VNIVERSITAT DE VALÈNCIA

Facultat de Física

Máster en Física Avanzada

Especialidad Física Teórica

Trabajo Fin de Máster

ONE-LOOP EFFECTIVE FIELD THEORY FROM MASSIVE SCALAR DARK MATTER: FUNCTIONAL METHODS

José Luis Oltra Calatayud

Tutor: Jorge Portolés Ibáñez

Curso académico 2015/16

Contents

1 Introduction

What the universe is made of seems like an important thing to know. Currently we only understand the stuff that makes up for 5% of the energy density of the universe. The rest we call dark matter and dark energy. Dark because we cannot see it, and matter or energy depending on their fundamental nature. Dark matter is thought to account for around 27% of the cosmic energy density, so understanding it is clearly important if we want to claim we have any knowledge about the universe we live in.

The first indications of the existence of dark matter from astrophysical observations are almost a hundred years old ([1] and references therein). Around the 1920s Öpik and then Kapteyn tried to estimate the density of matter around the Solar System. They found that the mass contributed by the known stars was sufficient to explain the dynamics observed. Later, Jeans introduced some corrections to the model used by Kapteyn and came to the conclusion that some extra matter exists in the galactic disk. He indicated that roughly two "dark" stars (i.e. not detectable via their luminosity) were needed for every bright one. Oort further developed the topic with an estimate of the contribution due to not-so-bright stars which suggested that no necessary extra matter was needed. Zwicky, in the early 1930s, was the first one to use the virial theorem to obtain the mass of bigger structures. He studied the Coma cluster and found that the speeds of the galaxies in the cluster were too high to be held only by their mutual gravitational attraction. His observations suggested that the total mass of the cluster was two orders of magnitude greater than the visible mass. He also was one of the first ones to suggest alternative methods of determining the mass of a big object, like gravitational lensing.

In the following years, Babcock and Oort analysed rotational curves of stars in a galaxy. Their measurements of the rotation curve of Andromeda showed a flat behaviour away from the galactic nucleus, in opposition to the expected decline in velocities as a consequence of the rapid fall in luminosity in the periphery of the neighbour galaxy. Babcock attributed this to a measurement problem, but a similar behaviour was observed in other spiral galaxies. As Oort described "...the distribution of mass appears to bear almost no relation to that of the light." The major breakthrough in the study of rotational curves came from Rubin and Ford, when they started using more sensitive spectrography. Rubin worked on rotational curves from the late 60s through the 70s. Her overall conclusion was that "rotation curves of high luminosity spiral galaxies are flat, at nuclear distances as great as 50 kpc". These suggest that more than half of the mass of galaxies is contained in some relatively dark galactic halo.

The dispersion in velocity of a stellar population in an elliptical galaxy or of galaxies in a cluster determines the kinetic energy of the structure. Assuming equilibrium, it is possible to estimate the total mass using the virial theorem. In a simple approach and assuming a spherical distribution, we have

$$\frac{GM}{R} = \sigma^2 \ , \tag{1.1}$$

where G is the gravitational constant, R is the radius and M the total mass of the structure inside of which the velocity dispersion σ is measured. Comparing this result to the mass estimated with the luminosity of an object one can obtain the dark matter present in a given structure. The results for elliptical galaxies tell us that dark matter is not the main mass component in the luminous region, although some elliptical galaxies show the presence of halos of dark matter.

An alternative method (and a more promising one) to study dark matter in elliptical galaxies and clusters of galaxies is the analysis of the interstellar (or intergalactic) medium, consisting of clouds of hot gas (with temperatures of the order of tens and hundreds of millions of degrees) radiating dominantly in X-ray frequencies and which represent the main component of the baryonic mass of such structures, being an order of magnitude larger than the mass due to stars. Through careful estimations of the distribution and temperature of these clouds, one can measure the gravitational potential of a cluster or a galaxy. The results obtained show that for clusters of galaxies 84% of the energy density comes from dark matter, 14% corresponds to the clouds of hot gas and the rest to stars and galaxies.

Another way of measuring the matter distribution of large structures is gravitational lensing. It is possible that the image of a distant object is distorted on its way to our detectors as an effect of the bending of light due to gravity. Thus knowing the distance to the object whose image has been distorted and to whatever is causing the distortion (the lens), it is possible to measure the total mass of the object acting as lens. Comparing this result with the visible mass, one can obtain the amount of dark matter in the object.

Dark matter has been estimated to comprise 27% of the cosmic energy density. Visible matter (or baryonic matter) amounts to less than 5% of this energy density. It is often characterized by the baryon-to-photon ratio [2],

$$Y_B = (8.59 \pm 0.11)10^{-11} \ . \tag{1.2}$$

This number is clearly important in our understanding of the universe. The Standard Model (SM) tells us it should be some orders of magnitude smaller than what has been measured. Therefore, explaining the relative abundance of the visible and dark matter components is a great motivation for physics beyond the Standard Model (BSM).

This missing mass that is the dark matter could have different origins. There could be a big number of very small and cold stars, too small and cold to emit any significant amount of radiation we can observe from Earth, or there could be many more black holes (not very massive ones) distributed throughout galaxies than previously thought. Also there could be clouds of gas or dust which cannot be detected with present technology. In addition it could be that there is some sort of cosmic background of some particle (or particles). This particle would have to interact very weakly with ordinary matter, or it would be part of the stars and clouds of gas and dust that one can found in a regular galaxy. This cosmic background (more probably just a cloud or halo surrounding the galaxy) cannot be made up of very hot (highly energetic) particles, for the gravity of a regular galaxy would not be enough to hold it together. Therefore it would have to be made of particles moving not too fast.

The dynamics related to the dark matter, which might have as a consequence the baryon asymmetry unexplained by the SM, have evaded us either because the scales associated are too high for present experiments or because their interactions with the SM particles are too weak. The discovery of the Higgs boson (H) at the LHC makes us wonder whether this particle might provide a window (or portal) of sorts into new particles which are presently hidden from our view. More refined studies of the properties and interactions associated with the Higgs boson might show us a rich landscape of new physics.

Clearly the nature and amount of dark matter is relevant to cosmology. If we are to understand how the universe has evolved from its first moments after the Big Bang all the way to the present era, and if we want to know how the structures we observe today (such us galaxies, clusters and superclusters of galaxies) formed, a deep knowledge of the building blocks of the universe is mandatory. Such a deep knowledge can be gained with the aid of colliders.

Dark matter is expected to bring change to modern physics, from particle physics to cosmology. In the same way as Copernicus displaced us from the center of the universe, Einstein showed that time and space are not absolute or quantum mechanics that knowledge is probable instead of certain, dark matter has already told us that what we are made of is not what most of the universe is made of. We still do not know how much of a revolution its final understanding will be, but it is never a bad thing to be humbled by nature.

The non-vanishing value of Y_B might be a result of unknown physics during the Big Bang or from dynamics of grand unified theories at very high energy scales. In all cases, the existence of new scalar particles whose interactions are somewhat analogous to those of the H is required [2]. If one of these particles is stable on cosmological time scales it may serve as a candidate for the dark matter. Also, the dark matter field may not be a scalar itself, but interact with the Higgs boson (and thus the SM) through the exchange of a scalar particle. We consider the first possibility: that the dark matter component is given by scalar particles. Even though there is no evidence for the scale of the dark matter masses , we consider the case in which the mass of this new particle is larger than those of the particles of the SM, which is actually less constrained by present data [2], so that we can build an effective theory in which the dark matter is integrated out [3]. The connection of this scalar dark matter candidate we will be considering to the SM comes from interactions with the Higgs boson, through a Higgs portal. Portals in particle physics and specifically the Higgs portal will be treated more thoroughly in section 3. The construction of such an effective field theory can be achieved by the construction of an effective Lagrangian in which only the light particles (lighter than the dark matter candidate considered) are present. This clearly only gives us an approximation of an underlying theory working at higher energies. Therefore starting from models of a Higgs portal with heavy dark matter, we build the corresponding EFT to one loop. The goal is to be able to study the effect of the existence of DM in low-energy collider physics. We will study this effect by considering how the presence of this DM affects the stability of the Higgs potential.

There are two possibilities when considering the construction of such a low-energy effective theory. It is possible to integrate out the heavy field (or fields) with diagrammatic methods, in which the contributions of the Feynman diagrams which contain the heavy particle in internal lines are calculated to some loop order and then the free undetermined parameters of the effective theory are found by matching both the approximate and the full theory (as shown for example at the end of section 4.1 of [4]).

A different approach uses functional methods and the background-field method (BFM, explained in [3] and [4]), integrating out the heavy field in the path integral formalism. This gives a contribution which can be expanded in inverse powers of the mass of the heavy particle. This is the approach we will be using to integrate out our nondecoupling heavy field, obtaining a one-loop effective Lagrangian.

The idea of the background-field method is the following: splitting the field we want to integrate out of our theory into a classical background part, which gives us the tree-level contributions, and into a quantum part, which gives the contributions inside loops. For this purpose, and since we are only interested in constructing our EFT at the one-loop level, we will consider only the part of the Lagrangian quadratic in the quantum part of the fields. Then these can be integrated out using Gaussian integration. The effective Lagrangian resulting from this may still contain the classical fields, which can be eliminated using the classical equations of motion.

In Section 2 we give a quick overview of effective field theories and their role in modern particle physics. Section 3 will be devoted to introduce and understand the concept of portal in particle physics and its importance in present and future studies. Then we will proceed to build one-loop effective field theories in section 4, first for a stable scalar dark matter candidate and then for an unstable scalar candidate using functional methods. We will be using different but related methods to build each of the effective theories, which are thoroughly explained in the literature, although we have tried to give an outline here. Finally in section 5 we explore a consequence of considering the dimension-six operators obtained for our effective field theories: their effect in the behaviour of the instability that appears in the Standard Model from the Higgs potential for high enough energies.

2 Overview of effective field theories

As a way of considering the effects of New Physics, endowed at a higher energy scale, effective field theories have gained an increasing weight in our methodology of study of "low-energy" particle physics. Effective field theories cannot be seen as an arbitrarily simplified version of more complex theories, but rather as an ingenious way of approaching physical problems for which full understanding and use of the mechanism that makes them possible is not necessary and even prevents us from achieving any useful results, [5]. It is not necessary to summon the full predictive power of Quantum Electrodynamics in order to do chemistry, or to use General Relativity in order to calculate where a stone should land when thrown from a window by a curious child. When studying this kind of problems, one wishes to isolate the important variables and parameters which give the relevant physics needed to treat them and leaves the rest out. This is done by identifying those contributions which are significantly larger or smaller than the scale for a given problem and taking them to infinity or zero, respectively. When done correctly, such a procedure should carry with it a considerable simplification of the original theory. If one requires an improvement of the simplification it is possible to include corrections previously neglected.

Low-energy particle physics is clearly the perfect ground for exploring the full potential of effective field theories. In problems with two very distinct energy scales (for example a theory describing the interactions of two particles, with mass m and M, such that $m \ll M$) the construction of an EFT lets us work only with the light degrees of freedom, whereas the heavy ones are integrated out. Any information on these heavy degrees of freedom will be contained in the new parameters appearing in the effective Lagrangian of the theory constructed.

Historically there have been two different ways of constructing such an effective theory: by matching the results of the full theory to the effective theory, diagram by diagram, and using functional methods. Since we are going to be treating only the second option, we shall give a very simple example of the first method, following the example given in [5], in what follows.

