences must be absorbed in the definition of the renormalized physical mass m and the field as $\phi = \sqrt{Z}\phi_r$ such that the propagator takes the shape

$$= \frac{iZ}{p^2 - m^2 - \Sigma_r(p^2)}. \tag{3.2}$$

The pole of the propagator is a physical observable and must be indepent of arbitrary definitions so through an expansion of both (3.1) and (3.2) the value of the field-strength renormalization constant Z can be found. Expanding the denominator of the latter around the position of the pole one gets

$$p^2 - m^2 - \Sigma_r(p^2) = 0 + \left(1 - \frac{\mathrm{d}\Sigma_r(p^2)}{\mathrm{d}p^2}\right)\bigg|_{p^2=m^2} (p^2 - m^2) + ..., \tag{3.3}$$

where the derivative vanishes.The expansion of the bare denominator gives similarly

$$p^2 - m_0^2 - \Sigma(p^2) = \left(p^2 - m_0^2 - \Sigma(p^2)\right)\big|_{p^2=m^2} + \left(1 - \frac{\mathrm{d}\Sigma(p^2)}{\mathrm{d}p^2}\right)\bigg|_{p^2=m^2} (p^2 - m^2) + ... \tag{3.4}$$

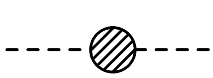
The first term in this expansion defines the physical mass through the condition

$$m^2 - \Sigma(m^2) = m_0^2. \tag{3.5}$$

In the second term the derivative of the unrenormalised self-energy does not vanish (in fact it generally contains divergences), leading to the identification of the wavefunction renormalization as

$$Z = \left[1 - \left.\frac{\mathrm{d}\Sigma(p^2)}{\mathrm{d}p^2}\right|_{p^2=m^2}\right]^{-1}. \tag{3.6}$$

In the Källén-Lehmann representation the full propagator then takes the form



$$= \frac{iZ}{p^2 - m^2 + i\varepsilon} + [\text{reg. at } p^2 = m^2]. \tag{3.7}$$

In calculating amplitudes that need renormalization this has the consequence that one must multiply the result with a factor of $\sqrt{Z}$ for each incoming or outgoing particle to account for the renormalization of the external legs. This can also be seen in the LSZ reduction formula, which for a two-point correlation function is given by

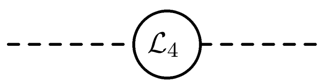
$$\int \mathrm{d}^4x \mathrm{d}^4y \; e^{i(qy-px)} \; \langle\Omega|T\{\phi(x)\phi(y)\}|\Omega\rangle = \frac{i\sqrt{Z}}{p^2 - m^2 + i\varepsilon} \frac{i\sqrt{Z}}{q^2 - m^2 + i\varepsilon} \; \langle\phi(q)|S|\phi(p)\rangle. \tag{3.8}$$

Since here on the left the fully renormalized amplitude is calculated also the right side of the equation must be renormalized which is being achieved with the wave-function renormalization factor [PS95].

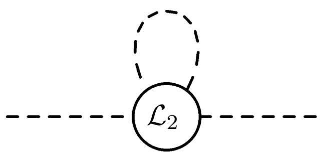
3.1. Calculation of the Pion Self-energy

The pion self-energy at leading order gets contributions from the contact terms of $\mathcal{L}_4$ as well as through loops stemming from the interaction terms of $\mathcal{L}_2$:

$$-i\Sigma(p^2) =$$



$+$



(3.9)

$$= -i\Sigma_4(p^2) - i\Sigma_2(p^2). \tag{3.10}$$

Of all the terms of the respective Lagrangians only those can contribute that have both a π^+ and a π^- field to be contracted with the incoming and outgoing pions. The additional fields in the $\mathcal{L}_2$ interaction terms must be of the form $\phi_a\phi_a^*$ to be able to contract in the shape of a loop. The fields appearing in the interaction terms may include two derivatives. Of these none or both must act on the pion field pair. If there was only one derivative acting on a field forming the loop, the integral over the loop momentum q would include a factor q_μ making it odd and thus zero. These considerations leave terms of the form

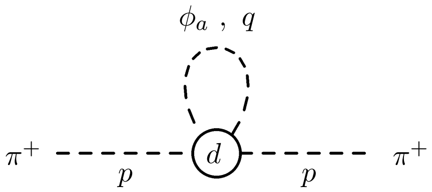
$$\mathcal{L}_2 \supset \; d \; \phi_a\phi_a^*\partial_\mu\pi^+\partial^\mu\pi^- + d' \; \pi^+\pi^-\partial_\mu\phi_a\partial^\mu\phi_a^* + d'' \; \pi^+\pi^-\phi_a\phi_a^* \tag{3.11}$$

$$\mathcal{L}_4 \supset \; c \; \partial_\mu\pi^+\partial^\mu\pi^- + c' \; \pi^+\pi^- \tag{3.12}$$

to be isolated from the Lagrangians.

3.1.1. $\mathcal{L}_2$ Contribution

In the general case for a term of the form $d\ \phi_a \phi_a^* \partial_\mu \pi^+ \partial^\mu \pi^-$ the Feynman rule for the loop vertex is $id\ p^2$ resulting in a contribution to the self-energy of

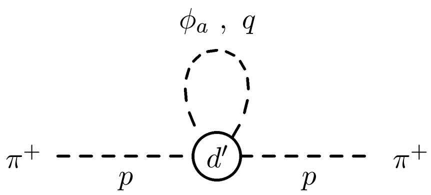


$$= id\ p^2 \int \frac{\mathrm{d}^4 q}{(2\pi)^4} \ \frac{i}{q^2 - m_\phi^2 + i\varepsilon} \equiv id\ p^2\ I_\phi. \qquad (3.13)$$

Here the notation I_ϕ for the divergent integral is introduced since it will appear frequently throughout the calculations. It can be solved employing the formalism of dimensional regularization to give an explicit expression for the contained divergence in terms of the parameter R:

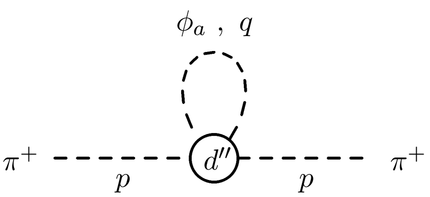
$$\begin{aligned} I_\phi &\equiv \int \frac{\mathrm{d}^4 q}{(2\pi)^4} \ \frac{i}{q^2 - m_\phi^2 + i\varepsilon} \\ &= \mu^{2\varepsilon} \left(\frac{m_\phi}{4\pi}\right)^2 \left[R + \ln \frac{m_\phi^2}{\mu^2}\right] \quad \text{where} \quad R \equiv \frac{1}{\varepsilon} + \gamma_E - 1 - \ln 4\pi. \end{aligned} \qquad (3.14)$$

For a term of the shape $d'\ \pi^+ \pi^- \partial_\mu \phi_a \partial^\mu \phi_a^*$ the Feynman rule depends on the loop momentum leading to



$$= id' \int \frac{\mathrm{d}^4 q}{(2\pi)^4} \ \frac{iq^2}{q^2 - m_\phi^2 + i\varepsilon} \equiv id'\ m^2\ I_\phi. \qquad (3.15)$$

In the third type of contributing Lagrangian term $d''\ \pi^+ \pi^- \phi_a \phi_a^*$ there is no momentum depence in the Feynman rule so the corresponding self energy diagram just gives



$$= id' \int \frac{\mathrm{d}^4 q}{(2\pi)^4} \ \frac{i}{q^2 - m_\phi^2 + i\varepsilon} \equiv id''\ I_\phi. \qquad (3.16)$$

The part of the $\mathcal{L}_2$ Lagrangian containing only terms with four Goldstone fields is

$$\mathcal{L}_{2,4\phi} = \frac{1}{6f^2} \left[\left\langle \partial_\mu \Phi\ \Phi\ \partial^\mu \Phi\ \Phi - \partial_\mu \Phi \partial^\mu \Phi\ \Phi^2 \right\rangle + B_0 \left\langle \mathcal{M} \Phi^4 \right\rangle\right], \qquad (3.17)$$

when $s = \mathcal{M}$ and without external fields. This includes terms generating loops of charged and uncharged pions and kaons:

$$\begin{aligned}
&- \frac{1}{6f^2}\left[2\pi^+\pi^- + 2\pi^0\pi^0 + K^+K^- + \bar{K}^0K^0\right]\partial_\mu\pi^+\partial^\mu\pi^- \\
&- \frac{1}{6f^2}\left[2\partial_\mu\pi^0\partial^\mu\pi^0 + \partial_\mu K^-\partial^\mu K^+ + \partial_\mu K^0\partial^\mu\bar{K}^0\right]\pi^-\pi^+ \\
&+ \frac{1}{6f^2}\left[M_\pi^2\pi^-\pi^+ + M_\pi^2\pi^0\pi^0 + (M_\pi^2+M_K^2)K^+K^- + (M_\pi^2+M_K^2)K^0\bar{K}^0 + M_\pi^2\eta_8^2\right]\pi^-\pi^+.
\end{aligned} \tag{3.18}$$

Applying the schematics developed above one needs to keep in mind that the term containing four charged pion fields $(\pi^-\pi^+)^2$ has four equivalent possibilities to contract, leading to an additional factor of four and further the term proportional to $\pi^-\pi^+\partial_\mu\pi^-\partial^\mu\pi^+$ has to be treated both as type d and as type d'.

Finally this leads the contribution to the self-energy stemming from the $\mathcal{L}_2$ loops to be

$$\Sigma_2(p^2) = \frac{1}{6f^2}\left(4p^2 - M_\pi^2\right)I_\pi + \frac{1}{3f^2}\left(p^2 - M_\pi^2\right)I_K - \frac{1}{6f^2}M_\pi^2 I_\eta. \tag{3.19}$$

3.1.2. $\mathcal{L}_4$ Contribution

The terms of $\mathcal{L}_4$ proportional to L_1, L_2 and L_3 contain a minimum of four fields since the expansion of $\partial_\mu U$ starts at $\mathcal{O}(\Phi)$. Without external fields the terms L_9 and L_{10} vanish (or do not couple to the Goldstone bosons in the case of H_1 and H_2).