Consider a theory consisting of two spinless fields, L and H, with respective masses m and M, such that $m \ll M$. The Lagrangian for such a theory would be

$$\mathcal{L} = \frac{1}{2}(\partial_\mu L \partial^\mu L - m^2 L^2) + \frac{1}{2}(\partial_\mu H \partial^\mu H - M^2 H^2) + \frac{\omega}{4!} L^4 + \frac{\tau}{2} L^2 H \ , \tag{2.1}$$

where there might be more interaction terms, which will have no effect in our discussion. Now, the amplitude corresponding to the process $LL \to LL$, which is given in the diagrams shown in

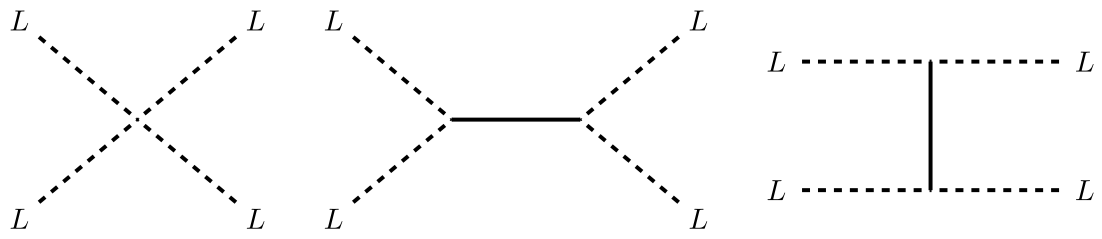

Figure 1: Diagrams for the tree level contribution to LL $\to$ LL in our effective theory.

figure 1, is

$$iA = -\omega + \tau^2 \Big[\frac{1}{s - M^2} + \frac{1}{t - M^2} + \frac{1}{u - M^2}\Big] \approx -\omega - \frac{3\tau^2}{M^2} - \frac{\tau^2(s+t+u)}{M^4} + O(M^{-6}) \ , \quad (2.2)$$

where in the last equality we have assumed all parameters: τ, m and s, t, u, to be much smaller than M and M^2 respectively. Therefore it is trivial to see that this result could have been obtained starting with a theory consisting of a Lagrangian containing only of the light L fields, with a kinetic term plus a potential of the form

$$\mathcal{L}_{eff} = \frac{1}{2}(\partial_\mu L \partial^\mu L - m^2 L^2) + \frac{L^2}{4!}\big(\omega - \frac{3\tau^2}{M^2} - \frac{\tau^2 \partial^2}{M^4}\big) L^2 \ . \quad (2.3)$$

We will not be using this method, which is extremely simple for toy theories like the one we just used, but gets rather involved for more complex ones. In Section 4 we will use some techniques which started to be developed during the past decades in the literature, such as those in references [6, 7] and some new ones recently developed, as those in [4].

3 Portals in particle physics

It is commonly expected that the next discoveries in particle physics should be found at higher energies than the ones that can be reached at present. This is a result from the strong constraints that exist on new physics charged under the SM gauge group. However, a clear indication for new physics is the lack of understanding of dark matter with present theories, suggesting the existence of a hidden sector, uncharged under the SM group. This would have as a consequence that the mass scale of the particles comprising this hidden sector would be less constrained. This is not something new. In the SM, there are matter fields uncharged under one or some of the color and electroweak gauge groups. Therefore including a new sector uncharged under the whole group is not something particularly daring from the theoretical point of view.

Now, given a hidden sector, which contains at low energies states that are necessarily electrically neutral and which we assume to be neutral with respect to the full SM gauge group, the interactions can be parameterized as [8]

$$\mathcal{L} = \sum_{k,l,n}^{k+l=n+4} \frac{\mathcal{O}_{NP}^{(k)} \mathcal{O}_{SM}^{(l)}}{\Lambda^n} \ , \quad (3.1)$$

where the $\mathcal{O}$ are operators for the SM and new physics (NP) with dimensions k and l, and Λ is a cutoff scale of O(1 TeV) at least. The SM-based operators of lowest dimension are often called portals. Some of the more representative are

$$\begin{array}{rl} B_{\mu\nu}V^{\mu\nu} & \text{Vector portal (dim= 4) ,} \\ \text{LHN} & \text{Neutrino portal (dim= 4) ,} \\ H^{\dagger}HS & \text{Higgs portal (dim= 4) .} \end{array}$$

Here H and L are the SM Higgs boson and lepton doublets, $B_{\mu\nu}$ is the hypercharge field strength tensor, $V_{\mu\nu}$ is the field strength tensor of a dark vector meson, N is a dark neutrino doublet and S a dark scalar boson. Such operators make couplings of the SM fields to new physics possible without compromising the renormalizability of the theory because they are dimension 4 and, correspondingly, $n = 0$ in Eq. (3.1), so there is no cutoff scale. Consequently there is no need to guess anything about the mass scale of these hidden-sector fields. As an example, neutrino oscillations can be understood as new physics beyond the SM, discovered through the neutrino portal, as a coupling to right-handed neutrinos. Therefore, the other portals are worth studying, in order to find out whether they are part of nature and our ability to probe them experimentally.

Recently there has been a considerable increase in the number and type of models pretending to explain the gravitational impact on cosmology and astrophysics of dark matter over a wide range of scales. There are some interesting possibilities which are not based on the typical candidate of a WIMP with a mass in the electroweak scale interacting with the SM. There are models of dark matter with masses of the order of the MeV which interact with the SM through a mediator with mass in the sub-GeV scale. Even though many astrophysical processes may be explained by other mechanisms, the possibilities that the low-mass dark matter phenomenology brings has inspired interest in their low and intermediate energy particle physics manifestations.

The vector portal

This portal gives us a simple way for the SM to interact with an abelian gauge sector, via a Lagrangian of the form

$$\mathcal{L} = \frac{\kappa}{2} V_{\mu\nu} F^{\mu\nu} \ , \tag{3.2}$$

where $F^{\mu\nu}$ and $V_{\mu\nu}$ are the SM electromagnetic and $U(1)$ field strengths. For high energies the hypercharge field strength should replace $F^{\mu\nu}$, but at low energies the coupling to the Z can be ignored. After the spontaneous breaking of the symmetry group U(1), the parameter κ gives the coupling of V_μ to the electromagnetic current,

$$\mathcal{L} = \kappa V_\mu J^\mu_{EM} + \dots \ . \tag{3.3}$$

Despite the simplicity of this model, there are many phenomenological consequences.

The neutrino portal

This portal permits the coupling of a singlet fermion N_j to the LH fermionic operator present in the SM, and results in a Yukawa interaction.

$$\mathcal{L} = y_{ij} L_i H N_j \ . \tag{3.4}$$

Combined with the Majorana mass terms, these interactions can explain observable neutrino masses and phenomenology of neutrino oscillations, meaning there is a real motivation for their study.

The Higgs portal

The Higgs portal has a distinct feature, and that is the fact that the Higgs sector has not been studied so thoroughly as other sectors and its details are not known. The Higgs portal can be seen as a parameterization of an extended Higgs sector where we take the new field to be a scalar singlet S, which couples to the Higgs portal as in

$$\mathcal{L} = (H^\dagger H)(\lambda S^2 + \beta S) \ . \tag{3.5}$$

When the coupling of the dimension 3 operator goes to zero, the model has a $\mathbb{Z}_2$ symmetry which gives the S particles stability, making them a viable candidate for dark matter. Also, the scalar S may not be itself the dark matter candidate, but might interact with the dark matter as in

$$\mathcal{L} = \bar{\chi}(a + b\gamma_5)S\chi + \dots \ , \tag{3.6}$$

where χ represents in this case fermionic dark matter.

If one is interested in low energy phenomenology, it is possible to integrate out the Higgs field:

$$\mathcal{L} = \mathcal{O}_{SM}^{(h)} \frac{\lambda S^2 + \beta S}{m_h^2} \ , \tag{3.7}$$

where $\mathcal{O}_{SM} = \sum_f m_f \bar{f} f$ is the familiar SM operator which describes the interaction of an off-shell Higgs with the SM light particles.

Future colliders, or perhaps the LHC, may discover the particles in the hidden sector

$$(S, \chi)$$

and study their interactions in different ways. They would be able to find modified couplings of the Higgs boson to itself or the other SM particles, new production mechanisms and decay channels of the H and also new scalar particles interacting with the SM either directly or through the Higgs boson.

From what is presently known, dark matter could exist in the form of scalar, fermionic or vector particles. Such dark matter candidates would be described by Lagrangians of the form [9]

$$\mathcal{L}_s = \frac{1}{2}\partial_\mu S \partial^\mu S - \frac{1}{2} m_s^2 S^2 - \frac{\lambda_{hs}}{2} H^\dagger H S^2 - \frac{\lambda_s}{4} S^4 \ , \tag{3.8}$$

$$\mathcal{L}_f = \bar{\chi}(i\not{\partial} - m_\chi)\chi - \frac{\lambda_{h\chi}}{\Lambda} H^\dagger H \bar{\chi}\chi \ , \tag{3.9}$$

$$\mathcal{L}_v = -\frac{1}{4} X_{\mu\nu} X^{\mu\nu} + \frac{m_x^2}{2} X_\mu X^\mu + \frac{\lambda_x}{4} (X_\mu X^\mu)^2 + \frac{\lambda_{hx}}{2} H^\dagger H X_\mu X^\mu \ , \tag{3.10}$$

The first Lagrangian is renormalizable and only requires a $\mathbb{Z}_2$ symmetry to ensure the stability of the scalar S. The second Lagrangian is non-renormalizable, which makes it not as likeable as the scalar option. The third Lagrangian seems renormalizable at first sight because it only has dimension 2 and dimension 4 operators, but is actually not renormalizable and violates unitarity, as happens with the intermediate vector boson model before introducing spontaneous symmetry breaking.

Here we are going to study the first case, the scalar dark matter candidate, in the following section.

4 One-loop Effective Theory: functional methods

For this scalar dark matter candidate we are going to consider two possibilities: a stable scalar and an unstable one. We are going to use different methods to study each possibility. For the stable candidate we will be following the work of [6]. This method permits the construction of a one-loop effective Lagrangian for theories which only have heavy fields in the propagators at the one-loop level. For the unstable candidate we will be following the work of [4], which presents a method more involved but which is also more general, for it allows the construction of effective field theories both with only heavy fields in the internal lines and with heavy and light fields.

We will consider, in both cases, that the scalar dark matter candidate S is a singlet under the SM group to which it couples only through the Higgs sector.

4.1 Stable scalar candidate

For the Lagrangian

$$\mathcal{L}(S) = \frac{1}{2}\partial_\mu S\partial^\mu S - \frac{1}{2}M^2S^2 + D_\mu H^\dagger D^\mu H - m^2(H^\dagger H) - \frac{\lambda}{2}(H^\dagger H)^2 - \frac{\kappa}{2}H^\dagger H S^2 - \frac{\lambda_S}{4}S^4 \ , \quad (4.1)$$

where we can separate the scalar field S into a classical background part, $\hat{S}$, and a quantum part, σ, so that $S = \hat{S} + \sigma$ and the Lagrangian is now

$$\begin{aligned}\mathcal{L}(S) \ &= \ \frac{1}{2}(\partial_\mu \hat{S}\partial^\mu \hat{S} - M^2\hat{S}^2) + D_\mu H^\dagger D^\mu H - m^2(H^\dagger H) + \frac{1}{2}(\partial_\mu\sigma\partial^\mu\sigma - M^2\sigma^2) + \\ &+ \ \partial^\mu\hat{S}\partial_\mu\sigma - 2M^2\hat{S}\sigma - \frac{\lambda}{2}(H^\dagger H)^2 - \frac{\kappa}{2}H^\dagger H\hat{S}^2 - \frac{\kappa}{2}(H^\dagger H)\sigma^2 - \kappa(H^\dagger H)\hat{S}\sigma - \\ &- \ \frac{\lambda_S}{4}\hat{S}^4 - \lambda_S\hat{S}^3\sigma - \frac{3}{2}\lambda_S\hat{S}^2\sigma^2 - \lambda_S\hat{S}\sigma^3 - \frac{\lambda_S}{4}\sigma^4 \ , \end{aligned} \quad (4.2)$$

where the terms containing any power of $\hat{S}$ can be eliminated using the classical equation of motion, obtained by applying the Euler-Lagrange equations to $\mathcal{L}(\hat{S})$,

$$\frac{\partial\mathcal{L}(S)}{\partial S} - \partial_\mu\frac{\partial\mathcal{L}(S)}{\partial_\mu S} = 0 \ . \quad (4.3)$$

It is trivial to obtain that

$$\frac{\partial\mathcal{L}(S)}{\partial S} = -M^2\hat{S} - \kappa H^\dagger H\hat{S} - \lambda_S\hat{S}^3 \quad ; \quad \frac{\partial\mathcal{L}(S)}{\partial_\mu S} = \partial^\mu\hat{S} \ . \quad (4.4)$$

Inserting this into the Euler-Lagrange equations gives

$$(\partial_\mu\partial^\mu + M^2 + \kappa H^\dagger H)\hat{S} + \lambda_S\hat{S}^3 = 0 \ , \quad (4.5)$$

which has as a solution $\hat{S} = 0$ and $\hat{S} = \frac{i}{\sqrt{\lambda_S}}(M^2 + \kappa H^\dagger H)^{\frac{1}{2}}$. We will only consider the first solution, $\hat{S} = 0$. Substituting this solution into equation (4.2) gives the tree-level effective Lagrangian (in which the terms $\propto \sigma$ are not included because the σ only contribute in loops)

$$\mathcal{L}_{eff}^{(0)} = D_\mu \hat{H}^\dagger D^\mu \hat{H} - m^2(\hat{H}^\dagger \hat{H}) - \frac{\lambda}{2}(\hat{H}^\dagger \hat{H})^2 \tag{4.6}$$

This separation into a classical background part and a quantum part leads to the generating functional

$$\begin{aligned} W(j) &= \int [dS] e^{i\int(\mathcal{L}(S(x))+j(x)S(x))d^4x} , \\ &= e^{i\int(\mathcal{L}(\hat{S}(x))+\mathcal{L}(\hat{H})+j(x)\hat{S}(x))d^4x} \int [d\sigma] e^{i\int d^4x(\frac{1}{2}\partial_\mu\sigma\partial^\mu\sigma - \frac{1}{2}M^2\sigma^2 - \frac{\kappa}{2}\hat{H}^\dagger\hat{H}\sigma^2 - \frac{\lambda_S}{4}\sigma^4)} . \end{aligned} \tag{4.7}$$

Since we are only interested in building an effective theory up to the one-loop level we only need to keep terms quadratic in the quantum fields σ. Integrating by parts we arrive at

$$W(j) = e^{i\int(\mathcal{L}(\hat{S}(x))+\mathcal{L}(\hat{H})+j(x)\hat{S}(x))d^4x} \int [d\sigma] e^{\frac{i}{2}\int d^4x\sigma(-\partial^2 - M^2 - \kappa\hat{H}^\dagger\hat{H})\sigma} . \tag{4.8}$$

Where now the integral is just a Gaussian integral, which we know how to solve [10]:

$$I_G \equiv \int [d\sigma] e^{-i\int d^4x\, \sigma \mathbf{O} \sigma} \propto (\det \mathbf{O})^{-\frac{1}{2}} , \tag{4.9}$$

where for our case

$$\mathbf{O} = -\partial^2 - M^2 - \kappa H^\dagger H . \tag{4.10}$$

This result is proved in section A.1 of Appendix A.

We now calculate the contribution to the effective Lagrangian coming from the term quadratic in the scalar quantum fields, following the techniques developed in [6], which was based on a work by Ball [6], and references therein. We will do the following calculations for a general case and then we will specify the D^μ and $U(x)$ corresponding to the theory under consideration here. For the generating functional

$$W(j) = e^{i\int(\mathcal{L}(\hat{S}(x))+j(x)\hat{S}(x)+\mathcal{L}_{eff}^{(1)}(x))d^4x} , \tag{4.11}$$

we have that

$$\begin{aligned} i\int d^4x \mathcal{L}_{eff}^{(1)}(x) &= \ln[(\det O)^{-1/2}] = -\frac{1}{2}\,\mathrm{Tr}[\ln O] = \\ &= -\frac{1}{2}\,\mathrm{Tr}[\ln(-D^2 - M^2 - U(x))] , \end{aligned} \tag{4.12}$$

where "Tr" is a trace over all degrees of freedom, D^μ is the covariant derivative of some gauge group and $U(x)$ is some general function of x. In general, we have

$$\begin{aligned} D_\mu &= \partial_\mu + A_\mu , \\ D_\mu U &= \partial_\mu U + [A_\mu, U] , \\ F_{\mu\nu} &= [D_\mu, D_\nu] . \end{aligned} \tag{4.13}$$

In the following we use the usual conventions for the momentum operator, $\hat{p}_\mu = i\partial_\mu$, and plane wave states, $\langle x|\, p\rangle = e^{-ipx}$. The normalizations of these states is such that, in $D = 4 - \epsilon$ dimensions,

$$\int d^D x \,|x\rangle \,\langle x| = 1 \qquad , \qquad \int \frac{d^D p}{(2\pi)^D} \,|p\rangle \,\langle p| = 1 \,.$$

The trace of an operator O is understood to be

$$\mathrm{Tr}[O] = \int d^D x \quad \mathrm{Tr}[\langle x|\, O\,|x\rangle] = \int \frac{d^D p}{(2\pi)^D} \quad \mathrm{Tr}[\langle p|\, O\,|p\rangle] \,. \tag{4.14}$$

Then, inserting a complete set of states in the last expression we have that

$$\mathrm{Tr}[O] = \int d^D x \int \frac{d^D p}{(2\pi)^D} \quad \mathrm{tr}[e^{ipx} \mathbf{O}_x e^{-ipx}] \,, \tag{4.15}$$

where "tr" is a trace over the internal degrees of freedom only. In the last equation $\mathbf{O}_x$ is the operator O in the representation of positions. In our case the operator O is

$$O = \ln[\Pi^2 - M^2 - U] \,, \tag{4.16}$$

where we have defined $\Pi_\mu \equiv iD_\mu$. Inserting this in equation (4.15) and using the operator identity $e^{ipx} f(\Pi) e^{-ipx} = f(\Pi + p)$, we get

$$i\mathrm{Tr}[\ln(\Pi^2 - M^2 - U)] = i \int d^D x \int \frac{d^D p}{(2\pi)^D} \mathrm{tr}[\ln(p^2 - M^2 + 2p\Pi + \Pi^2 - U)] \,. \tag{4.17}$$

We can re-express the logarithm as

$$\ln(p^2 - M^2) + \ln(1 + \frac{2p\Pi + \Pi^2 - U}{p^2 - M^2}) \,, \tag{4.18}$$

and expand the second part to finally obtain the following expression for (4.17),

$$i \int d^D x \int \frac{d^D p}{(2\pi)^D} \mathrm{tr} \left[\ln(p^2 - M^2) + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \frac{(2p\Pi + \Pi^2 - U)^n}{(p^2 - M^2)^n} \right] \,. \tag{4.19}$$

The first term is the Coleman-Weinberg term, which is constant and therefore only contributes to the energy density. This means it can be ignored for our calculation. We then have for the effective Lagrangian of (4.12)

$$\mathcal{L}^{(1)}_{eff} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2n} \mathrm{tr} \left[i \int \frac{d^D p}{(2\pi)^D} \frac{(2p\Pi + \Pi^2 - U)^n}{(p^2 - M^2)^n} \right] \,. \tag{4.20}$$

This last equation clearly gives an expansion in powers of M^{-2}. By dimensional inspection we see that

$$\int \frac{d^D p}{(2\pi)^D} \cdots \propto M^{D-2n} O_n \,, \tag{4.21}$$

where O_n are traces of operators of dimension $2n$. We do not know the form of the O_n, but in general we can write

$$O_n = \sum_i \gamma_i^{(n)} \tilde{O}_n^{(i)} \ , \tag{4.22}$$

where the $\tilde{O}_n^{(i)}$ form a linearly independent set of traces of operators. The only operators we can use for the construction of a basis of such traces are

$$U \qquad , \qquad F_{\mu\nu} \ , \tag{4.23}$$

and covariant derivatives of these. Using that

$$[U] = M^2 \qquad , \qquad [F_{\mu\nu}] = M^2 \ , \tag{4.24}$$

ensuring Lorentz invariance and excluding operators which can be eliminated through partial integration, we have up to third order (i.e. for $n \leq 3$)

$$\begin{aligned} \tilde{O}_1 &= \mathrm{tr}[U] \ , \\ \tilde{O}_2 &= \mathrm{tr}[U^2], \mathrm{tr}[F_{\mu\nu}F^{\mu\nu}] \ , \\ \tilde{O}_3 &= \mathrm{tr}[U^3], \mathrm{tr}[(D_\mu U)^2], \mathrm{tr}[F_{\mu\nu}UF^{\mu\nu}], \mathrm{tr}[D_\mu F^{\mu\nu} D^\sigma F_{\sigma\nu}], \\ & \ \mathrm{tr}[F_{\mu\nu}F^{\nu\sigma}F^{\mu}_{\sigma}] \ . \end{aligned} \tag{4.25}$$

Thus we would like to evaluate the coefficients $\gamma_i^{(n)}$. These can be evaluated directly by expanding (4.20) and performing the integrals. However there is a trick we can use which makes this calculation much easier. By noticing that equations (4.20), (4.21) and (4.22) are all valid for any A_μ and U and that the coefficients we are looking for are independent of A_μ and U, we can calculate them for a specific configuration. Therefore we can choose a configuration which simplifies greatly our calculation and then go back to the general case.

For the numerator of the integral of (4.20) we have

$$2p\Pi + \Pi^2 - U = 2ipA + 2p\hat{p} + \hat{p}^2 + i(\hat{p}A + A\hat{p}) - A^2 - U \ , \tag{4.26}$$

and it is easy to see that choosing $\partial_\mu A_\nu = 0$ and $U = -A^2$ we have

$$(2p\Pi + \Pi^2 - U)^n = (2ipA)^n \ . \tag{4.27}$$

We shall then choose this configuration. From now on we have that $A_\mu \equiv N_\mu$ such that

$$\partial_\mu N_\nu = 0, \quad U = -N^2, \quad F_{\mu\nu} = [N_\mu, N_\nu], \quad D_\mu F = [N_\mu, F] \ , \tag{4.28}$$

where F is any matrix function of A_μ and U in our special configuration. Equation (4.20) now reads

$$\mathcal{L}_{eff}^{(1)} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2n} \mathrm{tr}\left[i \int \frac{d^D p}{(2\pi)^D} \frac{(2ipN)^n}{(p^2 - M^2)^n} \right] \ . \tag{4.29}$$

Extracting the p's from the trace and using the fact that the integrals with an odd number of these cancel out, we redefine $n \to 2n$. Using the fact that $i^{2n} = (-1)^n = (-1)^{-n}$ we have

$$\mathcal{L}_{eff}^{(1)} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} 4^n}{4n} i \int \frac{d^D p}{(2\pi)^D} \frac{p_{\mu_1} p_{\mu_2} \ldots p_{\mu_{2n}}}{(p^2 - M^2)^{2n}} \mathrm{tr}[N^{\mu_1} N^{\mu_2} \ldots N^{\mu_{2n}}] \ . \tag{4.30}$$

We solve this equation in section A.2 of Appendix A and give the result directly here. We have for the effective Lagrangian at one loop

$$\mathcal{L}^{(1)}_{eff} = \sum_{n=1}^{\infty} \left(\frac{M^2}{4\pi\mu^2}\right)^{\frac{D}{2}-2} \frac{1}{(4\pi)^2} \frac{\Gamma(n-\frac{D}{2})}{M^{2n-4}} \frac{2^{n-1}}{(2n)!} S_n^{\mu_1\mu_2\ldots\mu_{2n}} \mathrm{tr}[N_{\mu_1}N_{\mu_2}\ldots N_{\mu_{2n}}] = \frac{c_n}{2M^{2n-4}} O_n \ . \tag{4.31}$$