The contributions from the remaining terms will be calculated in the following in the limit of isospin symmetry and the external fields set to zero except for $s = \mathcal{M}$.

$$\mathcal{L}_{4,4} = 2B_0L_4\left\langle\partial_\mu U^\dagger\partial^\mu U\right\rangle\left\langle\mathcal{M}\left(U^\dagger + U\right)\right\rangle \tag{3.20}$$

Since the first trace is already $\mathcal{O}(\Phi^2)$ only the constant term of the second trace has to be taken into account.

$$\begin{aligned}
&= \frac{4L_4}{f^2}(2M_K^2 + M_\pi^2)\langle\partial_\mu\Phi\partial^\mu\Phi\rangle + \mathcal{O}\left(\Phi^4\right) \\
&\supset \frac{8L_4}{f^2}(2M_K^2 + M_\pi^2)\,\partial_\mu\pi^-\partial^\mu\pi^+
\end{aligned} \tag{3.21}$$

In the last line all terms without pion fields were neglected. The next contribution stems from the L_5 part of the Lagrangian:

$$\begin{aligned}
\mathcal{L}_{4,5} &= 2B_0L_5\,\left\langle\partial_\mu U^\dagger\partial^\mu U\left[U^\dagger\mathcal{M} + \mathcal{M}U\right]\right\rangle \\
&= \frac{8B_0L_5}{f^2}\langle\partial_\mu\Phi\partial^\mu\Phi\cdot\mathcal{M}\rangle + \mathcal{O}\left(\Phi^4\right) \\
&\supset \frac{8L_5}{f^2}M_\pi^2\,\partial_\mu\pi^-\partial^\mu\pi^+
\end{aligned} \tag{3.22}$$

Another contribution comes from the L_6 part of $\mathcal{L}_4$:

$$\begin{aligned}
\mathcal{L}_{4,6} &= 4L_6B_0^2\left\langle\mathcal{M}\left(U^\dagger + U\right)\right\rangle^2 \\
&= \text{const.} - \frac{16L_6}{f^2}\left(2M_K^2 + M_\pi^2\right)\,B_0\langle\mathcal{M}\Phi^2\rangle + \mathcal{O}(\Phi^4) \\
&\supset -\frac{16L_6}{f^2}\left(2M_K^2 + M_\pi^2\right)\,M_\pi^2\,\pi^-\pi^+
\end{aligned} \tag{3.23}$$

Potentially there could also be a contribution from the L_7 term given by

$$\begin{aligned}\mathcal{L}_{4,7} &= 2B_0L_7\,\left\langle \mathcal{M}\left(U^\dagger - U\right)\right\rangle^2 \\ &= \frac{16B_0L_7}{f^2}\left\langle \mathcal{M}\Phi + \mathcal{O}(\Phi^3)\right\rangle^2 \\ &= \frac{16B_0L_7}{f^2}\left(\frac{1}{\sqrt{3}}(m-m_s)\eta_8 + \sqrt{2}m\pi^0\right)^2 + \mathcal{O}(\Phi^4),\end{aligned} \tag{3.24}$$

which turns out to be indepent of the charged pion fields at $\mathcal{O}(\Phi^2)$. The last source of contact terms is the Lagrangian containing L_8:

$$\begin{aligned}\mathcal{L}_{4,8} &= 4L_8B_0^2\left\langle \mathcal{M}\left(U\mathcal{M}U + U^\dagger\mathcal{M}U^\dagger\right)\right\rangle \\ &= \text{const.} - \frac{16L_8B_0^2}{f^2}\left\langle \Phi^2\mathcal{M}^2 + \Phi\mathcal{M}\Phi\mathcal{M}\right\rangle + \mathcal{O}(\Phi^4) \\ &\supset -\frac{16L_8}{f^2}M_\pi^4\pi^-\pi^+\end{aligned} \tag{3.25}$$

Combining these results the terms of $\mathcal{L}_4$ that contribute to the pion self-energy are

$$\begin{aligned}\mathcal{L}_4 \supset\, & \frac{8}{f^2}\left[L_4(2M_K^2 + M_\pi^2) + L_5M_\pi^2\right]\,\partial_\mu\pi^-\partial^\mu\pi^+ \\ & - \frac{16}{f^2}\left[L_6\left(2M_K^2 + M_\pi^2\right)\,M_\pi^2 + L_8M_\pi^4\right]\pi^-\pi^+.\end{aligned} \tag{3.26}$$

The Feynman rule for the charged pion contact term can be read off this expression leading to the self-energy contribution

$$\Sigma_4(p^2) = \frac{8}{f^2}\left[\left(2L_6M_\pi^2 - L_4p^2\right)\left(2M_K^2 + M_\pi^2\right) + 2L_8M_\pi^4 - L_5p^2M_\pi^2\right]. \tag{3.27}$$

Combining the previous results one gets for the full self-energy of the charged pion the expression

$$\boxed{\begin{aligned}\Sigma(p^2) = &-\frac{1}{f^2}\left[\frac{1}{6}I_\pi + \frac{1}{3}I_K + \frac{1}{6}I_\eta - 16L_6\left(2M_K^2 + M_\pi^2\right) - 16L_8M_\pi^2\right]M_\pi^2 \\ &+\frac{1}{f^2}\left[\frac{2}{3}I_\pi + \frac{1}{3}I_K - 8L_4\left(2M_K^2 + M_\pi^2\right) - 8L_5M_\pi^2\right]\,p^2.\end{aligned}} \tag{3.28}$$

3.2. Field-strength Renormalization

The field-strength renormalization factor Z needed to renormalize the external legs of the amplitudes to be calculated in the following chapters can be determined directly from equation (3.6) to be

$$Z = \left[1 - \frac{1}{f^2}\left(\frac{2}{3}I_\pi + \frac{1}{3}I_K - 8L_4\left(2M_K^2 + M_\pi^2\right) - 8L_5M_\pi^2\right)\right]^{-1}. \tag{3.29}$$

Keeping in mind that the self-energy is a correction, Z^{-1} can be expanded around 1 to give

$$= 1 + \frac{1}{f^2}\left(\frac{2}{3}I_\pi + \frac{1}{3}I_K - 8L_4\left(2M_K^2 + M_\pi^2\right) - 8L_5M_\pi^2\right) + \mathcal{O}(p^4) \tag{3.30}$$

3.3. Mass

The equation to determine the mass at $\mathcal{O}(p^4)$ was found in (3.5). In an effective field theory the divergences appearing at a certain order will always be renormalized by the tree-level amplitudes at that order, so the "bare" pion mass in this context is the pion mass determined at $\mathcal{O}(p^2)$, which has already been substituted into the results until now. Taking now the self-energy at $p^2 = m^2$ means substituting the pion mass at $\mathcal{O}(p^4)$, which will be called $M_{\pi,4}$, into the equation.

Separating the self-energy into a constant and a momentum dependent part one has

$$\Sigma(p^2) = C_1 + C_2 p^2, \tag{3.31}$$

which inserted into the equation for the mass gives

$$M_{\pi,4}^2 = \frac{M_\pi^2 + C_1}{1 - C_2}. \tag{3.32}$$

Since both the "bare" pion mass and the coefficient C_2 are of $\mathcal{O}(p^2)$ while C_1 is of $\mathcal{O}(p^4)$ the denominator can be expanded such that

$$\begin{aligned}\frac{M_{\pi,4}^2}{M_\pi^2} &= (1 + C_2) + \frac{C_1}{M_\pi^2}\\ &= 1 + \frac{1}{f^2}\left[\frac{1}{2}I_\pi - \frac{1}{6}I_\eta + 8(2M_K^2 + M_\pi^2)(2L_6 - L_4) + 8M_\pi^2(2L_8 - L_5)\right],\end{aligned} \tag{3.33}$$

where the infinities contained in the integrals I_π and I_η can be explicitly expressed using (B.1), inserting the Gell-Mann Okubo mass relation (1.26) for the η mass, to give

$$\begin{aligned}&= 1 + \frac{1}{f^2}\left[\frac{\mu^{2\varepsilon}}{(4\pi)^2}\left(\frac{5}{9}M_\pi^2 - \frac{2}{9}M_K^2\right)R + \frac{1}{2}\frac{M_\pi^2}{(4\pi)^2}\ln\frac{M_\pi^2}{\mu^2} - \frac{1}{6}\frac{M_{\eta_8}^2}{(4\pi)^2}\ln\frac{M_\eta^2}{\mu^2}\right.\\ &\quad\left. + 8(2M_K^2 + M_\pi^2)(2L_6 - L_4) + 8M_\pi^2(2L_8 - L_5)\right].\end{aligned} \tag{3.34}$$

Although these divergences can be perfectly absorbed in the four low-energy couplings appearing here, their exact definitions have to be determined elsewhere. Using the general expression

$$L_i = L_i^r(\mu) + \Gamma_i \frac{\mu^{2\varepsilon}}{2(4\pi)^2}R \tag{3.35}$$

the corrective divergence produced by the couplings is given by

$$\left[8M_K^2(2\Gamma_6 - \Gamma_4) + 4M_\pi^2(2\Gamma_8 - \Gamma_5 + 2\Gamma_6 - \Gamma_4)\right]\frac{\mu^{2\varepsilon}}{(4\pi)^2 f^2}R. \tag{3.36}$$

For the pion mass to take a finite value this leads to the conditions

$$2\Gamma_8 - \Gamma_5 + 2\Gamma_6 - \Gamma_4 = -\frac{5}{36} \quad \text{and} \quad 2\Gamma_6 - \Gamma_4 = \frac{2}{72}, \tag{3.37}$$

which are consistent with $\Gamma_4 = \frac{1}{8}$, $\Gamma_5 = \frac{3}{8}$, $\Gamma_6 = \frac{11}{144}$ and $\Gamma_8 = \frac{5}{48}$ given in [GL85].

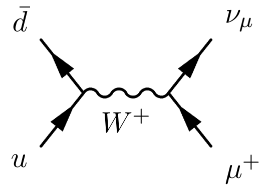
The renormalized pion mass at $\mathcal{O}(p^4)$ is then given by

$$\begin{aligned}M_{\pi,4}^2 = M_\pi^2\Bigg\{&1 + \frac{M_\pi^2}{32\pi^2 f^2}\ln\frac{M_\pi^2}{\mu^2} - \frac{M_{\eta_8}^2}{96\pi^2 f^2}\ln\frac{M_\eta^2}{\mu^2}\\ &+ \frac{8}{f^2}\Big((2M_K^2 + M_\pi^2)\left[2L_6^r(\mu) - L_4^r(\mu)\right] + M_\pi^2\left[2L_8^r(\mu) - L_5^r(\mu)\right]\Big)\Bigg\}.\end{aligned} \tag{3.38}$$

4. The Pion Decay Constant

4.1. Parametrization of Charged Pion Decay

Charged pions decay almost exclusively into a muon and anti-muon neutrino (or anti-muon and muon neutrino). On the level of the Standard Model this corresponds to the production of a W boson by the pion's constituent quarks and it's subsequent decay into the lepton-neutrino pair. In the case of a π^+ the leading order process is given by



and can in principle be calculated through the Standard Model Lagrangian in leading order as

$$i\mathcal{M}(\pi^+ \to \mu^+\nu_\mu) = \langle \mu^+\nu_\mu | \, T \exp\left(i \int \mathrm{d}^4x \; \mathcal{L}_{\mathrm{SM}}\right) |\pi^+\rangle \tag{4.1}$$

$$= -V_{ud}\frac{g^2}{16} \int \mathrm{d}^4x \int \mathrm{d}^4y \; \langle \mu^+\nu_\mu | \, W^\dagger_\mu(x) W_\nu(y) \bar{\nu}(x)\gamma^\mu(1-\gamma_5)\mu(x) \, |0\rangle \times$$

$$\times \langle \Omega | \, \bar{d}(y)\gamma^\nu(1-\gamma_5)u(y) \, |\pi^+\rangle \,, \tag{4.2}$$

where the matrix element has been pulled apart into the electroweak part that can be analytically determined and the hadronic part involving the non-perturbative QCD vacuum. Because of the parity of the pion only the axial vector component can contribute leaving the matrix element of the pion coupling to the axial vector quark current, that can be parametrized while defining the pion decay constant:

$$\langle \Omega | \, \bar{u}\gamma^\mu\gamma_5 d \, |\pi^+\rangle = \langle \Omega | \, A^\mu(0) \, |\pi^+\rangle \equiv i\sqrt{2} f_\pi p^\mu. \tag{4.3}$$

In the QCD Lagrangian used here the coupling to the W is introduced through the external field

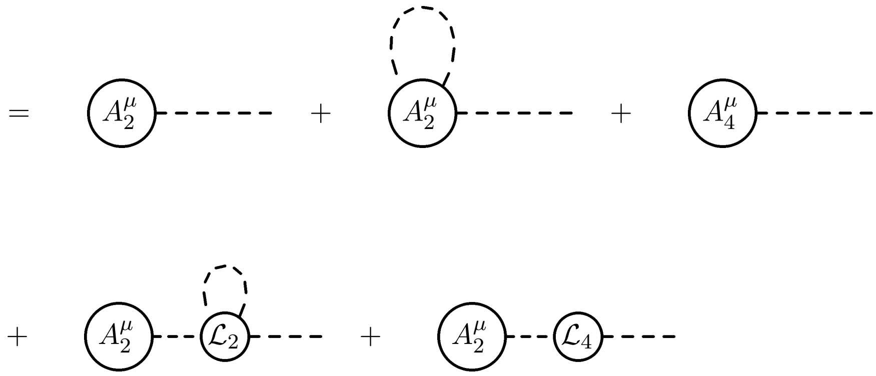
$$a_\mu = \frac{e}{2\sqrt{2}\sin\theta_W} \left[W^\dagger_\mu \begin{pmatrix} 0 & V_{ud} & V_{us} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + h.c. \right]. \tag{4.4}$$

Operators of Goldstone fields that correspond to the $u\bar{d}$-annihilation into a W^+ can thus be deduced from the effective Lagrangian through the functional derivative with respect to a^{12}_μ, which makes it clear that the relevant component of the matrix-valued axial vector current needed for the calculation of the pion decay is $[A^\mu]^{12}$.