With c_n and O_n given in (A.17). Thus in our special configuration we can write

$$O_n = \sum_i \delta_i^{(n)} \hat{O}_n^{(i)} \ , \tag{4.32}$$

where the $\hat{O}_n^{(i)}$ are traces of 2n N's contracted pairwise. Defining a set of linearly independent traces analogously to our foregoing definition, this time using N_μ, we have again up to third order

$$\begin{aligned} \hat{O}_1 &= \mathrm{tr}[N^2] \ , \\ \hat{O}_2 &= \mathrm{tr}[(N_\mu N_\nu)^2], \quad \mathrm{tr}[(N^2)^2] \ , \\ \hat{O}_3 &= \mathrm{tr}[(N_\mu N_\nu N_\sigma)^2], \quad \mathrm{tr}[(N^2)^3], \quad \mathrm{tr}[(N^2 N_\mu)^2], \quad \mathrm{tr}[(N_\mu N_\nu N^\mu)^2], \\ &\quad \mathrm{tr}[N^2 (N_\mu N_\nu)^2] \ . \end{aligned} \tag{4.33}$$

So now using that

$$\begin{aligned} \mathrm{tr}[S_1(N)] &= \mathrm{tr}[S_1^{\mu\nu} N_\mu N_\nu] = \mathrm{tr}[N^2] \ , \\ \mathrm{tr}[S_2(N)] &= \mathrm{tr}[S_2^{\mu\nu\sigma\rho} N_\mu N_\nu N_\sigma N_\rho] = \mathrm{tr}[N_\mu N_\nu N^\mu N^\nu] + 2\mathrm{tr}[(N^2)^2] \ , \\ \mathrm{tr}[S_3(N)] &= \mathrm{tr}[S_3^{\mu\nu\sigma\rho\alpha\beta} N_\mu N_\nu N_\sigma N_\rho N_\alpha N_\beta] = \mathrm{tr}[(N_\mu N_\nu N_\sigma)^2] + 2\mathrm{tr}[(N^2)^3] + \\ &+ 3\mathrm{tr}[(N^2 N_\mu)^2] + 3\mathrm{tr}[(N_\mu N_\nu N^\mu)^2] + 6\mathrm{tr}[N^2 (N_\mu N_\nu)^2] \ , \end{aligned} \tag{4.34}$$

and the definition of the O_n in (A.17) we can see that

$$\begin{aligned} O_1 &= \hat{O}_1 \ , \\ O_2 &= \frac{1}{6}\hat{O}_2^{(1)} + \frac{1}{3}\hat{O}_2^{(2)} \ , \\ O_3 &= \frac{1}{90}\hat{O}_3^{(1)} + \frac{1}{45}\hat{O}_3^{(2)} + \frac{1}{30}\hat{O}_3^{(3)} + \frac{1}{30}\hat{O}_3^{(4)} + \frac{1}{15}\hat{O}_3^{(5)} \ . \end{aligned} \tag{4.35}$$

And then, as we know that in general the O_n can be written in terms of the $\tilde{O}_n$, we have that in our special configuration

$$O_n = \sum_i \delta_i^{(n)} \hat{O}_n^{(i)} = \sum_i \gamma_i^{(n)} \tilde{O}_n^{(i)} \ . \tag{4.36}$$

Therefore we can relate the operators $\hat{O}$ with the $\tilde{O}$. In general we can write

$$\tilde{O}_n^{(i)} = \sum_j (M_n)_{ij} \hat{O}_n^{(j)} \quad \text{and} \quad \delta_i^{(n)} = \sum_j \gamma_i^{(n)} (M_n)_j i \ , \tag{4.37}$$

where the linear transformation given by M_n can be inverted, so that we can compute the γ's in terms of the δ's, which were just calculated above. We will then have the γ's given by

$\gamma^{(n)} = (M_n^T)^{-1}\delta^{(n)}$. We can obtain the elements of the matrix $(M_n^T)^{-1}$ from the elements of M_n, which are obtained directly by expressing each trace of the linearly independent sets of traces of operators constructed from U, $F_{\mu\nu}$ and their covariant derivatives, using the form of these in the special configuration in (4.28). We give the full calculation of these matrices in section A.3 of Appendix A. We obtain in (A.29) after some simple but tedious algebra:

$$(M_1^T)^{-1} = -1 \qquad ; \qquad (M_2^T)^{-1} = \begin{pmatrix} 1 & 1 \\ 1/2 & 0 \end{pmatrix} , \tag{4.38}$$

$$(M_3^T)^{-1} = -\frac{1}{4}\begin{pmatrix} 4 & 4 & 4 & 4 & 4 \\ -3 & 0 & -2 & -3 & -2 \\ 6 & 0 & 0 & 4 & 2 \\ -3 & 0 & 0 & 1 & 0 \\ 4 & 0 & 0 & 0 & 0 \end{pmatrix} . \tag{4.39}$$

After obtaining the γ's with these, we end up with

$$\begin{aligned} O_1 &= -\mathrm{tr}[U] , \\ O_2 &= \frac{1}{2}\mathrm{tr}[U^2] + \frac{1}{12}\mathrm{tr}[F_{\mu\nu}F^{\mu\nu}] , \\ O_3 &= -\frac{1}{6}\mathrm{tr}[U^3] + \frac{1}{12}\mathrm{tr}[(D_\mu U)^2] - \frac{1}{12}\mathrm{tr}[F_{\mu\nu}UF^{\mu\nu}] \\ &\quad + \frac{1}{60}\mathrm{tr}[D_\mu F^{\mu\nu}D^\sigma F_{\sigma\nu}] - \frac{1}{90}\mathrm{tr}[F_{\mu\nu}F^{\nu\sigma}F_\sigma^{\mu}] . \end{aligned} \tag{4.40}$$

For the theory we are studying here, we have that $D_\mu \equiv \partial_\mu$, and therefore $F_{\mu\nu} = 0$, and thus

$$\begin{aligned} O_1 &= -\mathrm{tr}[U] , \\ O_2 &= \frac{1}{2}\mathrm{tr}[U^2] , \\ O_3 &= -\frac{1}{6}\mathrm{tr}[U^3] + \frac{1}{12}\mathrm{tr}[(\partial_\mu U)^2] . \end{aligned} \tag{4.41}$$

The coefficients c_n, given in (A.17) are

$$\begin{aligned} c_1 &= -(\Delta + 1)\frac{1}{(4\pi)^2} , \\ c_2 &= \Delta\frac{1}{(4\pi)^2} , \\ c_3 &= \frac{1}{(4\pi)^2} , \end{aligned} \tag{4.42}$$

where

$$\Delta = \frac{2}{\epsilon} - \gamma + \ln(4\pi) + \ln(\mu^2/M^2) . \tag{4.43}$$

Now, comparing the different definitions of the operator O in equations (4.10) and (4.16) we have for U

$$U = \kappa(H^\dagger H) . \tag{4.44}$$

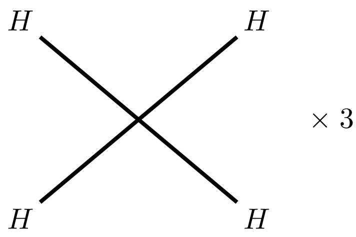

Figure 2: Diagrams for the tree level contribution to HH → HH in the effective theory.

Inserting all this into (4.31) we obtain

$$\begin{aligned}\mathcal{L}^{(1)}_{eff} &= \frac{1}{2}c_1 M^2\, O_1 + \frac{1}{2}c_2\, O_2 + \frac{c_3\, O_3}{2M^2} = \\ &= \frac{\kappa M^2}{2(4\pi)^2}(\Delta+1)H^\dagger H + \frac{\kappa^2\Delta}{4(4\pi)^2}(H^\dagger H)^2 + \\ &+ \frac{\kappa^2}{24M^2(4\pi)^2}[\partial_\mu(H^\dagger H)\partial^\mu(H^\dagger H) - \kappa(H^\dagger H)^3]\ .\end{aligned} \tag{4.45}$$

We would like to show now how our effective theory gives the appropriate results when compared to the full theory. In order to do this we calculate all the contributions for a given process to one-loop in the original theory and at tree level in our new theory. The process we will consider is $HH \to HH$. In the effective theory it comes from two contributions,

$$\frac{\kappa^2}{4}\frac{\Delta}{(4\pi)^2}(H^\dagger H)^2 + \frac{\kappa^2}{24}\frac{1}{M^2(4\pi)^2}\partial_\mu(H^\dagger H)\partial^\mu(H^\dagger H)\ , \tag{4.46}$$

and its diagram is given in figure 2, where the $\times\, 3$ means there is also a contribution from the s, t and u channels.

In the effective theory we simply have

$$iA_{eff} = \sum_j \frac{1}{4}\left(\frac{i\kappa^2}{2}\frac{3\Delta}{(4\pi)^2} + \frac{i\kappa^2}{12(4\pi)^2}\frac{q_j^2}{M^2}\right) = \frac{i\kappa^2}{8(4\pi)^2}\left(3\Delta + \frac{s+t+u}{6M^2}\right)\ , \tag{4.47}$$

where q_j, $j = s,t,u$, depends on the momentum of the incoming H, such that $q_s^2 \equiv s$, $q_t^2 \equiv t$ and $q_u^2 \equiv u$.

In the full theory it is given by the term $-\frac{\kappa}{2}H^\dagger H S^2$ whose corresponding diagrams are given in figure 3.

The one-loop amplitude is

$$iA_j^{full} = \frac{1}{2}\int \frac{d^Dp}{(2\pi)^D}\frac{i\kappa}{2}\frac{i}{p^2-M^2}\frac{i}{(q_j+p)^2-M^2}\frac{i\kappa}{2}\ . \tag{4.48}$$

We solve this integral in section A.4 of Appendix A, obtaining for the amplitude of the process $HH \to HH$ to one loop in the full theory

$$iA_j^{full} = \frac{i\kappa^2}{8(2\pi)^D}\left[\Delta + 2 - \sqrt{1-\frac{4M^2}{q_j^2}}\ln\left(\frac{\sqrt{1-\frac{4M^2}{q_j^2}}+1}{\sqrt{1-\frac{4M^2}{q_j^2}}-1}\right)\right]\ , \tag{4.49}$$

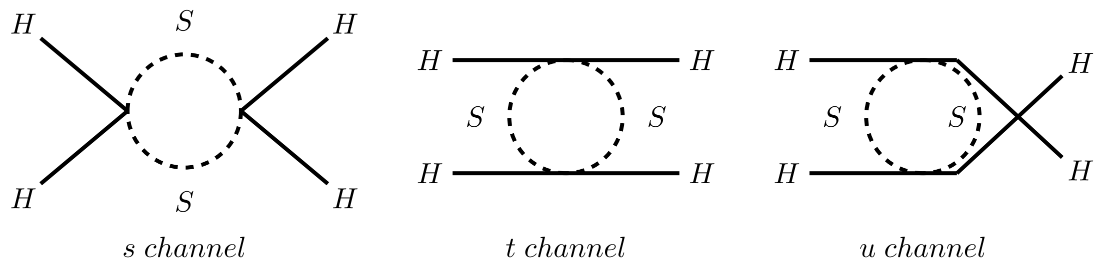

Figure 3: Diagrams for the one-loop contribution to HH → HH in the full theory.

where Δ has been defined in (4.43) and the complete amplitude is given by the sum

$$iA^{full} = i\sum_j A_j^{full} \; . \tag{4.50}$$

We cannot compare both results directly, because in the one-loop calculation we obtain logarithms which cannot arise in any way in the tree-level calculation of the effective theory. We can expand the result obtained from the full theory for small energies (that is, for $q^2/M^2 \to 0$) and then compare both expressions. Writing the Taylor series for

$$f(q_j^2/M^2) = 2 - \sqrt{1-\frac{4M^2}{q_j^2}}\ln\left(\frac{\sqrt{1-\frac{4M^2}{q_j^2}}+1}{\sqrt{1-\frac{4M^2}{q_j^2}}-1}\right) \; , \tag{4.51}$$

around $q_j^2/M^2 \equiv x_j = 0$ we obtain

$$f(x_j) = \frac{x_j}{6} + \frac{x_j^2}{60} + O(x_j^{5/2}) \; . \tag{4.52}$$

Retaining only the first term we have for the amplitude in the full theory:

$$iA_{full} = \frac{i\kappa^2}{8(4\pi)^2}\left(3\Delta + \frac{s+t+u}{6M^2}\right) \; . \tag{4.53}$$

Which clearly coincides with the result for the effective theory, (4.47).

4.2 Unstable scalar candidate

In this section we consider the case in which the dark matter candidate is an unstable particle, and the process $S \to HH$ can take place. For this case we will be using a different Lagrangian to the one we have been using up to this point. Now we have

$$\mathcal{L} = \frac{1}{2}(\partial_\mu S\partial^\mu S - M^2S^2) + (D_\mu H^\dagger D^\mu H - m^2 H^\dagger H) - \frac{\lambda}{2}(H^\dagger H)^2 - \frac{\lambda_S}{4}S^4 - \alpha H^\dagger H S \; . \tag{4.54}$$

Notice that $[\alpha] = M$. By solving the classical equations of motion for the field S, H and $H^\dagger$ we can construct the tree-level effective Lagrangian. We have

$$0 = \partial^2\hat{S} + M^2\hat{S} + \lambda_S\hat{S}^3 + \alpha\hat{H}^\dagger\hat{H} \; , \tag{4.55}$$

$$0 = D^2\hat{H}^\dagger + m^2\hat{H}^\dagger + \lambda\hat{H}^\dagger(\hat{H}^\dagger\hat{H}) + \alpha\hat{H}^\dagger\hat{S} \; , \tag{4.56}$$

$$0 = D^2\hat{H} + m^2\hat{H} + \lambda\hat{H}(\hat{H}^\dagger\hat{H}) + \alpha\hat{H}\hat{S} \; . \tag{4.57}$$

At leading order we can neglect the term with ∂^2 and any term not linear in $\hat{S}$ in (4.55). This leaves us with a solution to the equations of motion for $\hat{S}$ given by

$$\hat{S} = -\frac{\alpha}{M^2}(\hat{H}^\dagger \hat{H}) + O(M^{-4}), \tag{4.58}$$

which gives the following tree-level effective Lagrangian:

$$\mathcal{L}^{(0)}_{eff} = (D_\mu \hat{H}^\dagger D^\mu \hat{H} - m^2 \hat{H}^\dagger \hat{H}) + \left(\frac{\alpha^2}{2M^2} - \frac{\lambda}{2}\right)(\hat{H}^\dagger \hat{H})^2 + O(M^{-4}) \tag{4.59}$$

The equations of motion for $\hat{H}$ and $\hat{H}^\dagger$ will be useful later on.

We are going to obtain a one-loop effective Lagrangian for this theory using the method developed in [4], which is based on the works by Aitchison and Fraser, Chan, Gaillard and Cheyette, gathered in reference [7] and then obtain the amplitude for the process $HH \to HH$, without comparing it to the amplitude obtained from the full theory, because we consider it is too long a calculation and does not provide any insights. We first give an outline of the method and then proceed to do the calculations for our case. In general, this method considers a theory which contains a heavy and a light field and groups them in $f = (f_h, f_l)$. Each component is split into a background part ($\hat{f}_{h,l}$) and a quantum fluctuation ($f_{h,l}$), such that we can write, as before, $f \to \hat{f} + f$. At the one-loop level then we only need to consider terms quadratic in the quantum field. The part that only contains these quadratic terms can be parameterized as

$$\mathcal{L}_{quadratic} = \frac{1}{2} f^\dagger \mathcal{O} f \ , \tag{4.60}$$

with

$$\mathcal{O} = \begin{pmatrix} \Delta_h & X^\dagger_{hl} \\ X_{hl} & \Delta_l \end{pmatrix} . \tag{4.61}$$

The effective action is given then by

$$e^{iS} \propto \int df_l df_h e^{(i \int dx\, \mathcal{L}_{\text{quadratic}})} . \tag{4.62}$$

The presence of terms containing contributions from both heavy and light fields makes any calculations harder than necessary. It is possible to rewrite the operator $\mathcal{O}$ in block-diagonal form. This can be achieved with a shift in the quantum fields. We choose a transformation which redefines the part corresponding to only heavy particles, while leaving the light part without changes. The form of the transformation is

$$T = \begin{pmatrix} I & 0 \\ -\Delta_l^{-1} X_{hl} & I \end{pmatrix} . \tag{4.63}$$

This brings $\mathcal{O}$ to

$$T^\dagger \mathcal{O} T = \begin{pmatrix} \tilde{\Delta}_h & 0 \\ 0 & \Delta_l \end{pmatrix} , \tag{4.64}$$

where $\tilde{\Delta}_h = \Delta_h - X^\dagger_{hl} \Delta_l^{-1} X_{hl}$.

The integration over the heavy fields can be performed and we end up with

$$e^{iS} = (\det \tilde{\Delta}_h)^{-1/2} N \int df_l e^{i \int dx\, f_l^\dagger \Delta_l f_l} . \tag{4.65}$$

The integral gives us the one-loop contributions containing only light particles. Therefore we can define the one-loop action containing only heavy fields as:

$$S_h = ic\ \ln(\det \tilde{\Delta}_h) = ic\ \mathrm{Tr}[\ln \hat{\Delta}_h]\ . \tag{4.66}$$

We can rewrite the trace using momentum eigenstates as we did in the previous section, we arrive at

$$S_h = ic\ \mathrm{tr} \int d^D x \int \frac{d^D p}{(2\pi)^D} \ln\left(\tilde{\Delta}_h(x, \partial_x + ip)\right)\ , \tag{4.67}$$

with $\tilde{\Delta}_h = -D^2 - M^2 - U$. After performing the shift $\partial \to \partial + ip$ and neglecting the constant term as we did before we have

$$S_h = -\frac{i}{2} \int d^D x \sum_{n=1}^{\infty} \frac{1}{n} \int \frac{d^D}{(2\pi)^D} \mathrm{tr}\left[\left(\frac{2ipD + D^2 + U(x, \partial_x + ip)}{p^2 - M^2}\right)^n \mathbf{1}\right]\ . \tag{4.68}$$

This is the general result. For our particular case we have the Lagrangian in (4.54) and $f^T = (S, H, H^*)^T$. Now we separate these fields into a background and a quantum part: $S \to \hat{S} + \sigma$ and $H \to \hat{H} + \eta$. For the operator defined in (4.61) we have that for our Lagrangian, Δ_l is a 2×2 matrix while X_{hl} is a 2×1 vector. We have for the part of the Lagrangian quadratic in the quantum fields:

$$\mathcal{L}^{(f^2)} = \frac{1}{2}(\sigma\ \eta^\dagger\ \eta^T) \begin{pmatrix} \Delta_h & X^\dagger_{\sigma\eta} & X^\dagger_{\sigma\eta^*} \\ X_{\sigma\eta} & \Delta_{\eta\eta^*} & \delta^\dagger_{\eta\eta^*} \\ X_{\sigma\eta^*} & \delta_{\eta\eta^*} & \Delta^T_{\eta\eta^*} \end{pmatrix} \begin{pmatrix} \sigma \\ \eta \\ \eta^* \end{pmatrix}\ , \tag{4.69}$$

where of course

$$\begin{aligned} \Delta_l &= \begin{pmatrix} \Delta_{\eta\eta^*} & \delta^\dagger_{\eta\eta^*} \\ \delta_{\eta\eta^*} & \Delta^T_{\eta\eta^*} \end{pmatrix}\ , \\ X_{hl} &= \begin{pmatrix} X_{\sigma\eta} \\ X_{\sigma\eta^*} \end{pmatrix}\ . \end{aligned} \tag{4.70}$$

The fluctuations in (4.69) are defined as

$$\begin{aligned} \Delta_h &= -\partial^2 - M^2 - 3\lambda_S \hat{S}^2\ , \\ \Delta_{\eta\eta^*} &= -\hat{D}^2 - m^2 - \lambda(\hat{H}^\dagger \hat{H}) - \lambda \hat{H}\hat{H}^\dagger - \alpha\hat{S} \\ \delta_{\eta\eta^*} &= -\lambda \hat{H}^* \hat{H}^\dagger\ , \\ X_{\sigma\eta} &= -\alpha\hat{H}\ , \\ X_{\sigma\eta^*} &= -\alpha\hat{H}^*\ . \end{aligned} \tag{4.71}$$

Using that $\tilde{\Delta}_h = \Delta_h - X^\dagger_{hl}\Delta_l^{-1}X_{hl} = -\partial^2 - M^2 - U$ and the form of Δ_h we have that

$$U(x, \partial_x + ip) = \frac{3\lambda_S\alpha^2}{M^4}(H^\dagger H)^2 - X^\dagger_{hl}\Delta_l^{-1}X_{hl}\ . \tag{4.72}$$

The calculation of U is given in section A.5 of Appendix A.

At the order we are interested in we only need to take into account the $n = 1$ term in (4.68). We have

$$\mathcal{L}_{eff}^{(1)} = -\frac{i}{2}\int \frac{d^Dp}{(2\pi)^D}\frac{U(x,\partial_x + ip)}{p^2 - M^2} \ . \tag{4.73}$$

The calculation of $\mathcal{L}_{eff}^{(1)}$ is also given in section A.5 of Appendix A. With these results we finally have, as shown in (A.46)

$$\mathcal{L}_{eff}^{(1)} = \frac{\alpha^2}{32\pi^2}\left[\left(\frac{3}{M^2}(\lambda_S + 2\lambda) + \frac{2}{M^4}(13m^2\lambda - \alpha^2)\right)(H^\dagger H)^2 - \frac{\lambda}{2M^4}(H^\dagger H)\partial^2(H^\dagger H)\right] + O(M^{-6}) \ . \tag{4.74}$$

As is mentioned at the end of section A.5, this result has some simplifications. We have ignored terms proportional to $(H^\dagger H)$ since these would give a correction to m, something in which we are not interested here. We have ignored terms proportional to $(H^\dagger H)^3$, since their complete calculation is too involved for the time we have. Also, we have considered only terms with no gauge fields (coming from the covariant derivatives), that is only terms with four Higgs fields. In practice this means neglecting terms proportional to $H^\dagger D^\mu H D_\mu(H^\dagger H)$ and analogous. Finally, we have not included the terms proportional to M^{-6} in our result, because our result at this order is incomplete. We would need to consider higher-order terms in (4.68) to obtain their complete contribution.

5 Effects of higher-order corrections in vacuum stability

The Higgs effective potential is stable at the electroweak scale, but develops an instability at higher scales, coming from the fact that the quartic coupling λ (which is positive at the EW scale) goes to negative values at high enough energies, in which case the electroweak vacuum will be just a local minimum. When this happens for some energy scale Λ, the potential either is unbounded from bellow or presents a minimum deeper than the EW vacuum, coming from the dominant term $\propto \lambda(H^\dagger H)^2$. This predicted instability, and the fact that the universe has existed for some billions of years now, means there may be new physics at energies which cannot be taken to be unreasonably large compared to those typical of the SM, which should solve the problem of the instability. This depends on the idea that the SM can be understood as an approximation of a more complete theory.

The potential, in quantum field theory, is defined from the pieces of the interacting Lagrangian that contribute at zero momenta, i.e. they include only those operators without derivatives. The appearance of higher-order operators after integrating out some new physics of mass M, which are suppressed by powers of $1/M$, may affect the vacuum stability of the Higgs potential. Considering a non-renormalizable operator of the form

$$V_6 = -\frac{\omega}{\Lambda^2}(H^\dagger H)^3 \ , \tag{5.1}$$

may change the behaviour of the Higgs potential and the scale at which the instability appears. The Λ in V_6 may come from mass terms at tree level so that $\Lambda^2 \sim M^2$ or from one-loop contributions with $\Lambda^2 \sim (4\pi)^2 M^2$. Large changes in the instability scale indicate that the approximations used are no longer valid, rather than meaning there must be new physics at much lower energies than those obtained without considering these higher-order operators.