4.2. The Pion Decay Constant at $\mathcal{O}(p^4)$

The matrix element for the pion decay at $\mathcal{O}(p^4)$ will include the tree-level contributions of $\mathcal{L}_2$ and $\mathcal{L}_4$ as well as 1-loop diagrams of $\mathcal{L}_2$. Since the divergences of the loops make renormalization necessary also corrections to the external pion field stemming from the pion self-energy need to be included. Thus the full amplitude will involve the following contributions:

$\langle 0| \left[A_2^\mu(\Phi)\right]^{12} \left|\pi^+(p)\right\rangle =$



Here the last two diagrams represent the contribution from the renormalization of the external pion field.

Of the axial vector current derived in chapter 2 only the terms in the first line of equation 2.12 can contribute to the pion decay in 4.3, since the contraction of a field and it's derivative vanish (second line) or there is no possibility of contraction for the incoming pion (third line). The remaining terms of the axial vector current are

$$\left[A_2^\mu(\Phi)\right]^{12} \supset -\sqrt{2}f \left[1 - \frac{1}{3f^2}\left(2\pi^0\pi^0 + 2\pi^+\pi^- + K^+K^- + K^0\bar{K}^0\right)\right]\partial^\mu\pi^+. \tag{4.5}$$

Thus the evaluation of equation 4.3 at $\mathcal{O}(p^4)$ involves matrix elements of the shape

$$\begin{aligned}\langle 0| \phi(x)\phi^*(x)\partial^\mu\pi^+(x) \left|\pi^+(p)\right\rangle &= \langle 0| \phi(x)\phi^*(x) |0\rangle \underbrace{\langle 0| \partial^\mu\pi^+(x) \left|\pi^+(p)\right\rangle}_{-ip^\mu e^{-ipx}} \\ &= -ip^\mu e^{-ipx}\; I_\phi,\end{aligned} \tag{4.6}$$

where the definition of the integral B.1 is used. In the isospin limit ($m_u = m_d = m$) the axial vector current stemming from $\mathcal{L}_2$ acts in the pion decay amplitude 4.3 as

$$\begin{aligned}\langle 0| \left[A_2^\mu\right]^{12} \left|\pi^+\right\rangle_{\text{bare}} &= -\sqrt{2}f \langle 0| \left[1 - \frac{1}{3f^2}\left(2\pi^0\pi^0 + 2\pi^+\pi^- + K^+K^- + K^0\bar{K}^0\right)\right]\partial^\mu\pi^+ \left|\pi^+\right\rangle \\ &= i\sqrt{2}f \left(1 - \frac{4}{3f^2}I_\pi - \frac{2}{3f^2}I_K\right)\, p^\mu.\end{aligned} \tag{4.7}$$

The axial vector current stemming from $\mathcal{L}_4$ differs from the tree-level $\mathcal{L}_2$ result only by factors (see eq. 2.22) so the full, bare amplitude reads

$$\begin{aligned}\langle 0| \left[A^\mu\right]^{12} \left|\pi^+\right\rangle_{\text{bare}} = i\sqrt{2}f \Bigg(&1 - \frac{4}{3f^2}I_\pi - \frac{2}{3f^2}I_K \\ &+2\,\frac{8M_K^2 + 4M_\pi^2}{f^2}\; L_4 + 2\,\frac{4M_\pi^2}{f^2}\; L_5\Bigg)\; p^\mu.\end{aligned} \tag{4.8}$$

The renormalized amplitude is calculated by multiplying the bare one with a factor of $\sqrt{Z}$ for the external pion field as elaborated in section 3.2. The necessary term for the pion decay is taken directly from 3.30:

$$\begin{aligned}\sqrt{Z} &= \left(1 - \left.\frac{\mathrm{d}\Sigma(p^2)}{\mathrm{d}p^2}\right|_{p^2=m^2}\right)^{-1/2} = 1 + \frac{1}{2}\left.\frac{\mathrm{d}\Sigma(p^2)}{\mathrm{d}p^2}\right|_{p^2=m^2} + \mathcal{O}\left(\left(\frac{\mathrm{d}\Sigma(p^2)}{\mathrm{d}p^2}\right)^2\right) \\ &= 1 + \frac{1}{3f^2}I_\pi + \frac{1}{6f^2}I_K - \frac{8M_K^2+4M_\pi^2}{f^2}\,L_4 - \frac{4M_\pi^2}{f^2}\,L_5. \end{aligned} \tag{4.9}$$

All terms differing from one in this expression are of $\mathcal{O}(p^2)$. Since the contributions to the axial vector current from loops and the ones from $\mathcal{L}_4$ are already suppressed at $\mathcal{O}(p^2)$ with respect to the tree-level $\mathcal{L}_2$ result, multiplying those terms with the non-trivial terms of $\sqrt{Z}$ gives results that are suppressed at $\mathcal{O}(p^4)$ and have to be neglected in this calculation. Thus the final, renormalized expression for the pion decay matrix element becomes

$$\begin{aligned}\langle 0|\,[A^\mu(\Phi)]^{12}\,|\pi^+(p)\rangle &= i\sqrt{2}fp^\mu\left[\sqrt{Z} - \frac{4}{3f^2}I_\pi - \frac{2}{3f^2}I_K \right.\\ &\qquad\left. +2\,\frac{8M_K^2+4M_\pi^2}{f^2}\,L_4 + 2\,\frac{4M_\pi^2}{f^2}\,L_5\right] \\ &= i\sqrt{2}fp^\mu\left[1 - \frac{I_\pi}{f^2} - \frac{I_K}{2f^2} + \frac{8M_K^2+4M_\pi^2}{f^2}\,L_4 + \frac{4M_\pi^2}{f^2}\,L_5\right] \\ &= i\sqrt{2}f_\pi p^\mu,\end{aligned} \tag{4.10}$$

where a comparison with the last line immediately gives the full expression for the pion decay constant at $\mathcal{O}(p^4)$.

$$\begin{aligned}\frac{f_\pi}{f} &= 1 - \frac{I_\pi}{f^2} - \frac{I_K}{2f^2} + \frac{8M_K^2+4M_\pi^2}{f^2}\,L_4 + \frac{4M_\pi^2}{f^2}\,L_5 \\ &= 1 - 2\mu_\pi - \mu_K - \frac{\mu^{2\varepsilon}}{32\pi^2 f^2}\left(2M_\pi^2 + M_K^2\right)R + \frac{8M_K^2+4M_\pi^2}{f^2}\,L_4 + \frac{4M_\pi^2}{f^2}\,L_5\end{aligned} \tag{4.11}$$

The divergence parametrized in R can now be absorbed in a redefinition of the coupling constants L_4 and L_5 to give the final and finite result.

$$\boxed{\frac{f_\pi}{f} = 1 - 2\mu_\pi - \mu_K + \frac{8M_K^2+4M_\pi^2}{f^2}\,L_4^r(\mu) + \frac{4M_\pi^2}{f^2}\,L_5^r(\mu)} \tag{4.12}$$

$$L_4 \equiv L_4^r(\mu) + \frac{1}{8}\frac{\mu^{2\varepsilon}}{32\pi^2}R \qquad\qquad L_5 \equiv L_5^r(\mu) + \frac{3}{8}\frac{\mu^{2\varepsilon}}{32\pi^2}R \tag{4.13}$$

$$\mu_\phi \equiv \frac{m_\phi^2}{32\pi^2 f^2}\ln\frac{m_\phi^2}{\mu^2} \tag{4.14}$$

4.3. Determination of $L_5^r(\mu)$

Analogously to the shown derivation the kaon decay constant at $\mathcal{O}(p^4)$ is [Pic98]

$$\frac{f_K}{f} = 1 - \frac{3}{4}\mu_\pi - \frac{3}{2}\mu_K - \frac{3}{4}\mu_{\eta_8} + \frac{8M_K^2+4M_\pi^2}{f^2}\,L_4^r(\mu) + \frac{4M_K^2}{f^2}\,L_5^r(\mu). \tag{4.15}$$

Having in mind that $f_\pi = f(1+\mathcal{O}(p^2))$ and using the expansion $1/(1-\varepsilon) = 1+\varepsilon+\mathcal{O}(\varepsilon^2)$ one can simplify the ratio of the two decay constants to be

$$\begin{aligned}\frac{f_K}{f_\pi} &= \frac{1+\Delta f_K + \mathcal{O}(p^4)}{1+\Delta f_\pi + \mathcal{O}(p^4)} = 1 + \Delta f_K - \Delta f_\pi + \mathcal{O}(p^4) \\ &= 1 + \frac{1}{4}\left(5\mu_\pi - 2\mu_K - 3\mu_{\eta_8}\right) + \frac{4}{f^2}\left(M_K^2 - M_\pi^2\right)\ L_5^r(\mu) + \mathcal{O}(p^4),\end{aligned} \tag{4.16}$$

which may be used to determine the value of the renormalized coupling $L_5^r(\mu)$ to be

$$L_5^r(\mu) = \frac{1}{4(M_K^2 - M_\pi^2)}\left[f^2\left(\frac{f_K}{f_\pi} - 1\right) - \frac{1}{128\pi^2}\left(5M_\pi^2 \ln\frac{M_\pi^2}{\mu^2} - 2M_K^2 \ln\frac{M_K^2}{\mu^2} - 3M_{\eta_8}^2 \ln\frac{M_{\eta_8}^2}{\mu^2}\right)\right]. \tag{4.17}$$

Already in the calculation of the currents the terms in the final results involving the quark masses were substituted by the expressions for the meson masses found at $\mathcal{O}(p^2)$ since the error produced thereof would appear only at $\mathcal{O}(p^6)$. Applying the same concept here lets one exchange f for f_π.

The error being made by identifying the measured ratio f_K/f_π with the $\mathcal{O}(p^4)$ result can be estimated by introducing a theoretical uncertainty of

$$\Delta_{\text{th}} = \frac{p^4}{\Lambda_\chi^4}, \tag{4.18}$$

where $\Lambda_\chi \sim 4\pi f_\pi \sim 1.2\,\text{GeV}$ is the typical breakdown scale of ChPT and $p^2 = M_K^2$ may be fixed since the kaon mass is the highest energy scale in the contributing decays.

Using the experimental values [O+14]

$$\frac{f_K}{f_\pi} = 1.198 \pm 0.005, \tag{4.19}$$

$$f_\pi = 92.21 \pm 0.14\,\text{MeV} \tag{4.20}$$

and the pseudoscalar masses as given by the Particle Data Group, the value of the low-energy coupling constant L_5^r can be determined. At the mass of the ρ meson its value is given by

$$\boxed{L_5^r(M_\rho) = (1.2 \pm 0.3)\cdot 10^{-3}.} \tag{4.21}$$

The uncertainty of L_5^r is mainly due to the neglected contributions of the next orders in the chiral expansion. Ignoring the experimental uncertainties of the pion decay constant and f_K/f_π as well as the error in the meson masses due to isospin breaking effects, the uncertainty of L_5^r changes only insignificantly:

$$\Delta L_5^r = 0.277\cdot 10^{-3} \longrightarrow 0.272\cdot 10^{-3}. \tag{4.22}$$

Although the low-energy constants of ChPT are most commonly determined at the ρ mass this is pure convention. The scale-dependence of $L_5^r(\mu)$, including all mentioned error sources, is shown in figure (4.1) for energies up to Λ_χ.