For the stable scalar candidate we do not need to calculate the Lagrangian to a higher order than we did. This is because, if we were to calculate the operator O_n for $n = 4$ we would obtain terms proportional to the trace of

$$U^4 \quad , \quad \partial^2 U^3 \quad , \quad U\partial^2 U^2 \quad , \quad U^2\partial^2 U \quad , \quad \dots \tag{5.2}$$

and we can see that we would not obtain contributions with six Higgs fields ($\propto U^3$) without derivatives. Therefore we only need to consider the contribution of the term proportional to $(H^\dagger H)^3$ in (4.45), for instance,

$$-\frac{\kappa^3}{24M^2(4\pi)^2}(H^\dagger H)^3 \; . \tag{5.3}$$

We consider that the only non-renormalizable operator that appears is similar to the one given in equation (5.1) and that $\omega > 0$ (since this sign gives effects that lower the instability scale and not increase it). For a given Higgs mass in the pure SM (that is, with $\omega = 0$), the scale of instability Λ^* is obtained requiring

$$\lambda(\mu)|_{\mu=\Lambda^*} = 0 \; . \tag{5.4}$$

Nonetheless, for $\omega \neq 0$ we also need to consider the term in (5.1). There are three possibilities to consider, because the mass of the new physics M need not be related to the instability scale Λ^*. One possibility is to have $M \ll \Lambda^*$. In this case the new physics would appear below Λ^*, and it may solve the instability problem and mean that there may not be any new physics at Λ^* at all. A theory describing the physics at scale M should be worked out before reaching any conclusions on the physics at Λ^*. Another possibility is the case $M \simeq \Lambda^*$, for which the expansion in powers of H/M stops working. This limits greatly any possible analysis. The third possibility is $M \gg \Lambda^*$. For this case effective low-energy calculations can actually give us information about how terms $(H^\dagger H)^3$ may influence Λ^*. The instability scale will be that for which the $(H^\dagger H)^2$(quartic) and $(H^\dagger H)^3$(non-renormalizable) contributions are comparable. The quartic term for $|H| \sim \Lambda^*$, writing $H^\dagger H = |H|^2$ in what follows, is

$$V_4 = \frac{1}{2}\lambda(\mu)|H(\mu)|^4 \; , \tag{5.5}$$

where the dependence of λ and H in Q is given at one loop by

$$\frac{d\lambda}{d\ln\mu} \equiv \beta_\lambda = \frac{1}{16\pi^2}\left[12(\lambda^2 - h_t^2 + \lambda_t^2) - (9g^2 + 3g'^2)\lambda + \frac{3}{4}(3g^4 + 2g^2g'^2 + g'^4)\right] \; , \tag{5.6}$$

$$\frac{d\ln|H|}{d\ln\mu} \equiv \gamma_H = \frac{1}{16\pi^2}\left[\frac{3}{4}(3g^2 + g'^2 - 4h_t^2)\right] \; , \tag{5.7}$$

with β_λ and γ_H obtained from reference [13]. In Eq. (5.6) we have that h_t is the top-quark Yukawa coupling and g and g' are the $SU(2)_L$ and $U(1)_Y$ gauge couplings, respectively. From (5.6) we can see how at high energies λ can evolve to negative values. The Yukawa coupling for the top quark, which grows for growing energies [14], overcomes the rest of contributions to λ at some point, bringing β_λ below zero. This makes λ negative at some scale. It can either be unbounded from below or it can have a minimum which, when deeper than the SM minimum, creates an instability. Now, keeping only first-order corrections in the evaluation of $\lambda(\mu)$ and $H(\mu)$ around $\mu \sim \Lambda^*$ (with Λ^* defined by $\lambda(\Lambda^*) = 0$) we have

$$V_4 = \frac{1}{2}\beta_\lambda(\Lambda^*)|H|^4 \left(1 + 4\gamma_H(\Lambda^*) \ln \frac{|H|}{\Lambda^*}\right) \ln \frac{|H|}{\Lambda^*} \ , \tag{5.8}$$

and

$$V_6 = -|H|^6 \left(\omega(\Lambda^*) + \beta_\omega(\Lambda^*) \ln \frac{|H|}{\Lambda^*}\right) \left(1 + 6\gamma_H(\Lambda^*) \ln \frac{|H|}{\Lambda^*}\right) \ , \tag{5.9}$$

with $\beta_\omega = \frac{d\omega}{d \ln Q}$, which can be read from reference [14]. The new instability scale (i.e. for $\omega \neq 0$) Λ^*_ω is obtained from the condition

$$V_4 + V_6 \simeq 0 \, . \tag{5.10}$$

Again retaining only first-order terms, this condition means

$$\beta_\lambda(\Lambda^*) \ln \frac{\Lambda^*_\omega}{\Lambda^*} \simeq 2 \left(\frac{\Lambda^*_\omega}{\Lambda}\right)^2 \left(\omega(\Lambda^*) + (\beta_\omega + 2\omega\gamma_H) \ln \frac{\Lambda^*_\omega}{\Lambda^*}\right) \ . \tag{5.11}$$

From this we can obtain an expression of the new instability scale in terms of the old one,

$$\Lambda^*_\omega \simeq \Lambda^* e^{\frac{2\omega(\Lambda^*)}{\beta_\lambda}\left(\frac{\Lambda^*}{\Lambda}\right)^2} \ . \tag{5.12}$$

As we said earlier, the Higgs quartic coupling λ goes to negative values for large enough energies. This means that β_λ will be a negative quantity and therefore we can have two possibilities with respect to the effect of the dimension-six operators in the stability of the Higgs potential. For $\omega < 0$ we have that the new scale for the instability will be greater than it is for the Standard Model, which has not many implications. Nonetheless, for $\omega > 0$ we have the instability appearing for lower values of the energy: $\Lambda^*_\omega < \Lambda^*$. This may mean that the scale for new physics (which could solve the instability problem of our theory) is lower than it is for the SM without these dimension-six operators.

6 Conclusions

The main purpose of this work has been to study the construction of one-loop effective field theories using functional methods. Effective field theories contain, in their couplings, information about a spectrum of particles much heavier than those of the Standard Model and, therefore, they let us study the effects of high energy physics at lower energies, they are an excellent tool to find and study hints of new physics beyond the Standard Model.

It is already well known that cosmological observations along the last half century indicate that most of our Universe is not composed by the matter and energy that we know. The overwhelming presence of dark matter and energy are one of the most relevant challenges that our comprehension of the Universe faces in the next years and maybe decades. The foreseen wide spectra of dark matter candidates endow a great part of the New Physics models and have become one of the goals to uncover in present colliders LHC and the future ones. With this in mind and by assuming that those particles are much heavier than those in the SM spectra we have studied "low-energy" settings from an effective theory point of view.

We have considered two different models for a heavy scalar neutral boson dark matter candidate. We have assumed that it only interacts with the SM spectra through the Higgs boson and, in addition, is a singlet under the SM gauge group. One of the models used the concept of the Higgs portal, with a stable scalar dark matter candidate, while the other one considered an unstable scalar. Though this latter is less popular due to the seemingly overpopulation of dark matter, it is clearly not excluded by present data. Hence we have employed functional techniques in order to integrate out the dark matter candidate in both models, and more interestingly, using a different procedure in each case. There is a notable contrast between both models when studying the structure of the integration: in the portal (stable) case the one loop diagrams only include heavier DM states and there is no tree level contribution; in the unstable case the diagrams include both heavy and light degrees of freedom. This latter case is more involved and has been a heated subject of debate in the last two years. In this Thesis we have applied a classical technique for the Higgs portal case and one of the latest for the unstable DM candidate.

As a possible application of the construction of the one-loop effective theory that has been carried out, we end this Thesis by considering the effect of new physics linked to the Higgs field in the behaviour and stability of the Higgs potential, studying how the inclusion of higher-order corrections decrease or increase the scale at which the instability in the Higgs potential appears with respect to the pure SM case.

Appendix A

A.1 Gaussian integral

Theresult shown in (4.9) can be easily proved expanding $\sigma(x)$ in terms of the eigenfunctions of the operator O:

$$\sigma(x) = \sum_n \alpha_n \sigma_n(x) \ ,$$

where the σ_n satisfy

$$O\sigma_n(x) = \lambda_n \sigma_n(x) \quad \text{and} \quad \int d^4x \sigma_n(x)\sigma_m(x) = \delta_{nm} \ . \tag{A.1}$$

With λ_n the eigenvalues of the operator O. Then, inserting the form of $\sigma(x)$ in terms of the $\sigma_n(x)$ into the Gaussian integral (4.9) we have that

$$I_G \propto \left[\prod_n \int_{-\infty}^{\infty} da_n\right] e^{-i\int d^4x \sum_{l=1}^{\infty} a_l\sigma_l(x) \sum_{k=1}^{\infty} a_k\sigma_k(x)\lambda_k} =$$

$$= \prod \int_{-\infty}^{\infty} da_n e^{-i\lambda_n a_n^2} \propto (\det O)^{-\frac{1}{2}} \ . \tag{A.2}$$

Where we have used both relations in (A.1).

It is usually quite difficult to evaluate the determinant of an operator, but we can re-express it in terms of a trace. For basis of finite dimensional matrices it is obvious that

$$e^{\text{tr}[\ln O]} = e^{\sum_n \ln \lambda_n} = \prod_n e^{\ln \lambda_n} = \prod_n \lambda_n = \det O \ . \tag{A.3}$$

Therefore we have that

$$\mathrm{tr}[\ln O] = \ln(\det O) \ . \tag{A.4}$$

A.2 Momentum integral

In order to obtain the effective Lagrangian (4.30) we need to solve the following momentum integral,

$$i \int \frac{d^D p}{(2\pi)^D} \frac{p_{\mu_1} p_{\mu_2} \dots p_{\mu_{2n}}}{(p^2 - M^2)^{2n}} \equiv I \ . \tag{A.5}$$

First of all we express the product of p's as a product of p^2. We can write

$$p^{\mu_1} p^{\mu_2} \dots p^{\mu_{2n}} = \frac{1}{D(D+2)\dots(D+2n)} S_n^{\mu_1\mu_2\dots\mu_{2n}} (p^2)^n \ , \tag{A.6}$$

where $S_n^{\mu_1\mu_2\dots\mu_{2n}}$ is a tensor completely symmetric built only using the metric tensor. For $n=1$, $S_1^{\mu_1\mu_2} = g^{\mu_1\mu_2}$ and for $n=2$, $S_2^{\mu_1\mu_2\mu_3\mu_4} = g^{\mu_1\mu_2}g^{\mu_3\mu_4} + g^{\mu_1\mu_3}g^{\mu_2\mu_4} + g^{\mu_1\mu_4}g^{\mu_2\mu_3}$. The factor of $D(D+2)\dots(D+2n)$ comes from considering all the equivalent ways there are of writing each term in $S_n^{\mu_1\mu_2\dots\mu_{2n}}$. Therefore we now wish to solve

$$J \equiv \int \frac{d^D p}{(2\pi)^D} \frac{(p^2)^n}{(p^2 - M^2)^{2n}} = i \int \frac{d^D p}{(2\pi)^D} \frac{(-p^2)^n}{(-p^2 - M^2)^{2n}} = i(-1)^n \int \frac{d^D p}{(2\pi)^D} \frac{(p^2)^n}{(p^2 + M^2)^{2n}} \ , \tag{A.7}$$

where in the first equality we have redefined $p^0 \to ip^0$. Now, using $d^D p = dp p^{D-1} d\Omega_D$ and later on changing the integration variable from p to $t = p^2/M^2$, which implies

$$dp = \frac{dp}{dp^2} dp^2 = \frac{M^2 dt}{2p} = \frac{M\, dt}{2t^{1/2}} \ , \tag{A.8}$$

we have

$$J = i\frac{(-1)^n}{(2\pi)^D} \int_0^\infty \frac{dp\, p^{2n+D-1}}{(p^2/M^2+1)^{2n}(M^2)^{2n}} \int d\Omega_D = i\frac{(-1)^n}{(2\pi)^D} \int_0^\infty \frac{M\, dt}{2t^{1/2}} \frac{t^{n+\frac{D-1}{2}} (M^2)^{\frac{D-1}{2}-n}}{(1+t)^{2n}} \int d\Omega_D \ . \tag{A.9}$$

Taking out of the integral everything that are not t's we have

$$J = i\frac{(-1)^n}{2(2\pi)^D} (M^2)^{D/2-n} \int d\Omega_D \int_0^\infty dt\, \frac{t^{n+D/2-1}}{(1+t)^{2n}} = i\frac{(-1)^n}{2(2\pi)^D} (M^2)^{D/2-n} \frac{2\pi^{D/2}}{\Gamma(D/2)} \int_0^\infty dt\, \frac{t^{n+D/2-1}}{(1+t)^{2n}} \ , \tag{A.10}$$

where we have used

$$\int d\Omega_D = \frac{2\pi^{D/2}}{\Gamma(D/2)} \ . \tag{A.11}$$

Clearly the integration in t looks like a Beta function. Using its definition in terms of Gamma functions, that is

$$\int_0^\infty \frac{t^{\alpha-1} dt}{(1+t)^\beta} = \frac{\Gamma(\alpha)\,\Gamma(\beta-\alpha)}{\Gamma(\beta)} \ , \tag{A.12}$$

we obtain

$$J = i\frac{(-1)^n}{(4\pi)^{D/2}} (M^2)^{D/2-n} \frac{\Gamma(n+D/2)\,\Gamma(n-D/2)}{\Gamma(D/2)\,\Gamma(2n)} \ . \tag{A.13}$$

In addition it is trivial to check that, up to n=3, the following result holds

$$\frac{\Gamma(n+D/2)}{D(D+2)\dots(D+2n)\Gamma(D/2)} = \frac{1}{2^n} \ . \tag{A.14}$$

Finally we have, for the momentum integral (A.5)

$$I = (-1)^{n+1}\left(\frac{M^2}{4\pi\mu^2}\right)^{\frac{D}{2}-2}\frac{1}{(4\pi)^2}\frac{\Gamma(n-D/2)}{M^{(2n-4)}2^n\Gamma(2n)}S_n^{\mu_1\mu_2\dots\mu_{2n}} \ . \tag{A.15}$$

Inserting this last result into (4.30) we finally obtain

$$\mathcal{L}_{eff}^{(1)} = \sum_{n=1}^{\infty}\left(\frac{M^2}{4\pi\mu^2}\right)^{\frac{D}{2}-2}\frac{1}{(4\pi)^2}\frac{\Gamma(n-\frac{D}{2})}{M^{2n-4}}\frac{2^{n-1}}{(2n)!}S_n^{\mu_1\mu_2\dots\mu_{2n}}\mathrm{tr}[N_{\mu_1}N_{\mu_2}\dots N_{\mu_{2n}}] = \frac{c_n}{2M^{2n-4}}O_n \ , \tag{A.16}$$

where clearly

$$c_n = \left(\frac{M^2}{4\pi\mu^2}\right)^{\frac{D}{2}-2}\frac{1}{(4\pi)^2}\Gamma\left(n-\frac{D}{2}\right) \quad , \quad O_n = \frac{2^n}{(2n)!}\mathrm{tr}[S_n(N)] \ , \tag{A.17}$$

with $S_n(N) \equiv S_n^{\mu_1\mu_2\dots\mu_{2n}}N_{\mu_1}N_{\mu_2}\dots N_{\mu_{2n}}$ the sum of all possible products of 2n contracted N's.

A.3 Transformation matrices

We now proceed to the calculation of the transformation matrices that let us express the γ's in terms of the δ's and vice versa. For $n=1$, M_n will be just a number:

$$\mathrm{tr}[U] = \mathrm{tr}[-N^2] = -\mathrm{tr}[N^2] \ , \tag{A.18}$$

so $M_1 = -1$. For $n=2$ it will be a 2x2 matrix. For the first row:

$$\mathrm{tr}[U^2] = \mathrm{tr}[(N^2)^2] \ , \tag{A.19}$$

so that $(M_2)_{11} = 0$ and $(M_2)_{12} = 1$. For the second row we have:

$$\mathrm{tr}[F_{\mu\nu}F^{\mu\nu}] = \mathrm{tr}\,[[N_\mu, N_\nu][N^\mu, N^\nu]] = 2\mathrm{tr}[(N_\mu N_\nu)^2] - 2\mathrm{tr}[(N^2)^2] \ , \tag{A.20}$$

so that $(M_2)_{21} = 2$ and $(M_2)_{22} = -2$. Consequently, M_2 is

$$M_2 = \begin{pmatrix} 0 & 1 \\ 2 & -2 \end{pmatrix} \ . \tag{A.21}$$

For $n=3$, we have a 3x3 matrix. For the first row we have:

$$\mathrm{tr}[U^3] = \mathrm{tr}[-(N^2)^3] = -\mathrm{tr}[(N^2)^3] \ , \tag{A.22}$$

so $(M_3)_{11} = (M_3)_{13} = (M_3)_{14} = (M_3)_{15} = 0$ and $(M_3)_{12} = -1$. For the second row:

$$\mathrm{tr}[(D_\mu U)^2] = \mathrm{tr}\left[-[N_\mu, N^2]^2\right] = 2\mathrm{tr}[(N_\mu N_\nu)^2] - 2\mathrm{tr}[(N^2)^3] \ , \tag{A.23}$$

so $(M_3)_{21} = (M_3)_{24} = (M_3)_{25} = 0$, $(M_3)_{22} = -2$ and $(M_3)_{23} = 2$. For the third row:

$$\mathrm{tr}[F_{\mu\nu} U F^{\mu\nu}] = \mathrm{tr}[-[N_\mu, N_\nu] N^2 [N^\mu, N^\nu]] = 2\mathrm{tr}[(N^2 N_\mu)^2] - 2\mathrm{tr}[N^2 (N_\mu N_\nu)^2] \,, \tag{A.24}$$

so $(M_3)_{31} = (M_3)_{32} = (M_3)_{34} = 0$, $(M_3)_{33} = 2$ and $(M_3)_{35} = -2$. For the fourth row:

$$\begin{aligned} \mathrm{tr}[D_\mu F^{\mu\nu} D^\sigma F_{\sigma\nu} &= \mathrm{tr}\,[[N_\mu, [N^\mu, N^\nu]][N^\sigma, [N_\sigma, N_\nu]]] = \\ = 2\mathrm{tr}[(N^2)^3] + 2\mathrm{tr}[(N^2 N_\mu)^2] &+ 4\mathrm{tr}[(N_\mu N_\nu N^\mu)^2] - 8\mathrm{tr}[N^2 (N_\mu N_\nu)^2] \,, \end{aligned} \tag{A.25}$$

so $(M_3)_{41} = 0$, $(M_3)_{42} = (M_3)_{43} = 2$, $(M_3)_{44} = 4$ and $(M_3)_{45} = -8$. For the fifth and final row:

$$\begin{aligned} \mathrm{tr}[F_{\mu\nu} F^{\nu\sigma} F^\mu_\sigma] &= \mathrm{tr}\,[[N_\mu, N_\nu][N^\nu, N^\sigma][N_\sigma, N^\mu]] = -\mathrm{tr}[(N_\mu N_\nu N_\sigma)^2] + \\ &+ \mathrm{tr}[(N^2)^3] + 3\mathrm{tr}[(N_\mu N_\nu N^\mu)^2] - 3\mathrm{tr}[N^2 (N_\mu N_\nu)^2] \,, \end{aligned} \tag{A.26}$$

so $(M_3)_{51} = -1$, $(M_3)_{52} = 1$, $(M_3)_{53} = 0$, $(M_3)_{54} = 3$ and $(M_3)_{55} = -3$. Taking it all together we have for M_3:

$$M_3 = \begin{pmatrix} 0 & -1 & 0 & 0 & 0 \\ 0 & -2 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 & -2 \\ 0 & 2 & 2 & 4 & -8 \\ -1 & 1 & 0 & 3 & -3 \end{pmatrix} . \tag{A.27}$$

Now taking the transpose and then inverting the resulting matrices we obtain:

$$(M_1^T)^{-1} = -1 \qquad ; \qquad (M_2^T)^{-1} = \begin{pmatrix} 1 & 1 \\ 1/2 & 0 \end{pmatrix} , \tag{A.28}$$

$$(M_3^T)^{-1} = -\frac{1}{4} \begin{pmatrix} 4 & 4 & 4 & 4 & 4 \\ -3 & 0 & -2 & -3 & -2 \\ 6 & 0 & 0 & 4 & 2 \\ -3 & 0 & 0 & 1 & 0 \\ 4 & 0 & 0 & 0 & 0 \end{pmatrix} . \tag{A.29}$$

A.4 One-loop amplitude for HH to HH for the stable candidate

In order to solve equation (4.48) we can rewrite the factor to be integrated using a trick developed by Feynman (and explained, for example, in Chapter 6 of [11]):

$$\frac{1}{AB} = \int_0^1 dx \frac{1}{[Ax + (1-x)B]^2} \,. \tag{A.30}$$

Writing $A = [(q+p)^2 - M^2]$ and $B = p^2 - M^2$ we have that

$$\begin{aligned} \frac{1}{[(q+p)^2 - M^2][p^2 - M^2]} &= \int_0^1 dx \frac{1}{[(q+p)^2 x - M^2 x + (1-x)(p^2 - M^2)]^2} = \\ &= \int_0^1 dx \frac{1}{[(p+qx)^2 - L^2]^2} \,, \end{aligned} \tag{A.31}$$

where $L^2 = q^2x(x-1) + M^2$. Inserting this into our expression for the amplitude A and redefining the integration variable via the transformation $k + qx \to k$, we arrive at

$$iA_{full} = \frac{\kappa^2}{8(2\pi)^D} \int d^Dp \int_0^1 \frac{dx}{(p^2 - L^2)^2} \ . \tag{A.32}$$

Using a general result of dimensional regularization, we get

$$\begin{aligned}
iA_{full} &= \frac{i\kappa^2}{8(2\pi)^D} \int_0^1 dx \quad i\pi^{D/2} \frac{\Gamma(2 - D/2)}{\Gamma(2)} (L^2)^{D/2-2} \\
&= \frac{i\kappa^2}{8(4\pi)^2} \int_0^1 dx \Gamma\left(\frac{\epsilon}{2}\right) \left(\frac{L^2}{4\pi\mu^2}\right)^{-\frac{\epsilon}{2}} = \frac{i\kappa^2}{8(4\pi)^2} \int_0^1 dx \left(\frac{2}{\epsilon} - \gamma\right) \left(1 - \frac{\epsilon}{2} \ln \frac{L^2}{4\pi\mu^2}\right) \\
&= \frac{i\kappa^2}{8(4\pi)^2} \int_0^1 dx \left[\frac{2}{\epsilon} - \gamma + \ln 4\pi - \ln \frac{q^2}{\mu^2} - \ln\left(\frac{M^2}{q^2} - x(1-x)\right)\right] \ ,
\end{aligned} \tag{A.33}$$

where we have used the form of L^2 to write the last term. Using the result in equation (C.55) in [12] with n=0 and $u = M^2/q^2$, we obtain

$$iA_{full} = \frac{i\kappa^2}{8(4\pi)^2} \left[\frac{2}{\epsilon} - \gamma + \ln 4\pi - \ln \frac{M^2}{\mu^2} + 2 - \sqrt{1 - \frac{4M^2}{q^2}} \ln \left(\frac{\sqrt{1 - \frac{4M^2}{q^2}} + 1}{\sqrt{1 - \frac{4M^2}{q^2}} - 1}\right)\right] \ . \tag{A.34}$$

And finally we have for the amplitude of the process $HH \to HH$ to one loop in the full theory

$$iA_{full} = \frac{i\kappa^2}{8(4\pi)^2} \left[\Delta + 2 - \sqrt{1 - \frac{4M^2}{q^2}} \ln \left(\frac{\sqrt{1 - \frac{4M^2}{q^2}} + 1}{\sqrt{1 - \frac{4M^2}{q^2}} - 1}\right)\right] \ . \tag{A.35}$$

A.5 Calculation of U and the one-loop effective Lagrangian for the unstable candidate