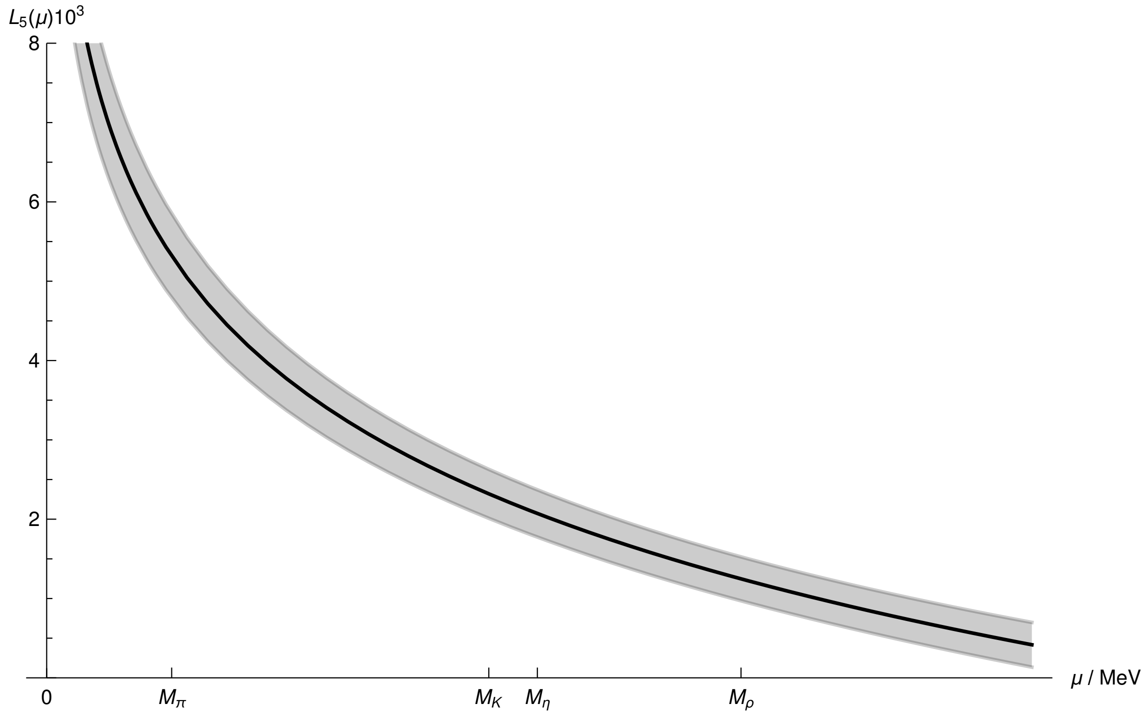


Figure 4.1.: The scale-dependence of the renormalized low-energy constant $L_5^r(\mu)$ with the uncertainty given by the shaded region.

5. Electromagnetic Form Factor

The coupling of the charged pions to the electromagnetic current, and as such to photons, is parametrized in the electromagnetic form factor $F(q^2)$ as

$$\left\langle \pi^+(p') \right| J^\mu(0) \left| \pi^+(p) \right\rangle = (p+p')^\mu \cdot F(q^2) \qquad , \text{ where } q = p' - p. \tag{5.1}$$

Due to Lorentz and gauge invariance other contributions are not possible and the form factor at zero momentum transfer is fixed at $F(0) = 1$ [DGH92].

5.1. Electromagnetic Current

The photon coupling is introduced in the Lagrangian through the external vector current $v_\mu = e\mathcal{A}_\mu Q$. In the QCD Lagrangian this creates the interaction term

$$\mathcal{L}_{\text{QCD}} \supset e\mathcal{A}_\mu \bar{q}\gamma^\mu Q q = e\mathcal{A}_\mu \left(\frac{2}{3}\bar{u}\gamma^\mu u - \frac{1}{3}\bar{d}\gamma^\mu d - \frac{1}{3}\bar{s}\gamma^\mu s \right), \tag{5.2}$$

allowing to read off the classical contribution to the electromagnetic current in terms of the quark fields as $J^\mu = (2\bar{u}\gamma^\mu u - \bar{d}\gamma^\mu d - \bar{s}\gamma^\mu s)/3$, which can also be calculated by taking the trace of the quark charge matrix and the vector-current. In this sense the electromagnetic current as given by the meson fields will take the form

$$\mathcal{L}_{\text{ChPT}} \supset \langle v_\mu V^\mu \rangle = e\mathcal{A}_\mu \langle Q V^\mu \rangle = e\mathcal{A}_\mu J^\mu \tag{5.3}$$

By determining this trace with all the contributions to the vector current calculated in section 2.3 the electromagnetic current stemming from $\mathcal{L}_2$ reads

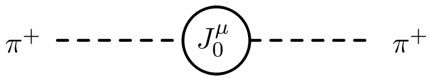
$$J_2^\mu = \langle Q V_2^\mu \rangle = \left[1 - \frac{1}{6f^2} \left(2\pi^0\pi^0 + 4\pi^+\pi^- + K^0\bar{K^0} + 4K^+K^- \right) \right] i \left(\pi^- \partial^\mu \pi^+ - \pi^+ \partial^\mu \pi^- \right), \tag{5.4}$$

where the leading order contribution may be defined as $J_0^\mu \equiv i\,(\pi^-\partial^\mu\pi^+ - \pi^+\partial^\mu\pi^-)$, and the contribution from $\mathcal{L}_4$ is given by

$$\begin{aligned} J_4^\mu &= \left\langle Q \left(V^\mu_{4,L_4} + V^\mu_{4,L_5} + V^\mu_{4,L_9} \right) \right\rangle \\ &= \frac{1}{f^2} \left[8L_4 \left(2M_K^2 + M_\pi^2 \right) + 8L_5 M_\pi^2 \right] J_0^\mu \\ &\quad - i\frac{4L_9}{f^2} \left[\partial_\nu \pi^- \partial^\nu \partial^\mu \pi^+ - \partial_\nu \pi^+ \partial^\nu \partial^\mu \pi^- + \partial^2 \pi^- \partial^\mu \pi^+ - \partial^2 \pi^+ \partial^\mu \pi^- \right]. \end{aligned} \tag{5.5}$$

5.2. Calculation of the Form Factor

The leading order contribution to the electromagnetic form factor of the charged pion is given by the matrix element



$$= i \left\langle \pi^+(p') \right| \pi^- \partial^\mu \pi^+ - \pi^+ \partial^\mu \pi^- \left| \pi^+(p) \right\rangle$$

$$= (p + p')^\mu, \tag{5.6}$$

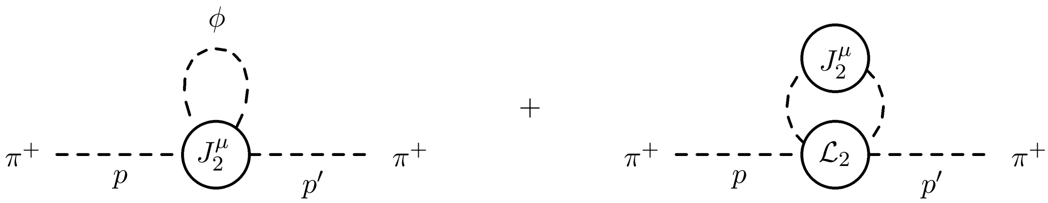
showing that the form factor will be of the form

$$F(q^2) = 1 + \Delta F(q^2). \tag{5.7}$$

This means that at very low energies the interaction of a charged pion with a photon can be reasonably well simplified by treating the pion as a point charge.

5.2.1. Loop Contribution from $\mathcal{L}_2$

For the form factor calculation at $\mathcal{O}(p^4)$ two different types of loop diagrams will be relevant. On the one hand there are the loops produced by calculating the current matrix element with the NLO expression for the electromagnetic current and on the other hand there are also diagrams that have a four meson interaction vertex producing a charged meson loop, which then interacts at leading order with the photon:



Inserting the electromagnetic current from $\mathcal{L}_2$ (5.4) at $\mathcal{O}(p^4)$ into expression 5.1 one needs to contract matrix elements of the form

$$\left\langle \pi^+(p') \right| \phi\phi^* (\pi^- \partial^\mu \pi^+ - \pi^+ \partial^\mu \pi^-) \left| \pi^+(p) \right\rangle . \tag{5.8}$$

If ϕ does not stand for the charged pion field the solution will be the loop integral I_ϕ (see appendix B) multiplied to the LO solution. In the case of the charged pion field there are two possibilities to contract the expression leading to an additional factor of 2. Summing up and going into the isospin limit again the contribution to the form factor is given by

$$\Delta F_{2,i}(q^2) = -\frac{5}{6f^2} \left[2I_\pi + I_K \right] . \tag{5.9}$$

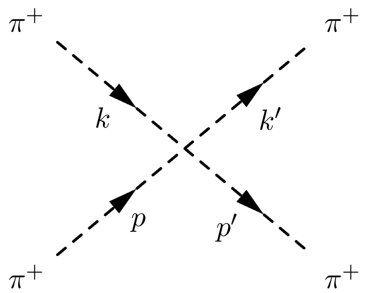
To identify the contribution stemming from the second type of loop diagram, first the relevant four-meson vertices need to be identified. Only terms of the Lagrangian that have both a π^+ and π^- field can contribute, as well as another charged particle-antiparticle pair to allow for the photon coupling. The relevant terms for the interaction of four charged pions are given by

$$\begin{aligned}\mathcal{L}_2 \supset \frac{1}{6f^2}\Big[&M_\pi^2\pi^+\pi^-\pi^+\pi^- + \pi^-\pi^-\partial_\mu\pi^+\partial^\mu\pi^+ + \pi^+\pi^+\partial_\mu\pi^-\partial^\mu\pi^- - 2\pi^+\pi^-\partial_\mu\pi^+\partial^\mu\pi^- \\ &+ \left(M_\pi^2 + M_K^2\right)K^+K^-\pi^+\pi^- - K^+K^-\partial_\mu\pi^+\partial^\mu\pi^- - \pi^+\pi^-\partial_\mu K^+\partial^\mu K^- \\ &+ 2\pi^-K^-\partial_\mu\pi^+\partial^\mu K^+ + 2\pi^+K^+\partial_\mu\pi^-\partial^\mu K^- - \pi^-K^+\partial_\mu\pi^+\partial^\mu K^- - \pi^+K^-\partial_\mu\pi^-\partial^\mu K^+\Big]\,.\end{aligned} \tag{5.10}$$

Keeping in mind that there are two possibilities each to contract the π^+ and π^- fields, which yealds a factor of four in each term, the matrix element for the interaction of four charged pions can be directly calculated.

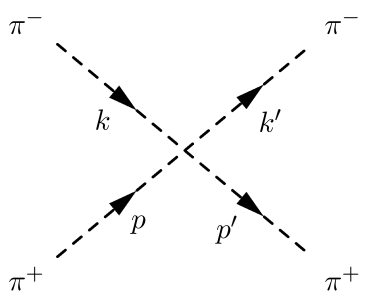
$$\begin{aligned}\int \mathrm{d}^4x\ \left\langle\pi^+(p')\pi^+(k')\middle| i\mathcal{L}_2\middle|\pi^+(p)\pi^+(k)\right\rangle &= \frac{i}{6f^2}\int \mathrm{d}^4x\ e^{i(p+k-p'-k')x}\ \times \\ &\quad\times 2\left[2M_\pi^2 - 2pk - 2p'k' - pk' - pp' - kk' - kp'\right] \\ &= (2\pi)^4\delta(p+k-p'-k')\ \frac{i}{3f^2}\left(t+u-2s+2M_\pi^2\right)\end{aligned} \tag{5.11}$$

Reading off the Feynman rule gives the result of



$$\frac{i}{3f^2}\left(t+u-2s+2M_\pi^2\right), \tag{5.12}$$

where the arrows denote the direction of the momenta flowing and $s=(p+k)^2$, $t=(p-p')^2$ and $u=(p-k')^2$ are the usual Mandelstam variables. Crossing symmetry immediately gives the Feynman rule for the $\pi^+\pi^- \to \pi^+\pi^-$ vertex to be



$$\frac{i}{3f^2}\left(t+s-2u+2M_\pi^2\right). \tag{5.13}$$

The Feynman rules for the corresponding diagrams involving kaons are calculated in the same way. With the momenta defined equivalently to the pion case the result reads

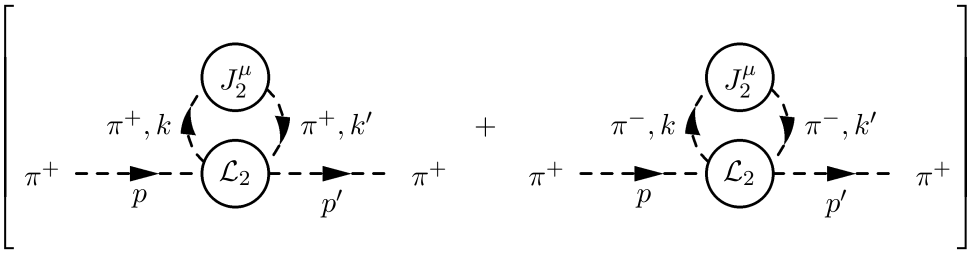
$$i\mathcal{M}\left(\pi^+K^+ \longrightarrow \pi^+K^+\right) = \frac{i}{6f^2}\left(t+u-2s+M_\pi^2+M_K^2\right) \qquad \text{and} \tag{5.14}$$

$$i\mathcal{M}\left(\pi^+K^- \longrightarrow \pi^+K^-\right) = \frac{i}{6f^2}\left(t+s-2u+M_\pi^2+M_K^2\right). \tag{5.15}$$

Noticing that the coupling of the leading order electromagnetic current to positively and negatively charged pions has exactly the opposite sign, one can simplify the calculation slightly by summing both over the π^+ as well as over the π^- loop. These are in principal equivalent, since, due to crossing symmetry, an amplitude depending on a incoming particle with momentum k is always equivalent to the same amplitude with the respective antiparticle going out with momentum $-k$ [PS95].

The full pion loop contribution $I_{\pi\pi}$ is thus equal to the sum of the π^+ and π^- loops divided by two:

$$i\mathcal{M}_{\text{loop}} = \frac{1}{2}$$



$$
\begin{aligned}
&= \frac{1}{2} \int \frac{\mathrm{d}^4 k}{(2\pi)^4} \frac{i}{k^2 - M_\pi^2 + i\varepsilon} \frac{i}{(k-q)^2 - M_\pi^2 + i\varepsilon} \frac{i}{3f^2} (k+k')^\mu \times \\
&\qquad \times \left[(t + u - 2s + 2M_\pi^2) - (t + s - 2u + 2M_\pi^2) \right] \\
&= -\frac{i}{6f^2} \int \frac{\mathrm{d}^4 k}{(2\pi)^4} \frac{(k+k')^\mu}{[k^2 - M_\pi^2 + i\varepsilon]\,[(k-q)^2 - M_\pi^2 + i\varepsilon]} \; 3(u-s) && (5.16) \\
&= i \frac{(p+p')_\nu}{f^2} \int \frac{\mathrm{d}^4 k}{(2\pi)^4} \frac{(2k-q)^\mu k^\nu}{[k^2 - M_\pi^2 + i\varepsilon]\,[(k-q)^2 - M_\pi^2 + i\varepsilon]} && (5.17)
\end{aligned}
$$

The divergences appearing in this integral need to be brought into a form that will allow the summation and cancellation of divergences appearing in the different $\mathcal{O}(p^4)$ contributions to the amplitude of the electromagnetic current. The rather lengthy calculation can be found in the appendix B and finally gives

$$i\mathcal{M}_{\text{loop}} = -(p+p')^\mu \left[\frac{1}{6f^2} \frac{q^2}{M_\pi^2} I_\pi - \frac{1}{f^2} I_\pi + \frac{1}{6f^2} \frac{q^2}{(4\pi)^2} A_\pi(q^2) \right], \tag{5.18}$$

where the function

$$A_\pi(q^2) \equiv \frac{8M_\pi^2}{q^2} - \frac{5}{3} + \sigma_\pi^3 \ln \frac{\sigma_\pi + 1}{\sigma_\pi - 1} \tag{5.19}$$

has been defined and $\sigma_\pi = \sqrt{1 - \frac{4M_\pi^2}{q^2}}$ is the pion phase space factor. Adding also the contribution from the kaon loop, which is the same expression divided by two and with the proper substitution of masses, one can read off the correction to the form factor from the second type of loop diagrams as

$$\Delta F_{2,ii}(q^2) = -\frac{q^2}{6f^2} \left(\frac{I_\pi}{M_\pi^2} + \frac{I_K}{2M_K^2} \right) + \frac{1}{f^2} \left(I_\pi + \frac{1}{2} I_K \right) - \frac{q^2}{96\pi^2 f^2} \left(A_\pi(q^2) + \frac{1}{2} A_K(q^2) \right). \tag{5.20}$$

5.2.2. Contributions from $\mathcal{L}_4$

The contributions to the form factor stemming from $\mathcal{L}_4$ are found by evaluating the matrix element of the electromagnetic current found as given in 5.5

$$\left\langle \pi^+(p') \right| J_4^\mu(0) \left| \pi^+(p) \right\rangle . \tag{5.21}$$

The first part of the current is directly proportional to the LO current and therefore the correction of the form factor is simply

$$\Delta F_{4,i}(q^2) = \frac{1}{f^2}\left[8L_4\left(2M_K^2 + M_\pi^2\right) + 8L_5 M_\pi^2\right] . \tag{5.22}$$

For the second part the matrix element is given by

$$\begin{aligned} -i\frac{4L_9}{f^2}&\left\langle \pi^+(p') \right| \partial_\nu\pi^-\partial^\nu\partial^\mu\pi^+ - \partial_\nu\pi^+\partial^\nu\partial^\mu\pi^- + \partial^2\pi^-\partial^\mu\pi^+ - \partial^2\pi^+\partial^\mu\pi^- \left|\pi^+(p)\right\rangle \\ &= -i\frac{4L_9}{f^2}\left[-ipp'(p+p')^\mu + ip'^2p^\mu + ip^2p'^\mu\right] \\ &= \frac{2}{f^2}L_9\left[q^2(p+p')^\mu - \frac{1}{2}(p'^2-p^2)(p'-p)^\mu\right] . \end{aligned} \tag{5.23}$$

The contribution to the form factor therefore becomes

$$\Delta F_{4,ii}(q^2) = \frac{2L_9}{f^2}q^2, \tag{5.24}$$

as the incoming and outgoing pion are on-shell.

5.2.3. Renormalization of the Amplitude

As already seen for the case of the pion decay constant, the full renormalised amplitude is calculated by multiplying the bare one with a factor of

$$\sqrt{Z}^2 = 1 + \left.\frac{\mathrm{d}\Sigma(p^2)}{\mathrm{d}p^2}\right|_{p^2=m^2} + \mathcal{O}\left(\left(\frac{\mathrm{d}\Sigma(p^2)}{\mathrm{d}p^2}\right)^2\right) \tag{5.25}$$

$$= 1 + \frac{2}{3f^2}I(M_\pi^2) + \frac{1}{3f^2}I(M_K^2) - 2\frac{8M_K^2 + 4M_\pi^2}{f^2}\,L_4 - \frac{8M_\pi^2}{f^2}\,L_5, \tag{5.26}$$

which leads to a correction of the form factor of

$$\Delta F_Z(q^2) = \frac{2}{3f^2}I_\pi + \frac{1}{3f^2}I_K - \Delta F_{4,i}(q^2), \tag{5.27}$$

where the last two terms have already been identified with the $\mathcal{L}_4$ contribution found in equation (5.22).

The full correction to the form factor is found by adding up the changes caused through the $\mathcal{O}(\Phi^4)$ terms of the electromagnetic current ($\Delta F_{2,i}$, 5.9), the $\mathcal{L}_2$ loop diagram containing the $\mathcal{O}(p^2)$ vertices ($\Delta F_{2,ii}$, 5.20), the tree-level $\mathcal{L}_4$ contributions ($\Delta F_{4,i} + \Delta F_{4,ii}$, 5.22 and 5.24) and the correction from the wave-function renormalization (ΔF_Z). The loop integrals I_π and I_K that are not proportional to q^2 as well as the terms containing L_4 or L_5 cancel directly, such that the remaining correction reads

$$\Delta F = \frac{2L_9}{f^2}q^2 - \frac{q^2}{6f^2}\left(\frac{I_\pi}{M_\pi^2} + \frac{I_K}{2M_K^2}\right) - \frac{q^2}{96\pi^2 f^2}\left(A_\pi(q^2) + \frac{1}{2}A_K(q^2)\right) . \tag{5.28}$$

Inserting the solutions for the integrals one can determine the necessary redefinition of L_9 to make the amplitude finite. Thus the final expression is found to be

$$= 2L_9 \frac{q^2}{f^2} - \frac{\mu^{2\varepsilon} R}{2 \cdot 32\pi^2} \frac{q^2}{f^2} - \frac{q^2}{96\pi^2 f^2} \left(\ln \frac{M_\pi^2}{\mu^2} + \frac{1}{2} \ln \frac{M_K^2}{\mu^2} + A_\pi(q^2) + \frac{1}{2} A_K(q^2) \right),$$

leading to

$$\boxed{F(q^2) = 1 + \frac{2L_9^r(\mu)}{f^2} q^2 - \frac{q^2}{96\pi^2 f^2} \left(\ln \frac{M_\pi^2}{\mu^2} + \frac{1}{2} \ln \frac{M_K^2}{\mu^2} + A_\pi(q^2) + \frac{1}{2} A_K(q^2) \right),} \tag{5.29}$$

where the renormalised coupling is defined as

$$L_9 = L_9^r(\mu) + \frac{1}{4} \frac{\mu^{2\varepsilon}}{32\pi^2} R. \tag{5.30}$$

5.3. The Electromagnetic Radius

The electromagnetic form factor $F(q^2)$ for on-shell pions is completely determined by the square of the transfered three-momentum and thus equal to the Fourier transform of the electric charge density $\rho(\mathbf{x})$ and can therefore be calculated for certain exemplary distributions. One sees that in the limit of a point-like particle the form factor is constant and (if properly normalised) equal to one. More extended charge distributions will lead to a decrease with $\mathbf{q}^2$. Since for wavelengths much smaller than the extension of the scattering potential the phases of the contributions will vary rapidly and cancel the form factor must be assumed to go to zero for very big momentum transfers, justifying a Talylor expansion around $q^2 = 0$ [PRSZ95].