First of all we need to obtain the inverse of the Δ_l. Using equation (16) of reference [4] we have

$$\Delta_l^{-1} = \tilde{\Delta}_l^{-1} - \tilde{\Delta}_l^{-1} X_l \tilde{\Delta}_l^{-1} + (\tilde{\Delta}_l^{-1} X_l)^2 \tilde{\Delta}_l^{-1} - \dots \tag{A.36}$$

where

$$\tilde{\Delta}_l^{-1} = \begin{pmatrix} \Delta^{-1} & 0 \\ 0 & (\Delta^T)^{-1} \end{pmatrix} \quad X_l = \begin{pmatrix} 0 & \delta^\dagger \\ \delta & 0 \end{pmatrix} \ , \tag{A.37}$$

with $\Delta \equiv \Delta_{\eta\eta^*}$ and $\delta \equiv \delta_{\eta\eta^*}$, as defined in (4.71). Therefore we have for U, given in (4.72),

$$\begin{aligned}
U &= X_\eta^\dagger \Delta^{-1} X_\eta + X_{\eta^*}^\dagger (\Delta^T)^{-1} X_{\eta^*} - X_\eta^\dagger \Delta^{-1} \delta^\dagger (\Delta^T)^{-1} X_{\eta^*} - X_{\eta^*}^\dagger (\Delta^T)^{-1} \delta \Delta^{-1} X_\eta + \\
&+ X_\eta^\dagger \Delta^{-1} \delta^\dagger (\Delta^T)^{-1} \delta \Delta^{-1} X_\eta + X_{\eta^*}^\dagger (\Delta^T)^{-1} \delta \Delta^{-1} \delta^\dagger (\Delta^T)^{-1} X_{\eta^*} + \frac{3\lambda_S \alpha^2}{M^4} (H^\dagger H)^2 \ .
\end{aligned} \tag{A.38}$$

Using equation (B.11) from [4] in order to write Δ^{-1}, with

$$\Omega = \hat{D}^2 + \left(\lambda - \frac{\alpha^2}{M^2}\right) (\hat{H}^\dagger \hat{H}) + \lambda \hat{H} \hat{H}^\dagger \ , \tag{A.39}$$

we can see that

$$-X_\eta^\dagger \Delta^{-1}\delta^\dagger (\Delta^T)^{-1} X_{\eta^*} = \alpha^2\lambda \hat{H}^\dagger [\frac{1}{p^4}\left(1+2\frac{m^2+\Omega}{p^2}\right)(\hat{H}\hat{H}^\dagger) + 2\frac{p_\mu}{p^6}(\hat{H}^\dagger\hat{H}\hat{D}^{*\mu} + \hat{D}^\mu \hat{H}^\dagger \hat{H}) - $$
$$- 4\frac{p_\mu p_\nu}{p^8}(\hat{H}^\dagger\hat{H}\hat{D}^{*\mu}\hat{D}^{*\nu} + \hat{D}^\mu\hat{H}^\dagger\hat{H}\hat{D}^{*\nu} + \hat{D}^\mu\hat{D}^\nu\hat{H}^\dagger\hat{H})]\hat{H} \, , \tag{A.40}$$

and $X_{\eta^*}^\dagger (\Delta^T)^{-1}\delta\Delta^{-1}X_\eta$ has the same expression but doing the substitution $\hat{D} \leftrightarrow \hat{D}^*$. The terms corresponding to $n=2$ will each only contribute with

$$\alpha^2\lambda^2\frac{1}{p^6}(\hat{H}^\dagger\hat{H})^3 \, . \tag{A.41}$$

Taking it all together we have for U

$$U = \alpha^2\hat{H}^\dagger\Bigg[\frac{3\lambda_S}{M^4}(H^\dagger H) + \frac{2}{p^2} + \frac{1}{p^4}\Bigg((2m^2 + 2\Omega - 4\frac{p_\mu p_\nu}{p^2}(\hat{D}^\mu\hat{D}^\nu + \hat{D}^{*\mu}\hat{D}^{*\nu}) + \lambda(\hat{H}^\dagger\hat{H} + \hat{H}\hat{H}^\dagger\Bigg) +$$
$$+ \frac{1}{p^6}\Big(2m^4 + 2\Omega^2 + 4m^2\Omega + 2\lambda(m^2+\Omega)(\hat{H}^\dagger\hat{H} + \hat{H}\hat{H}^\dagger) + \lambda^2(\hat{H}^\dagger\hat{H})^2 -$$
$$- 12\frac{p_\mu p_\nu}{p^2}m^2(\hat{D}^\mu\hat{D}^\nu + \hat{D}^{*\mu}\hat{D}^{*\nu}) + 16\frac{p_\mu p_\nu p_\rho p_\sigma}{p^4}(\hat{D}^\mu\hat{D}^\nu\hat{D}^\rho\hat{D}^\sigma + \hat{D}^{*\mu}\hat{D}^{*\nu}\hat{D}^{*\rho}\hat{D}^{*\sigma}) -$$
$$- 4\frac{p_\mu p_\nu}{p^2}(\hat{D}^\mu\hat{D}^\nu\Omega + \Omega\hat{D}^\mu\hat{D}^\nu + \hat{D}^\mu\Omega\hat{D}^\nu + \hat{D}^{*\mu}\hat{D}^{*\nu}\Omega + \Omega\hat{D}^{*\mu}\hat{D}^{*\nu} + \hat{D}^{*\mu}\Omega\hat{D}^{*\nu}) -$$
$$- 4\lambda\frac{p_\mu p_\nu}{p^2}(\hat{D}^\mu\hat{D}^\nu(\hat{H}^\dagger\hat{H}) + (\hat{H}^\dagger\hat{H})\hat{D}^\mu\hat{D}^\nu + \hat{D}^\mu(\hat{H}^\dagger\hat{H})\hat{D}^\nu +$$
$$+ \hat{D}^{*\mu}\hat{D}^{*\nu}(\hat{H}\hat{H}^\dagger) + (\hat{H}\hat{H}^\dagger)\hat{D}^{*\mu}\hat{D}^{*\nu} + \hat{D}^{*\mu}(\hat{H}\hat{H}^\dagger)\hat{D}^{*\nu})\Big)\Bigg]\hat{H} + O(\zeta^{-7}) \tag{A.42}$$

In order to simplify U and the calculation of $\mathcal{L}^{(1)}$ we are going to ignore all terms in U containing only two Higgs fields, since these would only give a correction to the mass, something in which we are not interested. We are also going to ignore the terms with six Higgs fields since the calculation of all the contributions containing this amount of H fields is too long for the time we have available. Also, we are only going to consider terms with four Higgs fields and no gauge fields (which come from the covariant derivatives present in U). In practice this means ignoring terms proportional to $H^\dagger D_\mu H D^\mu (H^\dagger H)$ and changing all the covariant derivatives for partial derivatives once all the simplifications and integrations occur.

In the simplification of U we have also used the following relation

$$D_\mu^* = -D_\mu^T \, , \tag{A.43}$$

in order to deal with the different D^*'s. The transpose of D_μ acts on everything it has on its left, in the same way as D_μ itself acts on everything to the right of it. Also we have used the fact that the trace of a number and its transpose are the same in order to group terms which seemed different at first sight.

The momentum integrals appearing in the calculation of $\mathcal{L}^{(1)}$ can be easily performed using the following result from reference [4]

$$\int \frac{d^D p}{(2\pi)^D}\frac{p_{\mu_1}\dots p_{\mu_{2k}}}{(p^2)^\alpha (p^2 - M^2)^\beta} = \frac{i(-1)^{\alpha+\beta+k}}{(4\pi)^{\frac{D}{2}}}\frac{\Gamma(\frac{D}{2}+k-\alpha)\Gamma(-\frac{D}{2}-k+\alpha+\beta)}{\Gamma(\beta)\Gamma(\frac{D}{2}+k)} \times$$
$$\times \frac{S_{\mu_1\dots\mu_{2k}}}{2^k}\left(M^2\right)^{\frac{D}{2}+k-\alpha-\beta} \, . \tag{A.44}$$

Now, using (A.44), and using the $\bar{MS}$ regularization scheme, setting our scale as $\mu = M$, we have for each one of the momentum integrals (writing $\int \frac{d^D p}{(2\pi)^D} = \int_p$):

$$
\begin{aligned}
\int_p \frac{1}{(p^2 - M^2)} &= \frac{iM^2}{(4\pi)^2}(\Delta + 1) \, , \\
\int_p \frac{1}{p^4(p^2 - M^2)} &= \frac{i}{(4\pi)^2 M^2}(\Delta + 1) \, , \\
\int_p \frac{1}{p^6(p^2 - M^2)} &= \frac{i}{(4\pi)^2 M^4}(\Delta + 1) \, , \\
\int_p \frac{p_\mu p_\nu}{p^8(p^2 - M^2)} &= \frac{g_{\mu\nu}}{D} \int_p \frac{1}{p^6(p^2 - M^2)} = \frac{i g_{\mu\nu}}{(4\pi)^2 M^4} \frac{1}{4} \left(\Delta + \frac{3}{2} \right) \, .
\end{aligned}
\tag{A.45}
$$

Taking all this into account we arrive at

$$
\mathcal{L}^{(1)}_{eff} = \frac{\alpha^2}{32\pi^2} \left[\left(\frac{3}{M^2}(\lambda_S + 2\lambda) + \frac{2}{M^4}(13m^2\lambda - \alpha^2) \right) (H^\dagger H)^2 - \frac{\lambda}{2M^4}(H^\dagger H)\partial^2(H^\dagger H) \right] + O(M^{-6}) \, . \tag{A.46}
$$

We have not given the terms proportional to M^{-6} because they are incomplete since we have not considered higher orders in the expansion of (4.68).

References

[1] V. Lukovic, P. Cabella and N. Vittorio, Int. J. Mod. Phys. A **29** (2014)

[2] K. Assamagan *et al.*, arXiv:1604.05324 [hep-ph].

[3] S. Dittmaier and C. Grosse-Knetter, Phys. Rev. D **52** (1995)

[4] J. Fuentes-Martin, J. Portoles and P. Ruiz-Femenia, arXiv:1607.02142 [hep-ph].

[5] C. P. Burgess, Ann. Rev. Nucl. Part. Sci. **57** (2007) 329

[6] M. S. Bilenky and A. Santamaria, Nucl. Phys. B **420** (1994)
R. D. Ball, Phys. Rept. **182** (1989) 1.

[7] C. M. Fraser, Z. Phys. C **28** (1985) 101.
I. J. R. Aitchison and C. M. Fraser, Phys. Lett. B **146** (1984) 63.
I. J. R. Aitchison and C. M. Fraser, Phys. Rev. D **31** (1985) 2605.
I. J. R. Aitchison and C. M. Fraser, Phys. Rev. D **32** (1985) 2190.
L. H. Chan, Phys. Rev. Lett. **54** (1985) 1222
L. H. Chan, Phys. Rev. Lett. **57** (1986) 1199.
M. K. Gaillard, Nucl. Phys. B **268** (1986) 669.
O. Cheyette, Phys. Rev. Lett. **55** (1985) 2394.

[8] B. Batell, M. Pospelov and A. Ritz, Phys. Rev. D **80** (2009)

[9] S. Baek, P. Ko, W. I. Park and E. Senaha, JHEP **1305** (2013)

[10] J. F. Donoghue, E. Golowich and B. R. Holstein, Camb. Monogr. Part. Phys. Nucl. Phys. Cosmol. **2** (1992)

[11] M. E. Peskin and D. V. Schroeder, "An Introduction to quantum field theory," Reading, USA: Addison-Wesley (1995) 842 p

[12] P. Pascual and R. Tarrach, "Qcd: Renormalization For The Practitioner," Lect. Notes Phys. **194** (1984) 1.

[13] C. P. Burgess, V. Di Clemente and J. R. Espinosa, JHEP **0201** (2002) 041

[14] B. Grzadkowski and J. Wudka, Phys. Rev. Lett. **88** (2002) 041802