Further assuming ρ to be spherically symmetric the form factor is given by

$$\begin{aligned} F(q^2) &= \int \mathrm{d}^3x \; e^{i\mathbf{qx}} \rho(\mathbf{x}) \\ &= 4\pi \int \mathrm{d}|\mathbf{x}| \; \rho(|\mathbf{x}|) \; \underbrace{\frac{\sin |\mathbf{q}||\mathbf{x}|}{|\mathbf{q}||\mathbf{x}|}}_{1 - \frac{q^2}{6}|\mathbf{x}|^2 + \mathcal{O}(q^4)} \; \mathbf{x}^2 \\ &= 4\pi \int \mathrm{d}|\mathbf{x}| \; \rho(|\mathbf{x}|)\mathbf{x}^2 - \frac{q^2}{6} 4\pi \int \mathrm{d}|\mathbf{x}| \; \rho(|\mathbf{x}|)\mathbf{x}^4 + \ldots, \end{aligned}$$

which can be written as

$$= 1 + \frac{q^2}{6} \langle r^2 \rangle + \ldots \tag{5.31}$$

when choosing the proper normalisation and defining the mean square charge radius $\langle r^2 \rangle$. In order to identify the charge radius the formula for the form factor 5.29, specifically the term containing the logarithm of the phase space factor, needs to be expanded in terms of the square of the momentum transfer q^2. The relevant expansion reads

$$\sigma_\pi^3 \ln \frac{\sigma_\pi + 1}{\sigma_\pi - 1} = -\frac{8M_\pi^2}{q^2} + \frac{8}{3} - \frac{q^2}{10M_\pi^2} + \mathcal{O}(q^4), \tag{5.32}$$

leading to the expression

$$A_\pi(q^2) = 1 - \frac{q^2}{10M_\pi^2} + \mathcal{O}(q^4). \tag{5.33}$$

Inserting this in the form factor formula and neglecting all terms of higher order than q^2 one finds

$$F(q^2) = 1 + \frac{1}{6}\left[\frac{12L_9^r(\mu)}{f^2} - \frac{1}{32\pi^2 f^2}\left(2\ln\frac{M_\pi^2}{\mu^2} + \ln\frac{M_K^2}{\mu^2} + 3\right)\right]q^2 + \mathcal{O}(q^4), \tag{5.34}$$

which directly identifies the pion electromagnetic charge radius at $\mathcal{O}(p^4)$ of ChPT as

$$\boxed{\langle r^2\rangle = \frac{12L_9^r(\mu)}{f^2} - \frac{1}{32\pi^2 f^2}\left(2\ln\frac{M_\pi^2}{\mu^2} + \ln\frac{M_K^2}{\mu^2} + 3\right).} \tag{5.35}$$

Mainly through pion scattering from electrons the radius has been experimentally determined as [O+14]

$$\sqrt{\langle r^2\rangle}_{\text{exp}} = (0.672 \pm 0.008)\,\text{fm}, \tag{5.36}$$

leading to the value for the low-energy coupling of

$$\boxed{L_9^r(M_\rho) = (6.9 \pm 0.6)\cdot 10^{-3},} \tag{5.37}$$

where the error in the charge radius due to the truncation of the chiral expansion has been estimated as

$$\Delta_{\text{th}}\langle r^2\rangle = \frac{p^4}{\Lambda_\chi^4}\,\text{fm}^2. \tag{5.38}$$

The relevant energy scales in the pion electron scattering used to determine the electromagnetic form factor are the pion mass and the momentum transfer q^2. In [A+86], one of the more precise sources for the charge radius cited by the PDG, the form factor is measured for q^2 up to $0.26\,\text{GeV}^2 \approx M_K^2$, thus justifying to set $p^2 = M_K^2$ for the theoretical error of identifying the measured charge radius with the $\mathcal{O}(p^4)$ result.

As already seen for L_5^r, the uncertainty is almost entirely due to the neglected $\mathcal{O}(p^6)$ contributions to the form factor. Ignoring isospin breaking and the experimental uncertainty of the pion decay constant ΔL_9^r changes only in the third significant figure.

The value of $L_5^r(\mu)$ for energies up to Λ_χ is shown in figure (5.1).

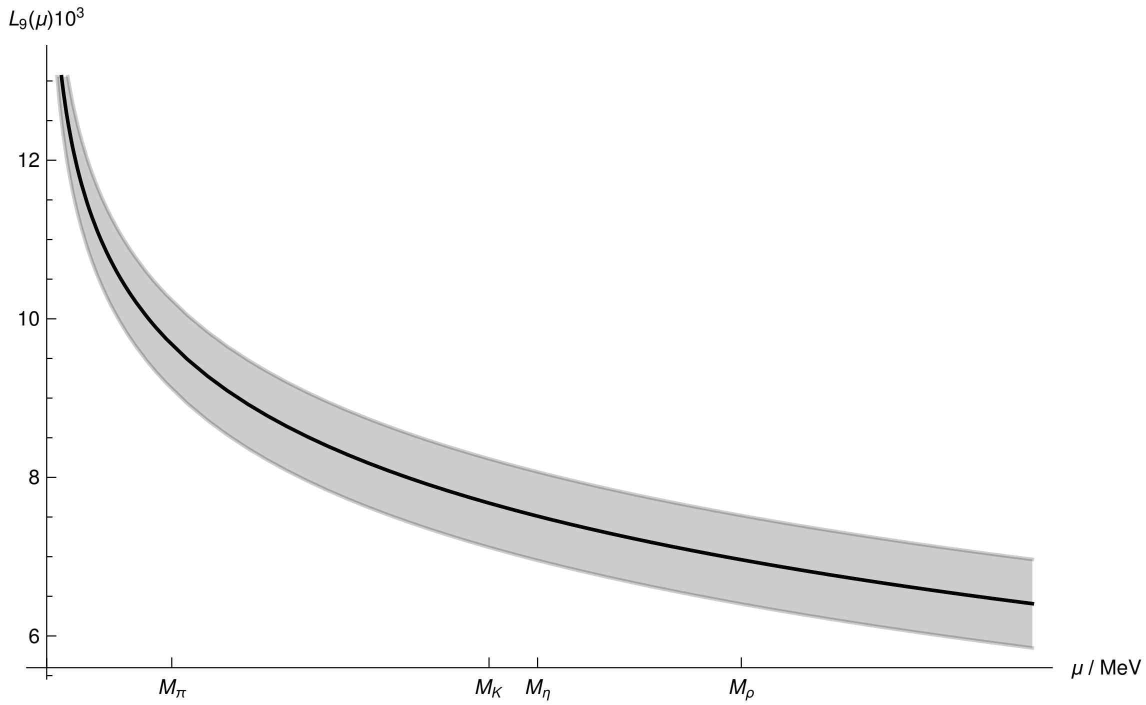


Figure 5.1.: The scale-dependence of the renormalized low-energy constant $L_9^r(\mu)$ with the uncertainty given by the shaded region.

6. Conclusion

The strong interaction is described in the Standard Model through quantum chromodynamics (QCD), which is a gauge theory in terms of quarks and gluons. Although its predictions in the high-energy regime are all consistent with experimental data, the dynamics in the low-energy regime cannot be calculated analytically due to its running coupling constant, which increases dramatically when lowering the energy scale. In the limit of massless quarks the QCD Lagrangian is symmetric under the chiral group, which consists of separate $SU(3)$ transformations for the left- and the right-handed quarks. This symmetry is spontaneously broken leading to eight Goldstone bosons, which can be identified with the pseudoscalar meson octet.

In terms of these mesons chiral perturbation theory (ChPT) is constructed as a low-energy effective field theory for the strong interaction. The explicit chiral symmetry breaking through quark masses and charges, as well as the coupling to the non-hadronic sector of the Standard Model, is implemented through the introduction of external classical fields into both the QCD and the ChPT Lagrangians.

Using the path integral formalism, operators involving quarks can be related to those containing meson fields. This is being done through functional derivatives of the generating functional with respect to the external fields. In this way both the axial vector current and the vector current have been determined up to an order in the meson fields sufficient for the $\mathcal{O}(p^4)$ calculations performed afterwards. In the case of the axial vector current, the contribution from $\mathcal{L}_2$, the leading-order term of the effective Lagrangian, has been calculated up to cubic order in the meson fields, as needed to calculate loop corrections to the pion decay constant later on. The contribution stemming from the next-to-leading order term $\mathcal{L}_4$ was only needed in tree-level amplitudes and has therefore only been determined up to linear order in the meson fields.

As a further preliminary step, the pion self-energy has been calculated including loop contributions from $\mathcal{L}_2$ and tree-level contributions stemming from $\mathcal{L}_4$. From this result the field-strength renormalization Z has been determined and its necessity for the renormalization of physical amplitudes explained. As a byproduct the $\mathcal{O}(p^4)$ expression for the pion mass has been found.

It was shown that the amplitude for the decay of a charged pion into a lepton-antineutrino pair can be factorized into a non-hadronic part and the hadronic part, where the latter involves the axial vector current and the decaying pion. This hadronic part cannot be calculated analytically within QCD and is parametrized in terms of the pion decay constant. In ChPT though, the amplitude involves the axial vector current in terms of pseudoscalar meson fields, as found before. Using this current, the amplitude has been calculated which involved solving loop diagrams and the inclusion of the field-strength renormalization. The appearing divergences have been absorbed in a redefinition of the low-energy constants L_4 and L_5 which fixed the infinite parts of these constants. Thus the next-to-leading order result for the pion decay constant was found. By taking the ratio of the expressions for the kaon and pion decay constants, the renormalized low-energy constant L_5^r was determined to be $L_5^r(M_\rho) = (1.2 \pm 0.3) \cdot 10^{-3}$.

Finally, the electromagnetic form factor of the pion was calculated using the vector current determined before. The part of the vector current stemming from $\mathcal{L}_2$ led to two different types of loop diagrams: On the one hand it involved terms quadratic in the fields and on the other hand its leading order contribution had to be combined with leading order interaction vertices. Through the renormalization of this amplitude the infinite part of L_9 has been fixed and the renormalized L_9^r has been determined from the electromagnetic charge radius to be $L_9^r(M_\rho) = (6.9 \pm 0.6) \cdot 10^{-3}$.

A. Expansions in Terms of the Meson Fields

A.1. Expansion of U

In the calculations an expansion of the unitary matrix U and its derivative in terms of the meson fields was needed, which are given here:

$$U(\Phi) = \exp\left\{ i\frac{\sqrt{2}}{f}\Phi \right\} = \mathbf{1} + i\frac{\sqrt{2}}{f}\Phi - \frac{1}{f^2}\Phi^2 - i\frac{\sqrt{2}}{3f^3}\Phi^3 + \frac{1}{6f^4}\Phi^4 + \mathcal{O}\left(\frac{\Phi}{f}\right)^5 \tag{A.1}$$

$$U^\dagger(\Phi) = \exp\left\{ -i\frac{\sqrt{2}}{f}\Phi \right\} = \mathbf{1} - i\frac{\sqrt{2}}{f}\Phi - \frac{1}{f^2}\Phi^2 + i\frac{\sqrt{2}}{3f^3}\Phi^3 + \frac{1}{6f^4}\Phi^4 - \mathcal{O}\left(\frac{\Phi}{f}\right)^5$$

$$\begin{aligned}\partial_\mu U(\Phi) = {} & i\frac{\sqrt{2}}{f}\partial_\mu\Phi - \frac{1}{f^2}\left[\Phi\ \partial_\mu\Phi + \partial_\mu\Phi\ \Phi\right] - i\frac{\sqrt{2}}{3f^3}\left[\partial_\mu\Phi\ \Phi^2 + \Phi\ \partial_\mu\Phi\ \Phi + \Phi^2\ \partial_\mu\Phi\right] \\ & + \frac{1}{6f^4}\left[\partial_\mu\Phi\ \Phi^3 + \Phi\ \partial_\mu\Phi\ \Phi^2 + \Phi^2\ \partial_\mu\Phi\ \Phi + \partial_\mu\Phi\ \Phi^3\right] + \mathcal{O}\left(\frac{\Phi}{f}\right)^5\end{aligned}$$

$$\begin{aligned}\partial_\mu U^\dagger(\Phi) = {} & -i\frac{\sqrt{2}}{f}\partial_\mu\Phi - \frac{1}{f^2}\left[\Phi\ \partial_\mu\Phi + \partial_\mu\Phi\ \Phi\right] + i\frac{\sqrt{2}}{3f^3}\left[\partial_\mu\Phi\ \Phi^2 + \Phi\ \partial_\mu\Phi\ \Phi + \Phi^2\ \partial_\mu\Phi\right] \\ & + \frac{1}{6f^4}\left[\partial_\mu\Phi\ \Phi^3 + \Phi\ \partial_\mu\Phi\ \Phi^2 + \Phi^2\ \partial_\mu\Phi\ \Phi + \partial_\mu\Phi\ \Phi^3\right] - \mathcal{O}\left(\frac{\Phi}{f}\right)^5\end{aligned}$$

A.2. Expansion of $\mathcal{L}_2$

The interaction terms of $\mathcal{L}_2$ involve a big quantity of terms, of which only few were needed for the calculations of the thesis. The mass-independent interaction term depends on the following trace, as seen in (1.22).

$$\big\langle (\Phi \overset{\leftrightarrow}{\partial_\mu} \Phi)(\Phi \overset{\leftrightarrow}{\partial^\mu} \Phi)\big\rangle = \tag{A.2}$$

$$\begin{array}{llll}
+\,2\ \partial_\mu\pi^+\partial^\mu\pi^+\pi^-\pi^- & +\,4\ \partial_\mu\pi^+\partial^\mu\pi^0\pi^-\pi^0 & -\,4\ \partial_\mu\pi^-\partial^\mu\pi^+\pi^0\pi^0 & -\,4\ \partial_\mu\pi^0\partial^\mu\pi^0\pi^-\pi^+ \\
+\,2\ \partial_\mu\pi^-\partial^\mu\pi^-\pi^+\pi^+ & +\,4\ \partial_\mu\pi^-\partial^\mu\pi^0\pi^+\pi^0 & -\,4\ \partial_\mu\pi^-\partial^\mu\pi^+\pi^-\pi^+ &
\end{array}$$

$$\begin{array}{llll}
-\,2\ \partial_\mu\pi^-\partial^\mu\pi^+K^-K^+ & -\,2\ \partial_\mu K^-\partial^\mu\pi^+K^+\pi^- & -\,\partial_\mu\pi^0\partial^\mu\pi^0K^-K^+ & -\,3\sqrt{2}\ \partial_\mu\bar{K}^0\partial^\mu\pi^-K^+\pi^0 \\
-\,2\ \partial_\mu\pi^-\partial^\mu\pi^+K^0\bar{K}^0 & +\,4\ \partial_\mu\bar{K}^0\partial^\mu\pi^+K^0\pi^- & -\,\partial_\mu\pi^0\partial^\mu\pi^0K^0\bar{K}^0 & -\,3\sqrt{2}\ \partial_\mu K^+\partial^\mu\pi^0\bar{K}^0\pi^- \\
-\,2\ \partial_\mu K^0\partial^\mu\bar{K}^0\pi^-\pi^+ & +\,4\ \partial_\mu K^0\partial^\mu\pi^-\bar{K}^0\pi^+ & +\,\partial_\mu K^-\partial^\mu\pi^0K^+\pi^0 & -\,3\sqrt{2}\ \partial_\mu K^-\partial^\mu\pi^0K^0\pi^+ \\
-\,2\ \partial_\mu K^-\partial^\mu K^+\pi^-\pi^+ & +\,4\ \partial_\mu K^-\partial^\mu\pi^-K^+\pi^+ & +\,\partial_\mu\bar{K}^0\partial^\mu\pi^0K^0\pi^0 & +\,3\sqrt{2}\ \partial_\mu K^0\partial^\mu\pi^0K^-\pi^+ \\
-\,2\ \partial_\mu\bar{K}^0\partial^\mu\pi^-K^0\pi^+ & +\,4\ \partial K^+\partial\pi^+K^-\pi^- & +\,\partial_\mu K^+\partial^\mu\pi^0K^-\pi^0 & +\,3\sqrt{2}\ \partial_\mu\bar{K}^0\partial^\mu\pi^0K^+\pi^- \\
-\,2\ \partial_\mu K^0\partial^\mu\pi^+\bar{K}^0\pi^- & -\,\partial_\mu K^-\partial^\mu K^+\pi^0\pi^0 & +\,\partial_\mu K^0\partial^\mu\pi^0\bar{K}^0\pi^0 & +\,3\sqrt{2}\ \partial_\mu K^+\partial^\mu\pi^-\bar{K}^0\pi^0 \\
-\,2\ \partial_\mu\pi^-\partial^\mu K^+K^-\pi^+ & -\,\partial_\mu K^0\partial^\mu\bar{K}^0\pi^0\pi^0 & -\,3\sqrt{2}\ \partial_\mu K^0\partial^\mu\pi^+K^-\pi^0 & +\,3\sqrt{2}\ \partial_\mu K^-\partial^\mu\pi^+K^0\pi^0
\end{array}$$

$$\begin{array}{llll}
-\,2\ \partial_\mu K^-\partial^\mu K^+K^0\bar{K}^0 & -\,2\ \partial_\mu K^0\partial^\mu\bar{K}^0K^-K^+ & -\,2\ \partial_\mu K^+\partial^\mu\bar{K}^0K^-K^0 & -\,2\ \partial_\mu K^-\partial^\mu K^0K^+\bar{K}^0 \\
+\,2\ K^-K^-\partial_\mu K^+\partial^\mu K^+ & +\,2\ K^+K^+\partial_\mu K^-\partial^\mu K^- & +\,2\ \partial_\mu K^0\partial^\mu K^0\bar{K}^0\bar{K}^0 & +\,2\ \partial_\mu\bar{K}^0\partial^\mu\bar{K}^0K^0K^0 \\
-\,4\ \partial_\mu K^0\partial^\mu\bar{K}^0K^0\bar{K}^0 & -\,4\ K^-K^+\partial_\mu K^-\partial^\mu K^+ & +\,4\ \partial_\mu K^-\partial^\mu\bar{K}^0K^+K^0 & +\,4\ \partial_\mu K^+\partial^\mu K^0K^-\bar{K}^0
\end{array}$$

$$\begin{array}{llll}
-\,\sqrt{3}\ \partial_\mu\eta\partial^\mu K^0\bar{K}^0\pi^0 & -\,\sqrt{3}\ \partial_\mu K^0\partial^\mu\pi^0\eta\bar{K}^0 & +\,\sqrt{3}\ \partial_\mu K^-\partial^\mu\pi^0\eta K^+ & +\,\sqrt{3}\ \partial_\mu\eta\partial^\mu K^+K^-\pi^0 \\
-\,\sqrt{3}\ \partial_\mu\eta\partial^\mu\bar{K}^0K^0\pi^0 & -\,\sqrt{3}\ \partial_\mu\bar{K}^0\partial^\mu\pi^0\eta K^0 & +\,\sqrt{3}\ \partial_\mu K^+\partial^\mu\pi^0\eta K^- & +\,\sqrt{3}\ \partial_\mu\eta\partial^\mu K^-K^+\pi^0 \\
+\,\sqrt{6}\ \partial_\mu\eta\partial^\mu\bar{K}^0K^+\pi^- & +\,\sqrt{6}\ \partial_\mu\eta\partial^\mu K^+\bar{K}^0\pi^- & +\,\sqrt{6}\ \partial_\mu K^-\partial^\mu\pi^+\eta K^0 & +\,\sqrt{6}\ \partial_\mu K^0\partial^\mu\pi^+\eta K^- \\
+\,\sqrt{6}\ \partial_\mu\eta\partial^\mu K^0K^-\pi^+ & +\,\sqrt{6}\ \partial_\mu\eta\partial^\mu K^-K^0\pi^+ & +\,\sqrt{6}\ \partial_\mu K^+\partial^\mu\pi^-\eta\bar{K}^0 & +\,\sqrt{6}\ \partial_\mu\bar{K}^0\partial^\mu\pi^-\eta K^+ \\
+\,2\sqrt{3}\ \partial_\mu\eta\partial^\mu\pi^0K^0\bar{K}^0 & +\,2\sqrt{3}\ \partial_\mu K^0\partial^\mu\bar{K}^0\eta\pi^0 & -\,2\sqrt{3}\ \partial_\mu\eta\partial^\mu\pi^0K^-K^+ & -\,2\sqrt{3}\ \partial_\mu K^-\partial^\mu K^+\eta\pi^0 \\
-\,2\sqrt{6}\ \partial_\mu K^-\partial^\mu K^0\eta\pi^+ & -\,2\sqrt{6}\ \partial_\mu\eta\partial^\mu\pi^+K^-K_0 & -\,2\sqrt{6}\ \partial_\mu\eta\partial^\mu\pi^-K^+\bar{K}^0 & -\,2\sqrt{6}\ \partial_\mu\eta\partial^\mu\pi^-K^+\bar{K}^0
\end{array}$$

$$\begin{array}{llll}
-\,3\ \eta\eta\partial_\mu K^-\partial^\mu K^+ & +\,3\ \eta K^-\partial_\mu\eta\partial^\mu K^+ & -\,3\ K^-K^+\partial_\mu\eta\partial^\mu\eta & +\,3\ \partial_\mu\eta\partial^\mu K^0\eta\bar{K}^0 \\
+\,3\ \eta K^+\partial_\mu\eta\partial^\mu K^- & -\,3\ \partial_\mu K^0\partial^\mu\bar{K}^0\eta\eta & -\,3\ \partial_\mu\eta\partial^\mu\eta K^0\bar{K}^0 & +\,3\ \partial_\mu\eta\partial^\mu\bar{K}^0\eta K^0
\end{array}$$

The mass-dependent interaction terms in $\mathcal{L}_2$ are proportional to the following trace:

$$\left\langle \mathcal{M}\Phi^4 \right\rangle = \tag{A.3}$$

$$
\begin{aligned}
&+ (m_u + m_d)\ \pi^+\pi^-\pi^+\pi^- && + (m_u + m_d)\ \pi^0\pi^0\pi^+\pi^- \\
&+ \frac{1}{4}(m_u + m_d)\ \pi^0\pi^0\pi^0\pi^0 && \\
&&& \\
&+ \frac{2}{\sqrt{3}}(m_u - m_d)\ \pi^+\pi^-\pi^0\eta && + \frac{1}{\sqrt{3}}(m_u - m_d)\ \pi^0\pi^0\pi^0\eta \\
&&& \\
&+ (2m_u + m_d + m_s)\ \pi^+\pi^-K^+K^- && + (2m_d + m_u + m_s)\ \pi^+\pi^-K^0\bar{K}^0 \\
&+ \frac{1}{2}(3m_u + m_s)\ \pi^0\pi^0K^+K^- && + \frac{1}{2}(3m_d + m_s)\ \pi^0\pi^0K^0\bar{K}^0 \\
&+ \frac{1}{\sqrt{2}}(m_u - m_d)\ \pi^+\pi^0K^-K^0 && + \frac{1}{\sqrt{2}}(m_u - m_d)\ \pi^-\pi^0K^+\bar{K}^0 \\
&+ (m_u + m_d)\ \pi^+\pi^-\eta\eta && + \frac{1}{2}(m_u + m_d)\ \pi^0\pi^0\eta\eta \\
&&& \\
&+ \frac{1}{\sqrt{3}}(m_u - m_s)\ \pi^0\eta K^+K^- && - \frac{1}{\sqrt{3}}(m_d - m_s)\ \pi^0\eta K^0\bar{K}^0 \\
&+ \frac{1}{\sqrt{6}}(m_u + m_d - 2m_s)\ \pi^+\eta K^-K^0 && + \frac{1}{\sqrt{6}}(m_u + m_d - 2m_s)\ \pi^-\eta K^+\bar{K}^0 \\
&+ (m_u - m_d)\ \pi^0\eta\eta\eta && \\
&&& \\
&+ \frac{1}{2}(m_u + 3m_s)\ \eta\eta K^+K^- && + \frac{1}{2}(m_d + 3m_s)\ \eta\eta K^0\bar{K}^0 \\
&+ (m_u + m_s)\ K^+K^-K^+K^- && + (m_d + m_s)\ K^0\bar{K}^0K^0\bar{K}^0 \\
&+ (m_u + m_d + 2m_s)\ K^+K^-K^0\bar{K}^0 && \frac{1}{36}(m_u + m_d + 16m_s)\ \eta\eta\eta\eta
\end{aligned}
$$

B. Loop Integrals

The basic integrals used are given by

$$I_\phi \equiv \int \frac{\mathrm{d}^D k}{(2\pi)^D} \; \frac{i}{k^2 - m^2 + i\varepsilon} \tag{B.1}$$
$$= \mu^{2\varepsilon} \left(\frac{m}{4\pi}\right)^2 \left[R + \ln\frac{m^2}{\mu^2}\right] \quad \text{where} \quad R \equiv \frac{1}{\varepsilon} + \gamma_E - 1 - \ln 4\pi$$

and [PT84]

$$I_{\phi\phi} = \int \frac{\mathrm{d}^D k}{(2\pi)^D} \; \frac{i}{[k^2 - m^2 + i\varepsilon]\,[(k-q)^2 - m^2 + \varepsilon]} \tag{B.2}$$
$$= \frac{\mu^{2\varepsilon}}{(4\pi)^2} \left\{ \frac{1}{\varepsilon} - \ln 4\pi + \gamma_E + \ln\frac{m^2}{\mu^2} + \sigma_\phi \ln\frac{\sigma_\phi + 1}{\sigma_\phi - 1} - 2 \right\}$$
$$\text{where } \sigma_\phi = \sqrt{1 - \frac{4m^2}{q^2}}$$
$$= \frac{\mu^{2\varepsilon}}{(4\pi)^2} \left\{ R + \ln\frac{m^2}{\mu^2} + \sigma_\phi \ln\frac{\sigma_\phi + 1}{\sigma_\phi - 1} - 1 \right\}$$
$$= \frac{I_\phi}{m^2} + \frac{\mu^{2\varepsilon}}{(4\pi)^2} \left\{ \sigma_\phi \ln\frac{\sigma_\phi + 1}{\sigma_\phi - 1} - 1 \right\}$$

The divergent integral found in the calculation of the pion form factor has to be expressed through the functions I_ϕ and $I_{\phi\phi}$. Further it will need to be separated into a part that is proportional to q^2 and will therefore be renormalized using the coupling L_9 and into a part that will be independent of q. The expression found was

$$f^2 \; i\mathcal{M}_{\text{loop}} = i(p + p')_\nu \int \frac{\mathrm{d}^4 k}{(2\pi)^4} \; \frac{(2k - q)^\mu k^\nu}{[k^2 - m^2 + i\varepsilon]\,[(k-q)^2 - m^2 + i\varepsilon]}. \tag{B.3}$$

The second summand in this expression can be simplified by substituting the integration variable and dropping all uneven functions that give zero when the integration over all momentum space takes place. Omitting the $i\varepsilon$ for shortness now and in the following the integral reads

$$-q^\mu \int \frac{\mathrm{d}^4 k}{(2\pi)^4} \; \frac{k^\nu}{[k^2 - m^2]\,[(k-q)^2 - m^2]} = -q^\mu \int \frac{\mathrm{d}^4 k}{(2\pi)^4} \; \frac{(k + \frac{q}{2})^\nu}{[(k + \frac{q}{2})^2 - m^2]\,[(k - \frac{q}{2})^2 - m^2]}$$
$$= -\frac{q^\mu q^\nu}{2} \int \frac{\mathrm{d}^4 k}{(2\pi)^4} \; \frac{1}{[k^2 - m^2]\,[(k-q)^2 - m^2]}$$
$$= \frac{i}{2} \, q^\mu q^\nu \; I_{\phi\phi}. \tag{B.4}$$

The remaining integral has the most general form

$$2 \int \frac{\mathrm{d}^4 k}{(2\pi)^4} \; \frac{k^\mu k^\nu}{[k^2 - m^2]\,[(k-q)^2 - m^2]} = 2 \left[q^\mu q^\nu A(q^2) + g^{\mu\nu} q^2 B(q^2) \right]. \tag{B.5}$$

Therefore the whole loop integral becomes

$$f^2\ i\mathcal{M}_{\text{loop}} = i(p+p')_\nu \left[q^\mu q^\nu \left(2A(q^2) + \frac{i}{2} I_{\phi\phi} \right) + 2g^{\mu\nu} q^2 B(q^2) \right]. \tag{B.6}$$

Since q is just the difference between the momenta of the incoming and outgoing pions and thus

$$(p'+p)_\nu q^\nu = (p'+p)(p'-p) = p'^2 - p^2 \tag{B.7}$$

the first term only contributes for off-shell pions. Neglecting that case the whole integral only depends on the function B as

$$f^2\ i\mathcal{M}_{\text{loop}} = 2i(p+p')^\mu q^2 B(q^2). \tag{B.8}$$

B can be determined by contracting B.5 with $g_{\mu\nu}$ to give

$$\begin{aligned} q^2\left[A(q^2) + DB(q^2)\right] &= \int \frac{\mathrm{d}^4k}{(2\pi)^4} \frac{k^2}{[k^2-m^2]\left[(k-q)^2-m^2\right]} \\ &= \int \frac{\mathrm{d}^4k}{(2\pi)^4} \frac{1}{(k-q)^2-m^2} + m^2 \int \frac{\mathrm{d}^4k}{(2\pi)^4} \frac{1}{[k^2-m^2]\left[(k-q)^2-m^2\right]} \\ &= -iI_\phi - im^2 I_{\phi\phi} \end{aligned} \tag{B.9}$$

and with $q_\mu q_\nu$ as

$$q^4\left[A(q^2) + B(q^2)\right] = \int \frac{\mathrm{d}^4k}{(2\pi)^4} \frac{(kq)^2}{[k^2-m^2]\left[(k-q)^2-m^2\right]}, \tag{B.10}$$

where the numerator can be rewritten as

$$\begin{aligned} (kq)^2 &= \frac{1}{4}\left[(k-q)^2 - k^2 - q^2\right]^2 \\ &= \frac{1}{4}\left[(k-q)^2-m^2\right]^2 + \frac{1}{4}\left[k^2-m^2\right]^2 + \frac{q^4}{4} - \frac{1}{2}\left((k-q)^2-m^2\right)\left(k^2-m^2\right) \\ &\quad - \frac{1}{2}\left((k-q)^2-m^2\right)q^2 + \frac{1}{2}\left(k^2-m^2\right)q^2. \end{aligned} \tag{B.11}$$

The last two terms give the same conribution but with the opposite sign canceling each other while the first two terms both result in the integral

$$\begin{aligned} \frac{1}{4}\int \frac{\mathrm{d}^4k}{(2\pi)^4} \frac{\left[(k-q)^2-m^2\right]^2}{[k^2-m^2]\left[(k-q)^2-m^2\right]} &= \frac{1}{4}\int \frac{\mathrm{d}^4k}{(2\pi)^4} \frac{(k-q)^2-m^2}{k^2-m^2} \\ &= \frac{1}{4}\int \frac{\mathrm{d}^4k}{(2\pi)^4} \frac{k^2-m^2+q^2-2kq}{k^2-m^2} \\ &= \frac{1}{4}\int \frac{\mathrm{d}^4k}{(2\pi)^4} + \frac{q^2}{4}\int \frac{\mathrm{d}^4k}{(2\pi)^4} \frac{1}{k^2-m^2}, \end{aligned} \tag{B.12}$$

where the first integral is canceled by the fourth term in B.11. Summing up one receives

$$\begin{aligned} q^2\left[A(q^2) + B(q^2)\right] &= \frac{1}{2}\int \frac{\mathrm{d}^4k}{(2\pi)^4} \frac{1}{k^2-m^2} + \frac{q^2}{4}\int \frac{\mathrm{d}^4k}{(2\pi)^4} \frac{1}{[k^2-m^2]\left[(k-q)^2-m^2\right]} \\ &= -\frac{i}{2} I_\phi - i\frac{q^2}{4} I_{\phi\phi}. \end{aligned} \tag{B.13}$$

Having now two equations for the unknown functions A and B these can be solved for B to give

$$B(q^2) = \frac{1}{D-1}\left[-i\frac{1}{2q^2}I_\phi - i\left(\frac{m^2}{q^2} - \frac{1}{4}\right)I_{\phi\phi}\right]. \tag{B.14}$$

The prefactor must be expanded around four dimensions by setting $D = 4 + 2\varepsilon$ such that

$$\frac{1}{D-1} = \frac{1}{3}\frac{1}{1+2/3\varepsilon} = \frac{1}{3}\left(1 - \frac{2}{3}\varepsilon + \mathcal{O}(\varepsilon^2)\right), \tag{B.15}$$

which leads to the final result

$$\begin{aligned} i\mathcal{M}_{\text{loop}} &= -\frac{2}{3f^2}(p+p')^\mu\left[\frac{1}{2}I_\phi + \left(m^2 - \frac{q^2}{4}\right)I_{\phi\phi} - \frac{1}{(4\pi)^2}\left(m^2 - \frac{q^2}{6}\right)\right] \\ &= -\frac{2}{3f^2}(p+p')^\mu\left[\left(\frac{3}{2} - \frac{q^2}{4m^2}\right)I_\phi - \frac{1}{(4\pi)^2}\left(2m^2 - \frac{5q^2}{12}\right) - \frac{1}{(4\pi)^2}\frac{q^2}{4}\sigma_\phi^3 \ln\frac{\sigma_\phi + 1}{\sigma_\phi - 1}\right] \end{aligned} \tag{B.16}$$

Bibliography

[A$^+$86] S. R. Amendolia et al. A Measurement of the Space - Like Pion Electromagnetic Form-Factor. *Nucl. Phys.*, B277:168, 1986.

[B$^+$12] J. Beringer et al. Review of Particle Physics (RPP). *Phys. Rev.*, D86:010001, 2012.

[DGH92] J. F. Donoghue, E. Golowich, and Barry R. Holstein. Dynamics of the standard model. *Camb. Monogr. Part. Phys. Nucl. Phys. Cosmol.*, 2:1–540, 1992.

[GL85] J. Gasser and H. Leutwyler. Chiral Perturbation Theory: Expansions in the Mass of the Strange Quark. *Nucl. Phys.*, B250:465–516, 1985.

[GM64] Murray Gell-Mann. A Schematic Model of Baryons and Mesons. *Phys. Lett.*, 8:214–215, 1964.

[Jon90] H. F. Jones. *Groups, representations and physics.* 1990.

[Kub07] Bastian Kubis. An Introduction to chiral perturbation theory. In *Workshop on Physics and Astrophysics of Hadrons and Hadronic Matter Shantiniketan, India, November 6-10, 2006*, 2007.

[O$^+$14] K. A. Olive et al. Review of Particle Physics. *Chin. Phys.*, C38:090001, 2014.

[Pic98] Antonio Pich. Effective field theory: Course. In *Probing the standard model of particle interactions. Proceedings, Summer School in Theoretical Physics, NATO Advanced Study Institute, 68th session, Les Houches, France, July 28-September 5, 1997. Pt. 1, 2*, pages 949–1049, 1998.

[Pic99] Antonio Pich. Aspects of quantum chromodynamics. In *Particle physics. Proceedings, Summer School, Trieste, Italy, June 21-July 9, 1999*, pages 53–102, 1999.

[Pic12] Antonio Pich. The Standard Model of Electroweak Interactions. In *Proceedings, High-energy Physics. Proceedings, 18th European School (ESHEP 2010): Raseborg, Finland, June 20 - July 3, 2010*, pages 1–50, 2012.

[PRSZ95] B. Povh, K. Rith, C. Scholz, and F. Zersche. *Particles and nuclei: An Introduction to the physical concepts.* 1995.

[PS95] Michael E. Peskin and Daniel V. Schroeder. *An Introduction to Quantum Field Theory.* 1995.

[PT84] P. Pascual and R. Tarrach. QCD: Renormalization for the Practitioner. *Lect. Notes Phys.*, 194:1–277, 1984.

[Sch03] Stefan Scherer. Introduction to chiral perturbation theory. *Adv. Nucl. Phys.*, 27:277, 2003.